ON MODELING AND CONTROL OF DISCRETE TIMED EVENT GRAPHS WITH MULTIPLIERS USING (MIN, +) ALGEBRA

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Abstract: Timed event graphs with multipliers, also called timed weighted marked graphs, constitute a subclass of Petri nets well adapted to model discrete event systems involving synchronization and saturation phenomena. Their dynamic behaviors can be modeled by using a particular algebra of operators. A just in time control method of these graphs based on Residuation theory is proposed.

1 INTRODUCTION

Petri nets are widely used to model and analyze discrete-event systems. We consider in this paper timed event graphs[1] with multipliers (TEGM's). Such graphs are well adapted for modeling synchronization and saturation phenomena. The use of multipliers associated with arcs is natural to model a large number of systems, for example when the achievement of a specific task requires several units of a same resource, or when an assembly operation requires several units of a same part. Note that TEGM's can not be easily transformed into (ordinary) TEG's. It turns out that the proposed transformation methods suppose that graphs are strongly connected under particular server semantics hypothesis (*single* server in (Munier, 1993), or *infinite* server in (Nakamura and Silva, 1999)) and lead to a duplication of transitions and places.

This paper deals with just in time control, *i.e.,* fire input transitions *at the latest* so that the firings of output transitions occur at the latest before the desired ones. In a production context, such a control input minimizes the work in process while satisfying the customer demand. To our knowledge, works on this tracking problem only concern timed event graphs without multipliers (Baccelli et al., 1992, §5.6), (Cohen et al., 1989), (Cottenceau et al., 2001).

[1]Petri nets for which each place has exactly one upstream and one downstream transition.

TEGM's can be handled in a particular algebraic structure, called *dioid*, in order to do analogies with conventional system theory. More precisely, we use an algebra of operators mainly inspired by (Cohen et al., 1998a), (Cohen et al., 1998b), and defined on a set of operators endowed with pointwise minimum operation as addition and composition operation as multiplication. The presence of multipliers in the graphs implies the presence of inferior integer parts in order to preserve integrity of discrete variables used in the models. Moreover, the resulting models are non linear which prevents from using a classical transfer approach to obtain the just in time control law of TEGM's. As alternative, we propose a control method based on "backward" equations.

The paper is organized as follows. A description of TEGM's by using recurrent equations is proposed in Section 2. An algebra of operators, inspired by (Cohen et al., 1998a), (Cohen et al., 1998b), is introduced in Section 3 to model these graphs by using a state representation. In addition to operators γ, δ usually used to model discrete timed event graphs (without multipliers), we add the operator μ to allow multipliers on arcs. The just in time control method of TEGM's is proposed in Section 4 and is mainly based on Residuation theory (Blyth and Janowitz, 1972). After recalling basic elements of this theory, we recall the residuals of operators γ, δ, and give the residual of operator μ which involves using the superior integer part. The just in time control is expressed as the great-

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est solution of a system of "backward" equations. We give a short example before concluding.

2 RECURRENT EQUATIONS OF TEGM's

We assume the reader is familiar with the structure, firing rules, and basic properties of Petri nets, see (Murata, 1989) for details.

Consider a TEGM defined as a valued bipartite graph given by a five-tuple (P, T, M, m, τ) in which:

- P represents the finite set of *places*, T represents the finite set of *transitions*;

- a *multiplier M* is associated with each *arc*. Given $q \in T$ and $p \in P$, the multiplier M_{pq} (respectively, M_{qp}) specifies the weight (in $\mathbb{N}$) of the arc from transition *q* to place *p* (respectively, from place *p* to transition *q*) (a zero value for M codes an absence of arc);

- with each place is associated an *initial marking* (m_p assigns an initial number of tokens (in $\mathbb{N}$) in place *p*) and a *holding time* (τ_p gives the minimal time a token must spend in place *p* before it can contribute to the enabling of its downstream transitions).

We denote by ${}^\bullet q$ (resp. $q^\bullet$) the set of places upstream (resp. downstream) transition q. Similarly, ${}^\bullet p$ (resp. $p^\bullet$) denotes the set of transitions upstream (resp. downstream) place p.

In the following, we assume that TEGM's function according to the *earliest firing* rule: a transition q fires as soon as all its upstream places $\{p \in {}^\bullet q\}$ contain enough tokens (M_{qp}) having spent at least τ_p units of time in place p (by convention, the tokens of the initial marking are present since time $-\infty$, so that they are immediately available at time 0). When the transition q fires, it consumes M_{qp} tokens in each upstream place p and produces $M_{p'q}$ tokens in each downstream place $p' \in q^\bullet$.

Remark 1 We disregard without loss of generality *firing times* associated with transitions of a discrete event graph because they can always be transformed into holding times on places (Baccelli et al., 1992, §2.5).

Definition 1 (Counter variable) With each transition is associated a *counter variable*: x_q is an increasing map from $\mathbb{Z}$ to $\mathbb{Z} \cup \{\pm\infty\}$, $t \mapsto x_q(t)$ which denotes the cumulated number of firings of transition q up to time *t*.

Assertion 1 The counter variables of a TEGM (under the earliest firing rule) satisfy the following *transition to transition* equation:

$$x_q(t) = \min_{p \in {}^\bullet q,\, q' \in {}^\bullet p} \lfloor M_{qp}^{-1}(m_p + M_{pq'} x_{q'}(t - \tau_p)) \rfloor. \tag{1}$$

Note the presence of inferior integer part to preserve integrity of Eq. (1). In general, a transition q may have several upstream transitions ($\{q' \in {}^{\bullet\bullet} q\}$) which implies that its associated counter variable is given by the $\min$ of *transition to transition* equations obtained for each upstream transition.

Example 1 The counter variable associated with transition q described in Fig. 1 satisfies the following equation:

$$x_q(t) = \lfloor a^{-1}(m + b\, x_{q'}(t - \tau)) \rfloor.$$

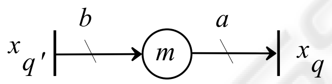

Figure 1: A simple TEGM.

Example 2 Let us consider TEGM depicted in Fig. 2. The corresponding counter variables satisfy the following equations:

$$\begin{cases} x_1(t) & = \min(3 + x_3(t-2), u(t)), \\ x_2(t) & = \min(\lfloor \frac{2x_1(t-2)}{3} \rfloor, \lfloor \frac{6+2x_3(t-2)}{3} \rfloor), \\ x_3(t) & = 3x_2(t-1), \\ y(t) & = x_2(t). \end{cases}$$

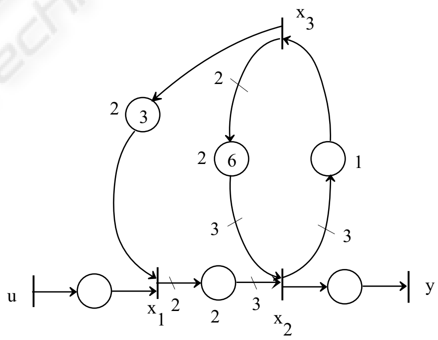

Figure 2: A TEGM.

3 DIOID, OPERATORIAL REPRESENTATION

Let us briefly define dioid and algebraic tools needed to handle the dynamics of TEGM's, see (Baccelli et al., 1992) for details.

Definition 2 (Dioid) A *dioid* $(\mathcal{D}, \oplus, \otimes)$ is a semiring in which the addition $\oplus$ is idempotent ($\forall a,\ a \oplus a = a$). Neutral elements of $\oplus$ and $\otimes$ are denoted ε and *e* respectively.

A dioid is *commutative* when the product $\otimes$ is commutative. Symbol $\otimes$ is often omitted.

Due to idempotency of $\oplus$, a dioid can be endowed with a natural order relation defined by $a \preceq b \Leftrightarrow b = a \oplus b$ (the least upper bound of {a,b} is equal to $a \oplus b$).

A dioid $\mathcal{D}$ is *complete* if every subset *A* of $\mathcal{D}$ admits a least upper bound denoted $\bigoplus_{x \in A} x$, and if $\otimes$ left and right distributes over infinite sums.

The greatest element noted $\top$ of a complete dioid $\mathcal{D}$ is equal to $\bigoplus_{x \in \mathcal{D}} x$. The greatest lower bound of every subset *X* of a complete dioid always exists and is noted $\bigwedge_{x \in X} x$.

Example 3
- The set $\mathbb{Z} \cup \{\pm\infty\}$, endowed with min as $\oplus$ and usual addition as $\otimes$, is a complete dioid noted $\overline{\mathbb{Z}}_{\min}$ with neutral elements $\varepsilon = +\infty$, $e = 0$ and $\top = -\infty$.
- Starting from a 'scalar' dioid $\mathcal{D}$, let us consider $p \times p$ matrices with entries in $\mathcal{D}$. The sum and product of matrices are defined conventionally after the sum and product of scalars in $\mathcal{D}$:
  $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$ and $(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} A_{ik} \otimes B_{kj}$.
  The set of square matrices endowed with these two operations is also a dioid denoted $\mathcal{D}^{p \times p}$.

Counter variables associated with transitions are also called *signals* by analogy with conventional system theory. The set of signal is endowed with a kind of module structure, called *min-plus semimodule*; the two associated operations are:

- pointwise minimum of time functions to add signals:
  $$\forall t, (u \oplus v)(t) = u(t) \oplus v(t) = \min(u(t), v(t));$$
- addition of a constant ($\in \overline{\mathbb{Z}}_{\min}$) to play the role of external product of a signal by a scalar:
  $$\forall t, \forall \lambda \in \mathbb{Z} \cup \{\pm\infty\},\ (\lambda.u)(t) = \lambda \otimes u(t) = \lambda + u(t).$$

A modeling method based on operators is used in (Cohen et al., 1998a), (Cohen et al., 1998b), a similar approach is proposed here to model TEGM's. Let us recall the definition of *operator*.

Definition 3 (Operator, linear operator) An *operator* is a map from the set of signals to the set of signals. An operator *H* is *linear* if it preserves the min-plus semimodule structure, *i.e.*, for all signals *u, v* and constant λ,

$$H(u \oplus v) = H(u) \oplus H(v) \quad \text{(additive property)},$$

$$H(\lambda \otimes u) = \lambda \otimes H(u) \quad \text{(homogeneity property)}.$$

Let us introduce operators γ, δ, μ which are central for the modeling of TEGM's:

1. Operator γ^ν represents a shift of ν units in counting ($\nu \in \mathbb{Z} \cup \{\pm\infty\}$) and is defined as $\gamma^\nu x(t) = x(t) + \nu$. It verifies the following relations: $\begin{cases} \gamma^\nu \oplus \gamma^{\nu'} = \gamma^{\min(\nu,\, \nu')}, \\ \gamma^\nu \otimes \gamma^{\nu'} = \gamma^{\nu+\nu'}. \end{cases}$
2. Operator δ^τ represents a shift of τ units in dating ($\tau \in \mathbb{Z} \cup \{\pm\infty\}$) and is defined as $\delta^\tau x(t) = x(t - \tau)$. It verifies the following relations: $\begin{cases} \delta^\tau \oplus \delta^{\tau'} = \delta^{\max(\tau,\, \tau')}, \\ \delta^\tau \otimes \delta^{\tau'} = \delta^{\tau+\tau'}. \end{cases}$
3. Operator μ_r represents a scaling of factor r and is defined as $\mu_r x(t) = \lfloor r \times x(t) \rfloor$ in which $r \in \mathbb{Q}^+$ (r is equal to a ratio of elements in $\mathbb{N}$). It verifies the following relation: $\mu_r \oplus \mu_{r'} = \mu_{\min(r,\, r')}$. Note that $\mu_r \otimes \mu_{r'}$ can be different from $\mu_{(r \times r')}$.

See Fig.3.a-3.c for a graphical interpretation of operators γ, δ, μ respectively. We note that operators γ, δ are linear while operator μ is only additive. We have the following properties:

1. $\gamma^\nu \delta^\tau = \delta^\tau \gamma^\nu$, $\mu_r \delta^\tau = \delta^\tau \mu_r$ (commutative properties),
2. Let $a, b \in \mathbb{N}$, we have $\mu_{a^{-1}} \mu_b = \mu_{(a^{-1}b)}$.

Let us introduce dioid $\mathcal{D}_{\min}[\![\delta]\!]$. First, we denote by $\mathcal{D}_{\min}$ the (noncommutative) dioid of finite sums of operators $\{\mu_r, \gamma^\nu\}$ endowed with pointwise min ($\oplus$) and composition ($\otimes$) operations, with neutral elements $\varepsilon = \mu_{+\infty}\gamma^{+\infty}$ and $e = \mu_1 \gamma^0$. Thus, an element in $\mathcal{D}_{\min}$ is a map $p = \bigoplus_{i=1}^{k} \mu_{r_i} \gamma^{\nu_i}$ such that

$$\forall t \in \mathbb{Z},\ p\,(x(t)) = \min_{1 \le i \le k} (\lfloor r_i(\nu_i + x(t)) \rfloor).$$

Operator δ is considered separately from the other operators in order to allow the definition of a dioid of formal power series. With each value of time delay τ (*i.e.,* with each operator δ^τ) is associated an element of $\mathcal{D}_{\min}$. More formally, we define a map $g : \mathbb{Z} \to \mathcal{D}_{\min}, \tau \mapsto g(\tau)$ in which $g(\tau) = \bigoplus_{i=1}^{k_\tau} \mu_{r_i^\tau} \gamma^{\nu_i^\tau}$.

Such an application can be represented by a formal power series in the indeterminate δ. Let the series $G(\delta)$ associated with map g defined by:

$$G(\delta) = \bigoplus_{\tau \in \mathbb{Z}} g(\tau)\, \delta^\tau.$$

The set of these formal power series endowed with the two following operations:

$$\begin{aligned} F(\delta) \oplus G(\delta) : (f \oplus g)(\tau) &= f(\tau) \oplus g(\tau) \\ &= \min(f(\tau), g(\tau)), \\ F(\delta) \otimes G(\delta) : (f \otimes g)(\tau) &= \bigoplus_{i \in \mathbb{Z}} f(i) \otimes g(\tau - i) \\ &= \inf_{i \in \mathbb{Z}} (f(i) + g(\tau - i)), \end{aligned}$$

is a dioid noted $\mathcal{D}_{\min}[\![\delta]\!]$ with neutral elements $\varepsilon = \mu_{+\infty}\gamma^{+\infty}\delta^{-\infty}$ and $e = \mu_1 \gamma^0 \delta^0$.

Elements of $\mathcal{D}_{\min}[\![\delta]\!]$ allow modeling the transfer between two transitions of a TEGM. A signal x can be also represented by a formal series of $\mathcal{D}_{\min}[\![\delta]\!]$ ($X(\delta) = \bigoplus_{\tau \in \mathbb{Z}} x(\tau)\ \delta^{\tau}$), simply due to the fact that it is also equal to $x \otimes e$ (by definition of neutral element e of $\mathcal{D}_{\min}$). For example, the graph depicted in Fig. 1 is represented by equation $X_q(\delta) = \mu_{a^{-1}}\gamma^m\delta^\tau\mu_b X_{q'}(\delta)$ where $X_q(\delta)$ and $X_{q'}(\delta)$ denote elements of $\mathcal{D}_{\min}[\![\delta]\!]$ associated with transitions q and q' respectively.

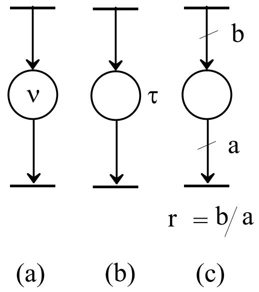

Figure 3: Graphs corresponding to operators γ, δ, μ.

In the following, matrices or scalars with elements in dioid $\mathcal{D}_{\min}[\![\delta]\!]$ are denoted by upper case letters, *i.e.,* X is a shorter notation for $X(\delta)$.

Let us extend the product notation to compose matrices of operators with vectors of signals (with compatible dimensions). Given a matrix of operators A and a vector of signals X with elements in $\mathcal{D}_{\min}[\![\delta]\!]$, we set $(AX)_i \overset{def}{=} \bigoplus_j A_{ij}(X_j)$.

Assertion 2 The counter variables of a TEGM satisfy the following state equations:

$$\begin{cases} X &= AX \oplus BU, \\ Y &= CX \oplus DU, \end{cases} \tag{2}$$

in which state X, input U and output Y vectors are composed of signals, entries of matrices A, B, C, D belong to dioid $\mathcal{D}_{\min}[\![\delta]\!]$.

Example 4 TEGM depicted in Fig. 2 admits the following state equations:

$$\begin{cases} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon & \gamma^3\,\delta^2 \\ \mu_{1/3}\,\delta^2\mu_2 & \varepsilon & \mu_{1/3}\,\gamma^6\,\delta^2\mu_2 \\ \varepsilon & \delta\mu_3 & \varepsilon \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \oplus \begin{pmatrix} e \\ \varepsilon \\ \varepsilon \end{pmatrix} U, \\ Y \quad = (\varepsilon \quad e \quad \varepsilon) \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \oplus \varepsilon U. \end{cases} \tag{3}$$

4 JUST IN TIME CONTROL

4.1 Residuation Theory

Laws $\oplus$ and $\otimes$ of a dioid are not reversible in general. Nevertheless *Residuation* is a general notion in lattice theory which allows defining "pseudo-inverses" of some *isotone* maps (*f* is isotone if $a \preceq b \Rightarrow f(a) \preceq f(b)$). Let us recall some basic results on this theory, see (Blyth and Janowitz, 1972) for details.

Definition 4 (Residual of map) An isotone map $f : \mathcal{D} \to \mathcal{C}$ in which $\mathcal{D}$ and $\mathcal{C}$ are ordered sets is *residuated* if there exists an isotone map $h : \mathcal{C} \to \mathcal{D}$ such that $f \circ h \preceq Id_{\mathcal{C}}$ and $h \circ f \succeq Id_{\mathcal{D}}$ ($Id_{\mathcal{C}}$ and $Id_{\mathcal{D}}$ are identity maps on $\mathcal{C}$ and $\mathcal{D}$ respectively). Map *h,* also noted $f^\sharp$, is unique and is called the *residual* of map *f.*

If *f* is residuated then $\forall y \in \mathcal{C}$, the least upper bound of subset $\{x \in \mathcal{D} \mid f(x) \preceq y\}$ exists and belongs to this subset. This *greatest* "subsolution" is equal to $f^\sharp(y)$.

Let $\mathcal{D}$ be a complete dioid and consider the isotone map $L_a : x \mapsto a \otimes x$ from $\mathcal{D}$ into $\mathcal{D}$. The greatest solution to inequation $a \otimes x \preceq b$ exists and is equal to $L_a^\sharp(b)$, also noted $\frac{b}{a}$. Some results related to this map and used later on are given in the following proposition.

Proposition 1 *((Baccelli et al., 1992, §4.4, 4.5.4, 4.6))*
Let maps $L_a : \mathcal{D} \to \mathcal{D}, x \mapsto a \otimes x$ and $L_b : \mathcal{D} \to \mathcal{D}, x \mapsto b \otimes x$.

1. $\forall a, b, x \in \mathcal{D}$,
$L_{ab}^\sharp(x) = (L_a \circ L_b)^\sharp(x) = (L_b^\sharp \circ L_a^\sharp)(x)$.
More generally, if maps $f : \mathcal{D} \to \mathcal{C}$ and $g : \mathcal{C} \to \mathcal{B}$ are residuated, then $g \circ f$ is also residuated and $(g \circ f)^\sharp = f^\sharp \circ g^\sharp$.
2. $\forall a, x \in \mathcal{D},\ x \succeq a\,x \Leftrightarrow x \preceq \frac{x}{a}$.
3. Let $A \in \mathcal{D}^{n \times p}, B \in \mathcal{D}^{n \times q}, \frac{B}{A} \in \mathcal{D}^{p \times q}$
and $(\frac{B}{A})_{ij} = \wedge_{l=1}^{n} \frac{B_{lj}}{A_{li}}, 1 \le i \le p, 1 \le j \le q$.

Proposition 2 The residuals of operators γ, δ, μ are given by:
$\gamma^{\nu^\sharp} : \{x(t)\}_{t \in \mathbb{Z}} \mapsto \{x(t) - \nu\}_{t \in \mathbb{Z}}$ in which $\nu \in \mathbb{Z} \cup \{+\infty\}$,
$\delta^{\tau^\sharp} : \{x(t)\}_{t \in \mathbb{Z}} \mapsto \{x(t + \tau)\}_{t \in \mathbb{Z}}$ in which $\tau \in \mathbb{Z}$,
$\mu_r^\sharp : \{x(t)\}_{t \in \mathbb{Z}} \mapsto \{\lceil \frac{1}{r} \times x(t) \rceil\}_{t \in \mathbb{Z}}$ in which $r \in \mathbb{Q}^+$ ($\lceil \alpha \rceil$ stands for the superior integer part of real number α).

Proof
Expressions of residuals of operators γ, δ are classical (Baccelli et al., 1992, Chap. 4), (Menguy, 1997, Chap. 4).

Relatively to residuation of operator μ, let us express that $\mu_r = P\mu'_r I$ in which
$I : (\mathbb{Z} \cup \{\pm\infty\})^{\mathbb{Z}} \to (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}}$,
$\{x(t)\}_{t\in\mathbb{Z}} \mapsto \{x(t)\}_{t\in\mathbb{Z}}$,
$\mu'_r : (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}} \to (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}}$,
$\{x(t)\}_{t\in\mathbb{Z}} \mapsto \{r \times x(t)\}_{t\in\mathbb{Z}}$
and $P : (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}} \to (\mathbb{Z} \cup \{\pm\infty\})^{\mathbb{Z}}$,
$\{x(t)\}_{t\in\mathbb{Z}} \mapsto \{\lfloor x(t) \rfloor\}_{t\in\mathbb{Z}}$.
Operator I is residuated, its residual is defined by $I^\sharp : (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}} \to (\mathbb{Z} \cup \{\pm\infty\})^{\mathbb{Z}}$,
$\{x(t)\}_{t\in\mathbb{Z}} \mapsto \{\lceil x(t) \rceil\}_{t\in\mathbb{Z}}$.
Operator P is residuated, its residual is defined by $P^\sharp : (\mathbb{Z} \cup \{\pm\infty\})^{\mathbb{Z}} \to (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}}$,
$\{x(t)\}_{t\in\mathbb{Z}} \mapsto \{x(t)\}_{t\in\mathbb{Z}}$, we have $P^\sharp = I$.
Residuations of I and P are proven directly from Def. 4. Indeed $I, P, I^\sharp$ and $P^\sharp$ are isotone, moreover, $\forall t \in \mathbb{Z}$,
$\forall x \in (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}}, I\,I^\sharp(x(t)) = I(\lceil x(t) \rceil) \preceq x(t)$ and $\forall x \in (\mathbb{Z} \cup \{\pm\infty\})^{\mathbb{Z}}, I^\sharp I(x(t)) = \lceil x(t) \rceil = x(t)$; $\forall x \in (\mathbb{Z} \cup \{\pm\infty\})^{\mathbb{Z}}, PP^\sharp(x(t)) = PI(x(t)) = \lfloor x(t) \rfloor = x(t)$ and $\forall x \in (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}}, P^\sharp P(x(t)) = IP(x(t)) = \lfloor x(t) \rfloor \succeq x(t)$.
Residual of operator μ' is classical, it is defined by ${\mu'_r}^\sharp : (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}} \to (\mathbb{R} \cup \{\pm\infty\})^{\mathbb{Z}}$,
$\{x(t)\}_{t\in\mathbb{Z}} \mapsto \{\frac{1}{r} \times x(t)\}_{t\in\mathbb{Z}}$. Hence, we can deduce the residuation of operator μ. We have $\mu_r^\sharp = (P\mu'_r I)^\sharp = I^\sharp {\mu'_r}^\sharp P^\sharp$ thanks to Prop. 1.1, *i.e.*, $\forall x \in (\mathbb{Z} \cup \{\pm\infty\})^{\mathbb{Z}}, \mu_r^\sharp x(t) = \lceil {\mu'_r}^\sharp I(x(t)) \rceil = \lceil \frac{1}{r} \times x(t) \rceil$.

4.2 Control Problem Statement

Let us consider a TEGM described by Eqs. (2). The just in time control consists in firing input transitions (u) *at the latest* so that the firings of output transitions (y) occur at the latest before the desired ones. Let us define reference input z as the *counter* of the desired outputs: $z_i(t) = n$ means that the firing numbered n of the output transition y_i is desired at the latest at time t. More formally, the just in time control noted u_{opt} is the *greatest* solution (with respect to the order relation $\preceq$) to Eqs. (2) such that $y \preceq z$ (with respect to the usual order relation $\leq$, u_{opt} is the lowest control such that $y \geq z$).

Its expression is deduced from the following result based on Residuation theory.

Proposition 3 Control u_{opt} of TEGM described by Eqs. (2) is the greatest solution (with respect to the order relation $\preceq$) to the following equations:

$$\begin{cases} \xi &= \frac{\xi}{A} \wedge \frac{Z}{C}, \\ U &= \frac{\xi}{B} \wedge \frac{Z}{D}. \end{cases}$$

ξ is the greatest solution of the first equation and corresponds to the latest firings of state transition X ($\xi \succeq X$).

Proof We deduce from Eqs. (2) that state X and output Y are such that $\begin{cases} X & \succeq AX \quad (i) \\ X & \succeq BU \quad (2i) \end{cases}$ and $\begin{cases} Y & \succeq CX \\ Y & \succeq DU \end{cases}$. Moreover, we look for control U such that $Y \preceq Z$ which leads to $\begin{cases} Z & \succeq CX \quad (3i) \\ Z & \succeq DU \quad (4i) \end{cases}$.
The greatest solution to Eq. (*3i*) is equal to $\frac{Z}{C}$. Hence we deduce thanks to Prop. 1.2 that the greatest solution noted ξ verifying Eqs. (*i*) and (*3i*) is equal to $\xi = \frac{\xi}{A} \wedge \frac{Z}{C}$ (sizes of ξ and X are equal). So the greatest solution verifying Eqs. (*2i*) and (*4i*) (in which ξ replaces X) is equal to $\frac{\xi}{B} \wedge \frac{Z}{D}$.

For example let us consider the TEGM depicted in Fig. 2 and modeled by Eqs. (3). Let us give the expression of the just in time control, which leads to calculating the greatest solution of the following equations:

$$\begin{cases} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \frac{\xi_2}{\mu_{(1/3)}\delta^2\mu_2} \\ \frac{\xi_3}{\delta\mu_3} \wedge Z \\ \frac{\xi_1}{\gamma^3\delta^2} \wedge \frac{\xi_2}{\mu_{(1/3)}\gamma^6\delta^2\mu_2} \end{pmatrix}, \\ U = \xi_1. \end{cases}$$

Let us express these equations in usual counter setting. The recursive equations are backwards in time numbering and are supposed to start at some finite time, noted t_f, which means that system is only controlled until this time. So let us consider the following "initial conditions":

$$z(t) = z(t_f) \text{ and } \xi(t) = \xi(t_f),\ \forall t > t_f.$$

For all $t \in \mathbb{Z}$, we have:

$$\begin{cases} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} = \begin{pmatrix} \lceil 3/2 \times \xi_2(t+2) \rceil \\ \max(\lceil 1/3 \times \xi_3(t+1) \rceil, z(t)) \\ \max(\xi_1(t+2) - 3, \lceil 3/2 \times \xi_2(t+2) - 3 \rceil) \end{pmatrix}, \\ u(t) = \xi_1(t). \end{cases}$$

Reference input z and output y are represented in Fig.4, z is such that $z(t) = z(t_f),\ \forall t > t_f = 15$. Control u is represented in Fig.5 and is as late as possible so that desired behavior of output transition is satisfied ($y \preceq z$). Moreover, control u is such that components of ξ are greater than or equal to those of x ($x \preceq \xi$).

5 CONCLUSION

Most works on dioid deal with discrete timed event graphs without multipliers. We aim at showing here the efficiency of dioid theory to also just in time control TEGM's without additional difficulties. The proposed method is mainly based on Residuation theory and the control is the greatest solution of "backward" equations. A possible development of this work would consist in considering hybrid systems or more complex control objectives.

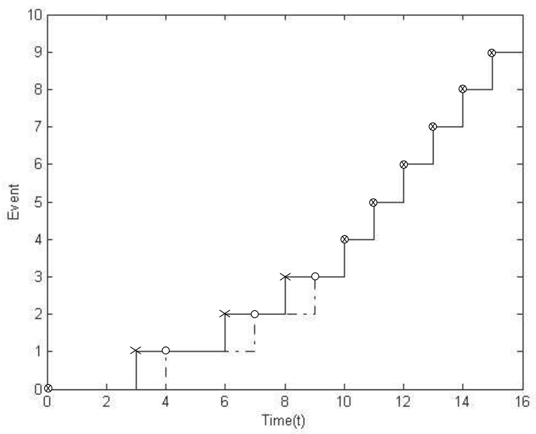

Figure 4: Output *y* (thin line) and reference input *z* (dotted line).

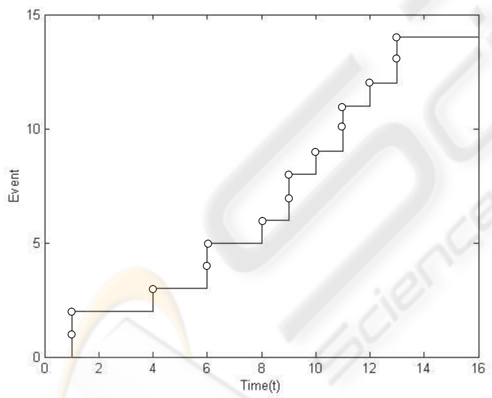

Figure 5: Control u.

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