H^∞ CONTROL THEORY ON THE INFINITE DIMENSIONAL SPACE

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Abstract: In this paper, the VH^∞ control theory on an infinite dimensional algebra to itself is presented. In order to establish the VH^∞ control theory, the concept and the properties of a meromorphic mapping and the theory of VH^∞ spaces on an infinite dimensional algebra to itself are founded.

1 INTRODUCTION

The theory of H^p-spaces and the H^∞ control theory on finite dimensional spaces have been summarized by C. G. Hu and C. C. Yang (Hu and Yang, 1992), and B. A. Francis and J. C. Doyle ((Francis, 1987), (Francis and Doyle, 1987)) respectively. In 1993, B. V. Keulen extended the H^∞ control theory on finite dimensional spaces to range in the infinite dimensional Hilbert space (Keulen, 1993). In 2002, C. G. Hu and L. X. Ma extended the result of Keulen to the locally convex space containing the Hilbert space (Hu and Ma, 2002). In this article, the VH^∞ control theory on an infinite dimensional algebra to itself is presented in Section 4. For this aim, the meromorphic mapping (in Section 2) and the theory of VH^p spaces on an infinite dimensional algebra without appearance in books ((Dineen, 1981) and (Mujica, 1986)) respectively, are given (in Section 3). The VH^∞ control theory can enlarge the scope of solutions in the control theory. So the research on these problems can develop and complete the control theory.

2 MEROMORPHIC MAPPINGS

Let $\mathcal{S}$ be the sequence space of all complex variables. Here $z = (z_1, z_2, \ldots, z_j, \ldots) \in \mathcal{S}$ and z_j is in the complex plane $\mathbb{C}_j$ for any j. If $z = (z_1, z_2, \ldots, z_j, \ldots) \in \mathcal{S}$, then the quasinorm over $\mathcal{S}$ is defined by

$$\|z\| = \sum_{j=1}^{\infty} \left(\frac{2}{3}\right)^j \frac{|z_j|}{1+|z_j|}.$$

The multiplication of z and w in $\mathcal{S}$ can be defined by

$$zw = (z_1w_1, z_2w_2, \ldots, z_jw_j, \ldots).$$

From the definition of the multiplication we may derive

$$\|zw\| \leq \|z\|\|w\|, \;\; z^k = (z_1^k, \ldots, z_j^k, \ldots)$$

for any $k > 0$. Thus $\mathcal{S}$ is a Fréchet algebra.

Assume for simplicity, that $L = \prod_{j=1}^{\infty} L_j$ is a manifold over $\mathcal{S}$, where each $L_j \subset \mathbb{C}_j$ is a simple path, and that $f(t) = (f_1(t), \ldots, f_j(t), \ldots) : L \to \mathcal{S}$, where t is over L and $t = (t_1, t_2, \ldots, t_j, \ldots) \in \mathcal{S}$. Let $D_s = \prod_{j=1}^{\infty} D_{sj}$ be a domain over $\mathcal{S}$, where D_{sj} is a domain over $\mathbb{C}_j$.

Theorem 2.1. *A mapping $f : D_s \to \mathcal{S}$ is holomorphic if and only if f can be denoted by*

$$f(z) = (f_1(z_1), f_2(z_2), \ldots, f_j(z_j), \ldots) \in \mathcal{S},$$

where $z = (z_1, z_2, \ldots, z_j, \ldots) \in \mathcal{S}$ and $f_j(z_j) : \mathbb{C}_j \to \mathbb{C}_j$ is a holomorphic function.

Proof. If f is a holomorphic mapping in D_s, then for any $z_0 = (z_{01}, z_{02}, \ldots, z_{0j}, \ldots) \in D_s$, there exists a

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neighborhood $U(z_0) \subset D_s$ such that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \alpha(k)(z-z_0)^k \\ &= \sum_{k=0}^{\infty}(\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{kj}, \ldots) \\ &\quad ((z_1 - z_{01})^k, \ldots, (z_j - z_{0j})^k, \ldots)) \\ &= \big(\sum_{k=0}^{\infty} \alpha_{k1}(z_1 - z_{01})^k, \ldots, \\ &\quad \sum_{k=0}^{\infty} \alpha_{kj}(z_j - z_{0j})^j, \ldots\big) \\ &= \big(f_{01}(z_1 - z_{01}), f_{02}(z_2 - z_{02}), \ldots, \\ &\quad f_{0j}(z_j - z_{0j}), \ldots\big), \end{aligned}$$

for $z \in U(z_0)$, where $\alpha(k) = (\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{kj}, \ldots) \in \mathcal{S}$ and

$$f_{0j}(z_j - z_{0j}) = \sum_{k=0}^{\infty} \alpha_{kj}(z_j - z_{0j})^k.$$

Analytic continuation and the Uniqueness Theorem of holomorphic mappings in complex analysis yield that $f(z)$ can be written as

$$f(z) = \big(f_1(z_1), f_2(z_2), \ldots, f_j(z_j), \ldots\big) \in \mathcal{S}.$$

This is just required conclusion.

Conversely, because the above each step is invertible, f is holomorphic in D_s.

This proof is ended.

A meromorphic mapping on $\mathcal{S}$ without appearance in books ((Dineen, 1981) and (Mujica, 1986)) respectively, may be defined as follows:

Definition 2.1. A mapping f on $\mathcal{S}$ is called meromorphic if its each component $f_j(z_j)$ is a meromorphic mapping of z_j over $\mathbb{C}_j$ for each j.

Remark. Using the similar method to Definition 2.1 we may define a meromorphic mapping on a domain $D_s = \prod_{j=1}^{\infty} D_{sj}$, where D_{sj} in $\mathbb{C}_j$ is a domain.

Let

$$\left\{z(k)^{(0)}\right\}_{k=1}^{\infty} = \left\{(z_{k1}^{(0)}, \ldots, z_{kj}^{(0)}, \ldots)\right\}_{k=1}^{\infty} \quad (2.1)$$

be an increasing sequence with distinct complex elements tending to the infinity $\infty = (\infty, \ldots, \infty, \ldots)$ in the sense of the quasinorm. From the above definition for convenience sake, without loss generality we may assume that every f_j is a meromorphic function of z_j which can be written for any j as

$$\sum_{k=1}^{\infty} \sum_{\imath_{k\varrho j}=-m_{kj}}^{\infty} \alpha_{k\varrho j}(z_j - z_{kj}^{(0)})^{\imath_{k\varrho j}},$$

where $-\infty < -m = \inf\{\imath_{k\varrho j} : k, \varrho, j = 1, 2, \ldots\} < 0$ and the following conditions are satisfied:

$(\alpha)\{z_{kj}^{(0)}\}$ for any k, j has no any finite limit point;

$(\beta) -\infty < \inf\{-m_{kj}\}$.

Under the preceding conditions $z(k)^{(0)}$ is called a pole of f.

On a meromorphic mapping $f(z)$ on $\mathcal{S}$ there is the following conclusion.

Theorem 2.2. *Let $\{z(k)^{(0)}\}_{k=1}^{\infty}$ satisfy (α)–(β), and let*

$$\begin{aligned} &\left\{\hbar_{(k)}(z - z(k)^{(0)})\right\}_{k=1}^{\infty} \\ &= \left\{(h_{k1}(z_1 - z_{k1}^{(0)}), \ldots, h_{kj}(z_j - z_{kj}^{(0)}), \ldots)\right\}_{k=1}^{\infty} \end{aligned}$$

be a sequence, where

$$h_{kj}(z_j - z_{kj}^{(0)}) = \sum_{-m_k}^{-1} \alpha_{k\varrho j}(z_j - z_{kj}^{(0)})^{\imath_{k\varrho j}},$$

and the coefficient of the first term after the equal-sign in the above formula is not 0*. Then there exists a meromorphic mapping*

$$f(z) = \sum_{k=1}^{\infty} \sum_{\imath_{k\varrho}=-m_k}^{\infty} \alpha(k\varrho)(z - z(k)^{(0)})^{\imath_{k\varrho}},$$

its poles coincide with (2.1)*, and its principal part at the pole $z(k)^{(0)}$ equals $\hbar_{(k)}$, for each $k = 0, 1, 2, \ldots$ and $\alpha(k\varrho) = (\alpha_{k\varrho 1}, \ldots, \alpha_{k\varrho j}, \ldots) \in \mathcal{S}$.*

Proof. In the proof of Theorem 2.1 we replace the power series $\sum_{k=0}^{\infty} \alpha_{kj}(z_j - z_{0j})^k$ by the Laurent series $\sum_{-m_k}^{\infty} \alpha_{k\varrho j}(z_j - z_{kj}^{(0)})^{\imath_{k\varrho j}}$. And using the famous Mittag–Leffler's theorem in complex analysis for each component we may obtain required result. This proof is finished.

Next, integrals can be defined on a manifold L in $\mathcal{S}$ as follows.

Definition 2.2. Let L_j be any closed rectifiable Jordan curve contained in a simply connected subdomain of a domain G_j in $\mathbb{C}_j$ and $L = \prod_{j=1}^{\infty} L_j$. Let D_j^+ be the interior of L_j and $D^+ = \prod_{j=1}^{\infty} D_j^+$. Then D^+ is called the interior of L, and

$$\int_L f(z)dz := \left(\int_{L_1} f_1(z_1)dz_1, \ldots, \int_{L_j} f_j(z_j)dz_j, \ldots\right)$$

in $\mathcal{S}$, where $f = (f_1, \ldots, f_j, \ldots)$ is defined on $\mathcal{S}$.

Theorem 2.3. *Let $f(z)$ be a single $\mathcal{S}$-valued holomorphic mapping on G. Then*
(A)(Cauchy's integral theorem).

$$\int_L f(z)dz = 0,$$

where $G = \prod_{j=1}^{\infty} G_j$;

(B)(Cauchy's integral formula).

$$\frac{1}{2\pi i}\int_L f(t)(t-z)^{-1}dz = f(z),$$

where $z \in D^+ \subset G$, *and* $(t-z)^{-1}$ *exists.*

Definition 2.3. A subset X_0 of X is called a base-real subspace of X and $\bar{u}$ is called the conjugate element of u if following conditions hold:

a) X is a vector space on $\mathbb{C}$.

b) X_0 is a vector subspace of X on $\mathbb{R}$.

c) For every $u \in X$, there exists an $\bar{u}(\in X)$ such that $u + \bar{u} \in X_0$ and $i(u - \bar{u}) \in X_0$ hold satisfying a unique decomposition $u = \xi + \eta i$ for $\xi, \eta \in X_0$.

d) $X_0 \cap iX_0 = \{0\}$, where 0 is the zero element.

Theorem 2.4. *If X is a complex vector space, then there exists a base-real subspace X_0 such that $X = X_0 + iX_0$, i.e. a complex vector space can be represented by a direct sum of two spaces which are generated by some real vector space.*

Proof. For any $x_0(\neq \theta) \in X$, let $M_0 = \mathbb{R}x_0 = \{tx_0 : t \in \mathbb{R}\}$. Then M_0 is a vector subspace of X on the restricted number field $\mathbb{R}$, and $M_0 \cap iM_0 = \{\theta\}$. Setting $\mathbb{C}M_0 = \{zm_0 : z \in \mathbb{C}, m_0 \in M_0\}$, we have that $\mathbb{C}M_0$ is a complex vector subspace of X. For any $x_1 \in X\backslash\mathbb{C}M_0$ we know that $M_1 = M_0 + \mathbb{R}x_1$ is also a vector subspace on a restricted number field $\mathbb{R}$ of X and $M_1 \cap iM_1 = \{\theta\}$. By induction we obtain a sequence $\{M_n\}$ of vector subspaces with $M_n \cap iM_n = \{\theta\}$. Assume that M is the family of all vector subspaces on $\mathbb{R}$, that $M' = \{M^0 \in M : M^0 \cap iM^0 = \{\theta\}\}$ (clearly, M' is nonempty), and that $\{M_\alpha\}_{\alpha\in J}$ is the family of totally ordered subsets of M', where J is an indexing set. Consequently, $X_M = \cup_{\alpha\in J} M_\alpha$ is a vector subspace on $\mathbb{R}$ and the supremum of $\{M_\alpha\}_{\alpha\in J}$. Further we have

$$\begin{aligned} X_M \cap iX_M &= \bigcup_{\alpha\in J} M_\alpha \cap i\bigcup_{\alpha\in J} M_\alpha \\ &= \bigcup_{\alpha\in J} M_\alpha \cap i\bigcup_{\beta\in J} M_\beta \\ &= \bigcup_{\alpha\in J}\bigcup_{\beta\in J}(M_\alpha \cap iM_\beta). \end{aligned}$$

Since $\{M_\alpha\}_{\alpha\in J}$ is a family of totally ordered subsets with $M_\alpha \cap iM_\alpha = \{\theta\}$ for any $\alpha \in J$ and $M_\alpha \cap iM_\beta = \{\theta\}$ for any $\alpha, \beta \in J$, we get $X_M \cap iX_M = \{\theta\}$. Now Zorn's lemma yields that M' has the maximum element X_0.

Next we shall show that X_0 is the required subspace. Firstly, $\mathbb{C}X_0 = X$, here $\mathbb{C}X_0$ is the smallest complex subspace containing X_0. In fact, if $\mathbb{C}X_0 \neq X$, then there exists an $x \in X\backslash\mathbb{C}X_0$. It follows that $X_0' = X_0 + \mathbb{R}x$ is a vector subspace on $\mathbb{R}$ containing X_0 with $X_0' \cap iX_0' = \{\theta\}$. This is in contradiction with the maximality of X_0. Obviously, $\mathbb{C}X_0 = X_0 + iX_0$. Since $X_0 \cap iX_0 = \{\theta\}$, $X = X_0 + iX_0$. If $u = x + iy \in X$ (here $x, y \in X_0$), then $\bar{u} = x - iy$ is the conjugate element of u, i.e. X_0 is a base real subspace of X.
This proof is finished.

Suppose that $\mathfrak{e} \in \mathcal{S}$ is the idempotent element. Theorem 5.3.2 (Hille and Phillips, 1957) may be extended to the Fréchet algebra $\mathcal{S}$ containing the Banach algebra. Then using the result after extending we can get

$$\exp(\mathrm{Log})z = z, \quad \exp(z + 2\pi i\mathfrak{e}) = \exp z,$$

where $\exp z = \sum_{j=1}^{\infty} z^j/j!$. If z and $\mathfrak{e}$ commute, then

$$\exp(z + 2\pi in\mathfrak{e}) = \exp z, \quad \text{for } n = 0, \pm1, \pm2, \ldots.$$

It follows that

$$\mathrm{Log}[\exp z] = z + 2\pi in\mathfrak{e} = z' + i(z'' + 2\pi n\mathfrak{e}), \quad (2.2)$$

for any integer n and $z', z'' + 2\pi n\mathfrak{e} \in \mathcal{S}_0$, where $\mathcal{S} = \mathcal{S}_0 + i\mathcal{S}_0$, $\mathcal{S}_0$ is a base-real Fréchet algebra. From (2.2) we can define the argument of $\exp z$ being $z'' + 2\pi in\mathfrak{e}$. Because

$$\exp z'z'' = (\exp z_1'z_1'', \ldots, \exp z_j'z_j'', \ldots),$$

$$\mathrm{Log}z = (\log|z_1|, \ldots, \log|z_j|, \ldots)$$

$$+i(\mathrm{Arg}z_1, \ldots, \mathrm{Arg}z_j, \ldots).$$

It follows that the argument $\mathrm{Arg}z$ of z may be defined by

$$(\mathrm{Arg}z_1, \ldots, \mathrm{Arg}z_j, \ldots). \quad (2.3)$$

3 THE VH^∞ SPACES

Let $|f(z)| = (|f_1(z_1)|, \ldots, |f_j(z_j)|, \ldots)$ be the vector modulus of f and $\|f\| = (\|f_1\|, \ldots, \|f_j\|, \ldots)$ the vector norm of f. For any $a, b \in \mathcal{S}$, $a \leq (<)b$ is $a_j \leq (<)b_j$ for each j. Let $\mathbb{C}_j^+ = \{z \in \mathbb{C}_j : \Re z > 0\}$ and $\mathcal{S}^+ = \prod_{j=1}^{\infty}\mathbb{C}_j^+$. The set $VH^p(\mathcal{S}^+)$ consists of all holomorphic mappings $f : \mathcal{S}^+ \to \mathcal{S}$ satisfying

$$\sup_{\Re z>0}\left\{\int_{\mathfrak{I}}|f(x+iy)|^p dy\right\}^{\frac{1}{p}}(=\|f\|_p)<\infty,$$

where $\mathfrak{I}=\prod_{j=1}^{\infty}\{(-\infty,\infty)\}$, $z=x+iy\in\mathcal{S}^+$, and $0<p<\infty$. The set $VH^{\infty}(\mathcal{S}^+)$ consists of all holomorphic mappings $f:\mathcal{S}^+\to\mathcal{S}$ satisfying

$$\sup_{\Re z>0}\{|f(x+iy)|\}(=\|f\|_{\infty})<\infty.$$

Theorem 3.1. It $f\in VH^p(\mathcal{S}^+)$ with $1\le p$, then f is written as

$$f(z)=\frac{1}{\pi}\int_{\mathfrak{I}}xf(it)[x^2+(y-t)^2]^{-1}dt, z\in\mathcal{S}^+$$

where $f(it)\in VL^p(\mathfrak{I})$.

In order to show Theorem 3.1, the following two lemmas are required.

Lemma 3.1. *If $f\in VH^p(\mathcal{S}^+)$, then there exists a constant element c such that*

$$|f(z)|\le cx^{-\frac{1}{p}}, z=x+iy\in\mathcal{S}^+.$$

Proof. Let $\mathfrak{L}=\prod_{j=1}^{\infty}\{(0,2\pi)\}$ and $\mathfrak{R}=\prod_{j=1}^{\infty}\{(0,r)\}$. Let $\mathfrak{X}=\prod_{j=1}^{\infty}\{(x_j-r_j,x_j+r_j)\}$ and $\mathfrak{Y}=\prod_{j=1}^{\infty}\{(y_j-r_j,y_j+r_j)\}$. Because $|f(z)|^p$ is a subharmonic mapping (Hoffman, 1962),

$$|f(z)|^p\le\frac{1}{2\pi}\int_{\mathfrak{L}}|f(z+\rho e^{i\theta})|^p d\theta, 0<\rho<x.$$

Product the two sides of the above formula by ρ and integrate on $\mathfrak{R}$ with respect to ρ. It follows that

$$\begin{aligned}\frac{r^2}{2}|f(z)|^p &\le \frac{1}{2\pi}\int_{\mathfrak{R}}\int_{\mathfrak{L}}|f(z+\rho\, e^{i\theta})|^p\rho d\theta d\rho\\ &\le \frac{1}{2\pi}\int_{\mathfrak{X}}\int_{\mathfrak{Y}}|f(\xi+i\eta)|^p d\eta d\xi\\ &\le \frac{1}{2\pi}\int_{\mathfrak{X}}\mathfrak{m}d\xi=\frac{\mathfrak{m}r}{\pi}.\end{aligned}$$

Therefore $|f(z)|^p\le\frac{c^p}{r}$ where $c^p=\frac{2\mathfrak{m}}{\pi}$. Lemma 3.1 is proved as $r\to x$.

Lemma 3.2. *If $f\in VH^p(\mathcal{S}^+)(p\ge 1)$ and $h>0$, then*

$$f(z+h)=\frac{1}{\pi}\int_{\mathfrak{I}}xf(it+h)[x^2+(y-t)^2]^{-1}dt,$$

for $z=x+iy$.

Proof. Take $\Gamma_R=\prod_{j=1}^{\infty}\Gamma_{Rj}$, where $\Gamma_{Rj}\subset\mathbb{C}_j$ is a closed lune path consisting of the line $x_j=h_j(>0)$ and the circular arc with the center at the origin and radius R sufficiently large in the right-half plane $\mathbb{C}_j^+$. Let $R\cos\theta_{0j}=h_j$. Since $\exp ix=(\exp ix_1,\ldots,\exp ix_j,\ldots)$, $\cos\theta_0=(\cos\theta_{01},\ldots,\theta_{0j},\ldots)$. So $R\cos\theta_0=h\in\mathcal{S}$. From Cauchy's integral formula we derive

$$f(z+h)=\frac{1}{2\pi i}\int_{\Gamma_R}f(\xi)[\xi-(z+h)]^{-1}d\xi,$$

where $z+h$ is in the interior of Γ_R. Because $h-\overline{z}$ lies the exterior of Γ_R, Cauchy's integral theorem yields

$$\frac{1}{2\pi i}\int_{\Gamma_R}f(\xi)[\xi-(-\overline{z}+h)]^{-1}d\xi=0.$$

It follows that

$$\begin{aligned}&f(z+h)|\\ =&\frac{1}{2\pi i}\int_{\Gamma_R}f(\xi)\{[\xi-(z+h)]^{-1}\\ &-[\xi-(-\overline{z}+h)]^{-1}\}d\xi\\ =&\frac{1}{\pi i}\int_{\Gamma_R}xf(\xi)[(\xi-h-iy)^2-x^2]^{-1}d\xi\\ =&\frac{1}{\pi}\int_{\mathfrak{T}}xf(it+h)[x^2+(y-t)^2]^{-1}dt\\ &+\frac{1}{\pi}\int_{\mathfrak{H}}xRe^{i\theta}f(Re^{i\theta})\\ &\{[Re^{i\theta}-h-iy]^2-x^2\}^{-1}\\ =&I_1+I_2,\end{aligned}$$

where $\mathfrak{T}=\prod_{j=1}^{\infty}(-R\sin\theta_{0j},R\sin\theta_{0j})$, and $\mathfrak{H}=\prod_{j=1}^{\infty}(-\theta_{0j},\theta_{0j})$. Obviously

$$\lim_{R\to\infty}I_1=\frac{1}{\pi}\int_{\mathfrak{I}}xf(it+h)[x^2+(y-t)^2]^{-1}dt.$$

Next we show $\lim_{R\to\infty}I_2=0$.

Lemma 3.1 implies $|f(Re^{i\theta})|\le c(R\cos\theta)^{-\frac{1}{p}}$. It follows that

$$\begin{aligned}&\left|xRe^{i\theta}[(Re^{i\theta}-h-iy)^2-x^2]^{-1}\right|\\ =&xR|Re^{i\theta}-h-iy+x|^{-1}\\ &|Re^{i\theta}-h-iy-x|^{-1}\\ \le&xR|R-h-y-x|^{-2}\end{aligned}$$

for R sufficiently large. Thus

$$|I_2|\le\frac{1}{\pi}\int_{\mathfrak{O}}c(R\cos\theta)^{-\frac{1}{p}}xR|R-h-y-x|^{-2}d\theta,$$

where $\mathfrak{O}=\prod\{(-\frac{\pi}{2},\frac{\pi}{2})\}$. Because the integral $\int_{\mathfrak{L}}(\cos\theta)^{-\frac{1}{p}}d\theta$ converges and

$$\lim_{R\to\infty} xR^{1-\frac{1}{p}}(R-h-y-x)^{-1}=0,\ \lim_{R\to\infty} I_2=0$$

if $p \geq 1$.

Combining the above, letting $R \to \infty$ we obtain required conclusion.

Proof of Theorem 3.1 The following two cases are discussed.

$\alpha)$ $p > 1$

Since $f \in VH^p(\mathcal{S}^+)$, there is an $M > 0$ such that $\int_{\mathfrak{I}} |f(it+h)|^p dt \leq M$, where $h > 0$ is any element in $\mathcal{S}$. It follows that $f(it+h)$ is weak convergence to $f(it) \in VL^p(\mathfrak{I})$. Lemma 3.2 yields $f(z+h) = \frac{1}{\pi}\int_{\mathfrak{I}} xf(it+h)[x^2+(y-t)^2]^{-1}dt$. Setting $h \to 0$ in the above formula we get the result of Theorem 3.1.

$\beta)$ $p = 1$

From Cauchy's integral theorem we derive $\int_{\mathfrak{I}} f(it+h)(it+\overline{z})^{-1}dt = 0$ for any $0 < h \in \mathcal{S}$. The mapping $f(it+h)dt$ is weak* convergence to $d\mu(t)$, where $d\mu(t)$ is a measure on $\mathfrak{I}_0 = \prod_{j=1}^{\infty} \{(-i\infty, i\infty)\}$ satisfying $\int_{\mathfrak{I}} |\mu(t)| < \infty$ if $h \to 0$. It follows that for any $0 < x \in \mathcal{S}, \int_{\mathfrak{I}}(it+x-iy)^{-1}d\mu(t) = 0$. Letting $y = 0$ in the above formula we get $\int_{\mathfrak{I}}(it+x)^{-1}d\mu(t) = 0$. Finding the Fréchet derivatives of each order we obtain $\int_{\mathfrak{I}}(it+x)^{-n}d\mu(t) = 0$ for $n = 0, 1, \ldots$. Specially there are $\int_{\mathfrak{I}}(it+I)^{-n}d\mu(t) = 0$ for $n = 0, 1, \ldots$ as $x = I$, where I is the unit element in $\mathcal{S}$. Define a measure $dv(\tau) = (it-I)^{-1}d\mu(t)$. The conformal mapping $w = (z-I)(z+I)^{-1}$ implies

$$\begin{aligned}\int_{\mathfrak{L}} e^{in\tau}dv(\tau) &= \int_{\mathfrak{I}}(it-I)^{n-1}(it+I)^n d\mu(t)\\ &= \int_{\mathfrak{I}}[(it+I)-2I]^{n-1}(it+1)^{-n}d\mu(t)\\ &= \sum_{k=1}^{n}\left[a_k\int_{\mathfrak{I}}(it+I)^{-k}d\mu(t)\right] = 0.\end{aligned}$$

From Riesz's theorem (Garnett, 1980) and the absolute continuity of $v(\tau)$ we derive that $\mu(t)$ is also absolutely continuous and $f(it) \in VL^1(\mathfrak{I})$ and that $d\mu(t) = f(it)dt$. Hence Lemma 3.2 yields

$$f(z) = \frac{1}{\pi}\int_{\mathfrak{I}} xf(it)[x^2+(y-t)^2]^{-1}dt.$$

This proof is finished.

Theorem 3.2. *Assume that $F(z) \in VH^p(\mathcal{S}^+)$, and that $f(w) = F(z)$, then $f(w) \in VH^p(D_s)$ where $w = (z-I)(z+I)^{-1}$, $D_s = (D_{s1}, \ldots, D_{sj}, \ldots)$ and D_{sj} is a unit disk in $\mathbb{C}_j$ for each j.*

Proof. The following two cases are discussed.

1) $p \geq 1$

From Theorem 3.1 we obtain

$$F(z) = \frac{1}{\pi}\int_{\mathfrak{I}} xF(it)[x^2+(y-t)^2]^{-1}dt,\ z \in \mathcal{S}^+,$$

where $F(it) \in VL^p(\mathfrak{I})$. Using $F(it) = f(e^{it})$, we can get

$$F(z) = \frac{1}{2\pi}(1-r^2)\int_{\mathfrak{L}} f(e^{i\tau})[1+r^2-2r\cos(\varphi-\tau)]^{-1}d\tau.$$

So

$$\int_{\mathfrak{L}} |f(e^{i\tau})|^p d\tau = \int_{\mathfrak{I}} 2|F(it)|^p(1+t^2)^{-1}dt \leq 2\int_{\mathfrak{I}} |F(it)|^p dt < \infty.$$

Hence $f(w) \in VH^p(D_s)$.

2) $0 < p < 1$.

Lemma 3.1 yields $|F(z)| \leq cx^{-\frac{1}{p}}$. Particularly $|F(z)|$ and $|F(z)|^p$ are bounded on the half space $\prod_{j=1}^{\infty}\{\Re z_j \geq h_j > 0\}$, thus $|F(z+h)|^p$ is a subharmonic mapping on $\mathcal{S}^+$. It follows that

$$|F(z+h)|^p \leq \frac{1}{\pi}\int_{\mathfrak{I}} x|F(it+h)|^p|x^2+(y-t)^2|^{-1}dt.$$

Since $\int_{\mathfrak{I}} |F(it+h)|^p dt \leq \mathfrak{m}$ for any $h > 0$, there is a measure μ such that $\int_{\mathfrak{I}} |d\mu(t)| < \infty$ and

$$|F(z)|^p \leq \frac{1}{\pi}\int_{\mathfrak{I}} x|x^2+(y-t)^2|^{-1}d\mu(t).$$

Let $d\nu(\tau) = -2(I+t^2)^{-1}d\mu(t)$. Then

$$|f(re^{i\varphi})|^p \leq \frac{1}{2\pi}\int_{\mathfrak{L}}(1-r^2)|1+r^2-2r\cos(\varphi-\tau)|^{-1}d\nu(\tau).$$

Fubini's theorem yields

$$\int_{\mathfrak{L}} |f(re^{i\varphi})|^p d\varphi \leq 2\int_{\mathfrak{I}} |1+t^2|^{-1}|d\mu(t)| < \infty,$$

so $f(w) \in VH^p(D_s)$.

Theorem 3.3. *If $F(z) \in VH^p(\mathcal{S}^+)$, then $F(z) = F_{[i]}(z)F_{[o]}(z)$, where*

$$F_{[i]}(z) = e^{i\gamma} B(z) \exp\left\{\frac{1}{\pi}\int_{\mathfrak{I}} \varpi(t,z) d\sigma(t)\right\} e^{i\alpha z}$$

is called the inner mapping of F, $\varpi(t,z) = (itz - I)[(it - z)(I + t^2)]^{-1}, \gamma \in \mathfrak{I}, B(z)$ is the Blaschke product of $F, \sigma(t)$ is a singular measure,

$$\int_{\mathfrak{I}} (I + t^2)^{-1} d\sigma(t) > -\infty, \alpha(\in \mathcal{S}) > 0$$

and the mapping

$$F_{[o]}(z) = \exp\left\{\frac{1}{\pi}\int_{\mathfrak{I}} \varpi(t,z) \log|F(it)| dt\right\}$$

is called the outer mapping of F.

Proof. By using the transform $w = (z - I)(z + I)^{-1}$ and Theorem 3.2 we obtain $f(w) \in VH^p(D_s)$, where $f(w) = F(z)$. Theorem 2.2.8 (Hu and Yang, 1992) implies $f(w) = f_{[i]} f_{[o]}$, where

$$f_{[i]}(w) = e^{i\gamma} B_f(w) \exp\left(-\frac{1}{2\pi}\int_{\mathfrak{L}} \vartheta(\tau,w) d\nu(\tau)\right),$$

$$B_f(w) = \prod_n \left(|w_{n]}|(w_{n]} - w)[w_{n]}(I - \overline{w}_{n]} w)]^{-1}\right),$$

$$f_{[o]}(w) = \exp\left(\frac{1}{2\pi}\int_{\mathfrak{L}} \vartheta(\tau,w) \log|f(e^{i\tau})| d\tau\right),$$

$\vartheta(\tau,w) = (e^{i\tau} + w)(e^{i\tau} - w)^{-1}, \gamma \in \mathfrak{I}, \nu \geq 0$ is a singular measure on $\mathfrak{L}$. It follows that

$$f_{[i]}(w) = e^{i\gamma} B_f(w) \exp\left(-\frac{1}{\pi}\int_{\mathfrak{I}} \varpi(t,z) d\sigma(t)\right) e^{-\alpha z},$$

$$f_{[o]}(w) = \exp\left(\frac{1}{\pi}\int_{\mathfrak{I}} \varpi(t,z) \log|F(it)| dt\right),$$

where $\alpha = \frac{1}{\pi}[\nu(0) + \nu(2\pi)]$, and $d\nu(\tau) = -2(I + t^2)^{-1} d\sigma(t)$. Let $B(z) = B_f(w), F_{[i]}(z) = f_{i]}(w)$, and $F_{[o]}(z) = f_{[o]}(w)$ via $w = (z - I)(z + I)^{-1}$. Then $F(z) = F_{[i]}(z)F_{[o]}(z)$. By using Theorem 2.2.8 (Hu and Yang, 1992) we can check that $F_{[i]}(z)$ is the inner mapping and $F_{[o]}(z)$ is the outer mapping. This ends the proof.

Theorem 3.4. *A mapping $f \in VH^\infty(l^\infty)$ is outer if and only if fVH^2 is dense in VH^2*

Proof. Let $\pounds$ is a shift operator on VH^2, i.e. $\pounds(f) = zf(z)$. Assume that Υ is a closure of fVH^2 in VH^2. Obviously, Υ is an invariant subspace with respect to $\pounds$. By using Beurling's Theoremin (Garnett, 1980), we know that there is an inter mapping g such that $\Upsilon = gVH^2$. Since $f \in \Upsilon$, the mapping f can be represented as $f = gh$, where $h \in VH^2$.

Necessity. If f is an inner mapping, then $g \equiv const$ by $f = gh$. It follows that $\Upsilon = gVH^2$, namely, fVH^2 is dense in VH^2.

Sufficiency. Suppose that f is not outer, and that $f = f_{[i]} f_{[o]}$, then $f_{[i]} \not\equiv const$. We can check that $f_{[i]}VH^2$ is an invariant subspace with respect to $\pounds$ and $f_{[i]}VH^2 \supset fVH^2$. Thus fVH^2 is not dense in VH^2. This is in contradiction with the hypothesis. This contradiction shows the sufficiency.

Therefore the result of this theorem holds.

4 THE VH^∞ CONTROL

In this section, we replace $\mathcal{S}$ by the complex bounded sequence space l^∞. The subset of VH^∞ consisting of all elements with every component being real-rational function, is denoted by VRH^∞. We call f to be strong proper if $f \in VRH^\infty$ and $\|f\|_\infty < \infty$, strictly strong proper if $f(\infty) = 0$. We call f to be stable if $f \in VRH^\infty$ and f has no poles in the domain $\mathfrak{D}^+(= \prod_{j=1}^\infty \mathfrak{D}_j^+)$, where $\mathfrak{D}_j^+ = \{\Re(s_j) \geq 0\}$.

From the above definitions and the corresponding conclusion (Hu and Ma, 2002) we derive $f \in VRH^\infty$ if and only if f is strong proper and stable.

The space $(l^\infty)^{n\times m}$ consists of all $n \times m$ complex matrices with each element being in l^∞. If $f(z) \in (l^\infty)^{n\times m}$, then f can be written as

$$\begin{aligned} f &= \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix} \\ &= (\mathfrak{f}_1, \ldots, \mathfrak{f}_j, \ldots), \end{aligned}$$

where

$$\mathfrak{f}_j = \begin{pmatrix} f_{11j} & \cdots & f_{1nj} \\ f_{21j} & \cdots & f_{2nj} \\ \vdots & \ddots & \vdots \\ f_{m1j} & \cdots & f_{mnj} \end{pmatrix}.$$

Let ϱ_j be the maximal singular value of $\mathfrak{f}_j$. Then $(\varrho_1, \ldots, \varrho_j, \ldots)$ is called the maximal singular value vector of f. Let VL^∞ be a space consisting of all mapping matrices $f(i\omega)$ with

$$\sup\{\bar{\sigma}[f(i\omega)] : \omega \in (-\infty, \infty)\} < \infty.$$

Here $\bar{\sigma}[f(i\omega)]$ is its maximal singular value vector for any fixed ω. The vector norm of $f \in VL^\infty$ is defined by

$$\|f\|_\infty = \sup\{\bar{\sigma}[f(i\omega)] : \omega \in (-\infty, \infty)\}.$$

The space VRL^∞ consists of all real-rational mapping matrices in VL^∞.

The space VL^2 consists of all mapping matrices $\{x(i\omega)\}$ which are in $(l^\infty)^n$ and satisfy

$$\int_{\Im} x^*(i\omega)x(i\omega)d\omega < \infty,$$

where x^* is the complex-conjugate transpose of x. The space VH^∞ consists of all holomorphic mapping matrices $\{F(s)\}$ satisfying

$$\sup\{\bar{\sigma}[F(s)] : \Re(s) > 0\} < \infty$$

and this sup is denoted by $\|F\|_\infty$ being the vector norm of $F \in VH^\infty$.

Three transfer matrices

$$T^{[l]} = (T_1^{[l]}, \ldots, T_j^{[l]}, \ldots), l = 1, 2, 3$$

are controllers. Similar to the classical method in [2], we define the transfer mapping matrix

$$G(s) := \begin{bmatrix} T^{[1]}(s) & T^{[2]}(s) \\ T^{[3]}(s) & 0 \end{bmatrix}, \quad K(s) = -Q(s)$$

where $T^{[l]} \in VH^\infty$ for $l = 1, 2, 3$ are given. Let

$$T^{[i]} = [\ T_{i1} \quad \cdots \quad T_{ij} \quad \cdots\ ]$$

for $i = 1, 2, 3$. Then G can be written as

$$\left[\begin{bmatrix} T_{11} & T_{21} \\ T_{31} & 0 \end{bmatrix} \cdots \begin{bmatrix} T_{1j} & T_{2j} \\ T_{3j} & 0 \end{bmatrix} \cdots \right].$$

For simplicity we only discuss under

$$T^{[l]} \in VRH^\infty(l^\infty), l = 1, 2, 3.$$

In VH^∞ control theory, the model-matching problem is to find an element $Q \in VRH^\infty$ or a matrix $Q \in VRH^\infty$ to minimize

$$\left\| T^{[1]} - T^{[2]}QT^{[3]} \right\|_\infty,$$

Q is the controller to be designed. Let

$$\alpha := \inf \left\{ \left\| T^{[1]} - T^{[2]}QT^{[3]} \right\|_\infty \right\}$$

be infimal model-matching error. The linear time invariant system ß in VRH^∞ is defined by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t).$$

Completely controllable (c.c.) and completely observable (c.o.) concepts and symbols (A, B) and (A, C) are similar to Definition 2.3 (Hu and Ma, 2002). The concept of the minimal realization is similar to Definition 2.4 (Hu and Ma, 2002). A matrix $A \in VRH^\infty$ is said to be antistable if all the generalized eigenvalue vectors consisting of its all eigenvalues, of A, are in $\prod_{j=1}^\infty \{\Re(s_j) > 0\}$.

From the H^∞-control theory we derive the following result.

Theorem 4.1. (i) *A realization $[A, B, C, 0]$ of a given transfer matrix $G(s) \in VRH^\infty$ is minimal if (A, B) is completely controllable and (A, C) is completely observable respectively.*

(ii) *If A is antistable, then the Lyapunov equations*

$$AL_c + L_cA^T = BB^T$$

$$A^TL_o + L_oA = C^TC$$

have the unique solutions respectively, where

$$L_c := \int_\Omega e^{-At}B\,B^Te^{-A^Tt}dt,$$

$$L_o := \int_\Omega e^{-A^Tt}C^TCe^{-At}dt,$$

where $\Omega = \prod_{j=1}^\infty \Omega_j, \Omega_j = [0, \infty)$.

Theorem 3.3 and Theorem 2.1 yield that a mapping T in VRH^∞ is inner if $T(-s)T(s) = I$, and outer if it has no zeros in $\prod_{j=1}^\infty \{\Re(s_j) > 0\}$, that $T(-s)T(s) = I$ if and only if each component

$$T_j(-s_j)T_j(s_j) = 1 \text{ for any } j,$$

that every mapping T in VRH^∞ has a factorization $T = T_{[i]}T_{[o]}$ with $T_{[i]}$ inner, $T_{[o]}$ outer, and $\left\|T_{[i]}(i\omega)\right\|_\infty$

$= I$ (the unit element), and that if $T_{[o]}(i\omega) \neq 0$ for all $\omega \in \Omega$, then $T_{[o]}^{-1}$ exists and $T_{[o]}^{-1} \in VRH^{\infty}$.

Returning to the model-matching problem, without loss generality we may assume $T^{[3]} = I$ and bring in an inner-outer factorization of $T^{[2]} : T^{[2]} = T_{[i]}^{[2]} T_{[o]}^{[2]}$. It follows that for Q in VRH^{∞} we have

$$\|T^{[1]} - T^{[2]}Q\|_{\infty} = \|T_{[i]}^{[2]^{-1}} T^{[1]} - T_{[o]}^{[2]} Q\|_{\infty}$$

$$= \|R - X\|_{\infty},$$

where $R = T_{[i]}^{[2]^{-1}} T^{[1]}, X = T_{[o]}^{[2]} Q$.

Let λ^2 be a generalized largest eigenvalue vector of $L_c L_o$ and w a corresponding vector matrix respectively. Define

$$f(s) = [A, w, C, 0], g(s) = [-A^T, \lambda^{-1} L_o w, B^T, 0]$$

and

$$X(s) = R(s) - \lambda f(s)[g(s)]^{-1}.$$

Let $F(s) \in VL^{\infty}$ and $g(s) \in VL^2$. Then the operator

$$\Lambda_{F(s)} : \Lambda_{F(s)} g(s) = F(s) g(s)$$

is called the Laurent operator. For $F(s)$ in VL^{∞}, the Hankel operator with symbol $F(s)$, denoted by $\Gamma_{F(s)}$, maps VH^2 to $VH^{2^{\perp}}$ and is defined as

$$\Gamma_{F(s)} := \Pi_1 \Lambda_{F(s)} | VH^2,$$

where Π_1 is the projection from VL^2 onto $VH^{2^{\perp}}$.

Let $\{s_j : \Re(s_j) = 0, \Im(s_j) \geq 0\} = \Theta_j$ and $\prod_{j=1}^{\infty} \Theta_j = \Theta$.

By using the preceding method we may obtain the following conclusions.

Theorem 4.2.

(a) *If the ranks of $T^{[2]}$ and $T^{[3]}$ are constant on Θ, then the optimal Q exists.*

(b) *There exists a closest VRH^{∞}-mapping $X(s)$ to a given VRL^{∞}-mapping $R(s)$, and $\|R - X\|_{\infty} = \|\Gamma_R\|$, where*

$$\|\Gamma_R\| = (\|\Gamma_{1R_1}\|, \ldots, \|\Gamma_{jR_j}\|, \ldots).$$

(c)*The infimal model-matching error α equals $\|\Gamma_R\|$ and the unique optimal X equals*

$$R(s) - \lambda f(s)[g(s)]^{-1}.$$

The optimal controller

$$Q = (Q_1, \ldots, Q_j, \ldots) = \left(T_{[o]}^{[2]}\right)^{-1} X \in VRH^{\infty}$$

is found via this theorem. Therefore the VH^{∞}-control theory is solved.

5 CONCLUSIONS

1) Theorem 4.2 gives the optimal solution Q and the infimal model-matching error α of the VH^{∞} control theory on the Banach algebra being isometric isomorphism l^{∞}. Section 2, Section 3 and Theorem 4.1 are the foundation of Theorem 4.2.

2) The concepts and the property of meromorphic mappings in Definition 2.1 and Theorem 2.2 are breakthroughs on infinite dimensional complex analysis. The argument on infinite dimensional spaces are defined in formula (2.3) being very important concept in the geometry.

3) All control theory on finite dimensional spaces can be extended that on infinite dimensional spaces to infinite dimensional spaces by using methods in this paper.

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