CONTINUOUS-TIME SIGNAL FILTERING FROM NON-INDEPENDENT UNCERTAIN OBSERVATIONS

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Abstract: Filtering algorithms are presented as solution of the least mean-squared error linear estimation problem of continuous-time wide-sense stationary scalar signals from uncertain observations perturbed by white and coloured additive noises. These algorithms, one of them based on Chandrasekhar-type equations and the other on Riccati-type ones, are derived assuming a specific type of dependence between the Bernoulli random variables describing the uncertainty and do not require the whole knowledge of the state-space model. By comparing both algorithms it is deduced that the Chandrasekhar-type one is more advantageous from a computational viewpoint.

1 INTRODUCTION

In the mid-seventies, the replacement of the Riccati-type equations by a set of Chandrasekhar-type ones in the algorithms proposed as solution of the least mean-squared error (LMSE) linear estimation problem led to more advantageous algorithms from a computational point of view since the Chandrasekhar-type algorithms contain, generally, less difference or differential equations than the ones based on Riccati-type equations. For continuous-time invariant systems, Kailath (1973) was the first author who proposed an algorithm of this kind to solve the LMSE linear estimation problem. This work was the starting point for many posterior contributions. We shall mention, among others, Sayed and Kailath (1994) who, assuming a full knowledge of the state-space model, obtained Chandrasekhar-type algorithms for a class of time-variant models. Recently, Nakamori (2000) has proposed a Chandrasekhar-type algorithm to estimate a continuous-time wide-sense stationary signal from observations perturbed by white and coloured additive noises but assuming, in contrast to the above papers, that the state-space model is not available and using covariance information.

On the other hand, in the last decades, considerable attention has been given to systems with uncertain observations, since they model many real situations. These systems are characterized by including in the observation model, besides additive noise, a multiplicative noise described by Bernoulli random variables, which determine the presence or absence of signal in the observations. These systems are then appropriate to model situations in which there exist intermittent failures in the observation mechanism and, hence, the observations may contain noise plus signal or only noise in a random manner; for example, communication systems with random interruptions.

The LMSE linear estimation problem of discrete signals from uncertain observations has been ap-

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proached under different hypotheses. For example, Hermoso and Linares (1994) proposed a Riccati-type algorithm in discrete-time systems, considering that the Bernoulli random variables are independent; on the other hand, Hadidi and Schwartz (1979) obtained an estimation algorithm based also on Riccati equations, but considering a specific type of dependence between the Bernoulli variables. Both papers are based on a full knowledge of the state-space model; recently, Nakamori et al. (2004) have proposed a Chandrasekhar-type filtering algorithm for wide-sense stationary signals from uncertain observations without using the state-space model but covariance information.

In this paper, we analyze the LMSE linear filtering problem of continuous-time wide-sense stationary scalar signals from uncertain observations perturbed by white and coloured additive noises. Assuming that the Bernoulli random variables present a type of dependence analogous to that considered by Hadidi and Schwartz (1979), we propose a Chandrasekhar and a Riccati-type algorithm, derived by using covariance information. The comparison between both algorithms shows the computational advantages of the Chandrasekhar-type one.

2 ESTIMATION PROBLEM

Let us consider a continuous-time scalar observation equation described by

$$y(t) = u(t)z(t)+v(t)+v_0(t), \;\; z(t) = Hx(t), \;\; t \geq 0 \tag{1}$$

where $y(t)$ represents the observation of the signal $z(t)$, perturbed by a multiplicative noise, $u(t)$, and by white and coloured additive noises, $v(t)$ and $v_0(t)$, respectively; the signal is expressed as a linear combination of the components of the n-dimensional state vector $x(t)$.

Denoting by Φ and Φ_0 the system matrices of the state and the coloured noise, respectively, we have assumed the following hypotheses on the processes appearing in equation (1):

(H.1) The signal process $\{z(t);\; t \geq 0\}$ is wide-sense stationary with zero mean, being its autocovariance function $K_z(t,s) = E[z(t)z(s)] = K_z(t-s)$, for $t, s \geq 0$. Moreover, the cross-covariance function of the state $x(t)$ and the signal $z(s)$, $K_{xz}(t,s)$, verifies the differential equation

$$\frac{\partial K_{xz}(t,s)}{\partial t} = \Phi K_{xz}(t,s), \quad s < t.$$

(H.2) The additive noise $\{v(t);\; t \geq 0\}$ is a zero-mean white process whose autocovariance function is given by $E[v(t)v(s)] = R\delta_D(t-s)$, for $t, s \geq 0$, being $R \neq 0$ and δ_D the Dirac delta function.

(H.3) The coloured noise $\{v_0(t);\; t \geq 0\}$ is a zero-mean wide-sense stationary process with autocovariance function $K_{v_0}(t,s) = E[v_0(t)v_0(s)] = K_{v_0}(t-s)$, for $t, s \geq 0$, which satisfies the differential equation

$$\frac{\partial K_{v_0}(t,s)}{\partial t} = \Phi_0 K_{v_0}(t,s), \quad s < t.$$

(H.4) The multiplicative noise $\{u(t);\; t \geq 0\}$ describing the uncertainty in the observations is modelled by identically distributed Bernoulli random variables with initial probability vector $(1-p,p)^T$ and conditional probability matrix $P(t/s)$. We assume that the $(2,2)$-element of this matrix is independent of t and s, that is,

$$P(u(t) = 1/u(s) = 1) = p_{22}$$

for $t \neq s$. Under these considerations, it is clear that

$$E\left[u(t)u(s)\right] = \begin{cases} p, & \text{if } t = s \\ p\,p_{22}, & \text{if } t \neq s \end{cases}$$

(H.5) The processes $\{x(t);\; t \geq 0\}$, $\{u(t);\; t \geq 0\}$, $\{v(t);\; t \geq 0\}$ and $\{v_0(t);\; t \geq 0\}$ are mutually independent.

Under these considerations, our aim consists of determining an algorithm to calculate the LMSE linear estimator of the signal $z(t)$ given the observations until time t, that is $\{y(s);\; 0 \leq s \leq t\}$. It is clearly observed that this estimator, denoted by $\widehat{z}(t)$, can be expressed as $\widehat{z}(t) = H\widehat{x}(t)$, where $\widehat{x}(t)$ is the LMSE linear filter of the state. For this reason, we have focussed our interest on obtaining an algorithm for $\widehat{x}(t)$, which can be expressed as

$$\widehat{x}(t) = \int_0^t h(t,\tau)y(\tau)d\tau \tag{2}$$

where $\{h(t,\tau), 0 \leq \tau \leq t\}$ denotes the impulse-response function.

As a consequence of the Orthogonal Projection Lemma (OPL) and the hypotheses on the model, $\widehat{x}(t)$ satisfies the Wiener-Hopf equation, given by

$$pK_{xz}(t,s) = \int_0^t h(t,\tau)E\left[y(\tau)y(s)\right]d\tau, \quad 0 \leq s \leq t \tag{3}$$

or, equivalently,

$$\begin{aligned} h(t,s)R &= pK_{xz}(t,s) - \int_0^t h(t,\tau)\overline{K}(\tau,s)d\tau, \\ \overline{K}(\tau,s) &= pp_{22}HK_{xz}(\tau,s) + K_{v_0}(\tau,s). \end{aligned} \tag{4}$$

In order to determine a differential equation for $\widehat{x}(t)$, we differentiate (2) with respect to t and so, we obtain

$$\frac{d\widehat{x}(t)}{dt} = \int_0^t \frac{\partial h(t,\tau)}{\partial t}y(\tau)d\tau + h(t,t)y(t). \tag{5}$$

On the other hand, differentiating (3) with respect to t, from (H.1) and the OPL, it is had that, for $0 < s < t$,

$$\int_0^t \Big(\Phi h(t,\tau) - \frac{\partial h(t,\tau)}{\partial t} - h(t,t)[p_{22}Hh(t,\tau) + g(t,\tau)]\Big) E[y(\tau)y(s)]\, d\tau = 0$$

where $\{g(t,\tau), 0 \leq \tau \leq t\}$ represents the impulse-response function of the coloured noise filter, $\widehat{v}_0(t)$. Then, it is clear that this integral equation will be satisfied if we consider a function h satisfying

$$\frac{\partial h(t,\tau)}{\partial t} = \Phi h(t,\tau) - h(t,t)[p_{22}Hh(t,\tau) + g(t,\tau)] \quad (6)$$

for $0 \leq \tau \leq t$. Hence, by substituting (6) in (5), the following differential equation for $\widehat{x}(t)$ is derived

$$\frac{d\widehat{x}(t)}{dt} = \Phi\widehat{x}(t) + h(t,t)\,[y(t) - p_{22}H\widehat{x}(t) - \widehat{v}_0(t)] \quad (7)$$

with initial condition $\widehat{x}(0) = 0$.

By following an analogous reasoning, it is obtained the Wiener-Hopf equation for $\widehat{v}_0(t)$,

$$g(t,s)R = K_{v_0}(t,s) - \int_0^t g(t,\tau)\overline{K}(\tau,s)d\tau,\ 0 \leq s \leq t \quad (8)$$

and the following differential equation,

$$\frac{\partial g(t,\tau)}{\partial t} = \Phi_0 g(t,\tau) - g(t,t)[p_{22}Hh(t,\tau) + g(t,\tau)]. \quad (9)$$

Then, $\widehat{v}_0(t)$ verifies the differential equation

$$\frac{d\widehat{v}_0(t)}{dt} = \Phi_0\widehat{v}_0(t) + g(t,t)\,[y(t) - p_{22}H\widehat{x}(t) - \widehat{v}_0(t)] \quad (10)$$

with initial condition $\widehat{v}_0(0) = 0$.

To complete the algorithm, in the following section we show two different ways to calculate the filtering gains, $h(t,t)$ and $g(t,t)$.

3 FILTERING ALGORITHMS

Next we derive two algorithms as a solution of the LMSE filtering problem: in one of them, the filtering gains are obtained from Chandrasekhar-type equations whereas, in the other, Riccati-type ones are used.

3.1 Chandrasekhar-type algorithm

Theorem 1. The filter of the signal, $\widehat{z}(t)$, is calculated from the relation $\widehat{z}(t) = H\widehat{x}(t)$ where the state filter, $\widehat{x}(t)$, satisfies the differential equation (7) and the coloured noise filter, $\widehat{v}_0(t)$, is given from (10).

The filtering gains are calculated as follows

$$\frac{dh(t,t)}{dt} = -h(t,0)\,[p_{22}Hh(t,0) + g(t,0)] \quad (11)$$

$$\frac{dg(t,t)}{dt} = -g(t,0)\,[p_{22}Hh(t,0) + g(t,0)] \quad (12)$$

where $h(t,0)$ and $g(t,0)$ satisfy the following differential equations

$$\frac{dh(t,0)}{dt} = \Phi h(t,0) - h(t,t)[p_{22}Hh(t,0) + g(t,0)] \quad (13)$$

$$\frac{dg(t,0)}{dt} = \Phi_0 g(t,0) - g(t,t)[p_{22}Hh(t,0) + g(t,0)] \quad (14)$$

being the initial conditions

$$h(0,0) = pR^{-1}K_{xz}(0). \quad (15)$$

$$g(0,0) = R^{-1}K_{v_0}(0). \quad (16)$$

Proof. Differentiating (4) with respect to t and s, we obtain the following expression, valid for $0 \leq s \leq t$,

$$\left(\frac{\partial h(t,s)}{\partial t} + \frac{\partial h(t,s)}{\partial s}\right)R = -h(t,0)\overline{K}(0,s) - \int_0^t \left(\frac{\partial h(t,\tau)}{\partial t} + \frac{\partial h(t,\tau)}{\partial \tau}\right)\overline{K}(\tau,s)d\tau$$

where we have used that, from the stationary property of the signal process, $\frac{\partial K_{xz}(t,s)}{\partial t} + \frac{\partial K_{xz}(t,s)}{\partial s} = 0$. Then, if we define a function J satisfying

$$J(t,s)R = \overline{K}(0,s) - \int_0^t J(t,\tau)\overline{K}(\tau,s)d\tau,\ 0 \leq s \leq t \quad (17)$$

it is immediately obtained that

$$\frac{\partial h(t,s)}{\partial t} + \frac{\partial h(t,s)}{\partial s} = -h(t,0)J(t,s), \quad 0 \leq s \leq t. \quad (18)$$

Next, if (4), replacing s by $t-s$, is multiplied on the left by $p_{22}H$ and the resultant expression is added to (8) replacing also s by $t-s$, we obtain

$$[p_{22}Hh(t,t-s) + g(t,t-s)]R = \overline{K}(s,0) - \int_0^t [p_{22}Hh(t,t-\tau) + g(t,t-\tau)]\overline{K}(s,\tau)d\tau \quad (19)$$

for $0 \leq s \leq t$. Then, by comparing (17) and (19),

$$J(t,s) = p_{22}Hh(t,t-s) + g(t,t-s), \quad 0 \leq s \leq t \quad (20)$$

and, consequently, from (18) and (20), (11) is derived.

On the other hand, the differential equation (13) and the initial condition (15) are respectively derived by taking $\tau = 0$ in (6) and $s = t = 0$ in (4).

By following an analogous reasoning, the differential equations (12) and (14) and the initial condition (16) are deduced. □

3.2 Riccati-type algorithm

Theorem 2. The filter of the signal, $\widehat{z}(t)$, is calculated from the relation $\widehat{z}(t) = H\widehat{x}(t)$ where the state filter, $\widehat{x}(t)$, satisfies the differential equation (7) and the coloured noise filter, $\widehat{v_0}(t)$, is given from (10).
The filtering gains are given by

$$h(t,t)=R^{-1}\big[pK_{xz}(0)-p_{22}S(t)H^T-T(t)\big] \quad (21)$$

$$g(t,t)=R^{-1}\left[K_{v_0}(0)-p_{22}T^T(t)H^T-U(t)\right] \quad (22)$$

where $S(t) = E[\widehat{x}(t)\widehat{x}^T(t)]$, $T(t) = E[\widehat{x}(t)\widehat{v_0}(t)]$ and $U(t) = E[\widehat{v_0}^2(t)]$ satisfy the following differential equations

$$\begin{aligned}\frac{dS(t)}{dt} &= \Phi S(t)+S(t)\Phi^T+ Rh(t,t)\,h^T(t,t),\\ \frac{dT(t)}{dt} &= \Phi T(t)+\Phi_0 T(t)+ Rg(t,t)h(t,t), \\ \frac{dU(t)}{dt} &= 2\Phi_0 U(t)+Rg^2(t,t).\end{aligned} \quad (23)$$

Proof. From the OPL and the hypotheses on the model, we have that

$$\begin{aligned}S(t) &= E[\widehat{x}(t)x^T(t)] = p\int_0^t h(t,\tau)K_{xz}^T(t,\tau)d\tau\\ T(t) &= E[\widehat{x}(t)v_0(t)] = \int_0^t h(t,\tau)K_{v_0}(\tau,t)d\tau\\ U(t) &= E[\widehat{v_0}(t)v_0(t)] = \int_0^t g(t,\tau)K_{v_0}(\tau,t)d\tau.\end{aligned} \quad (24)$$

Then by putting $s = t$ in (4) and (8) and using (24), equations (21) and (22) are obtained. Again, from (24), the differential equations given in (23) are derived by using (6) and (9). □

By comparing the algorithms proposed in the above theorems, it is observed that the Chandrasekhar-type one contains less differential equations than the Riccati-type algorithm; specifically, $3n + 3$ are the differential equations included in the Chandrasekhar-type algorithm, and $n^2 + 2n + 2$, in the Riccati-type one. So, if $n \geq 2$, there exists a reduction regarding the number of equations in the Chandrasekhar-type algorithm, which implies a decrease in the computation time. Hence, the Chandrasekhar-type algorithm is more advantageous than the Riccati-type one in a computational sense.

Finally, as a measure of the estimation accuracy, the filtering error variance, which is defined by $P(t) = E\left[\{z(t) - \widehat{z}(t)\}^2\right]$, can be calculated as

$$P(t,t) = H\left[K_{xz}(t,t) - S(t)H^T\right]$$

with $S(t)$ given in Theorem 2.

4 CONCLUSION

In this paper, the LMSE linear filtering problem of wide-sense stationary scalar signals in continuous-time systems with uncertain observations perturbed by white and coloured additive noises is analyzed. Assuming uncertainty non-independent we derive two algorithms without requiring the whole knowledge of the state-space model but using covariance information. Both algorithms differs in the way of calculating the filtering gains. From the comparison between them it is deduced that the Chandrasekhar-type one is computationally more appropriate than the Riccati-type algorithm.

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