AN EVOLUTIONARY APPROACH TO NONLINEAR DISCRETE-TIME OPTIMAL CONTROL WITH TERMINAL CONSTRAINTS

Yechiel Crispin
Department of Aerospace Engineering
Embry-Riddle University
Daytona Beach, FL 32114

Keywords: Optimal Control, Genetic Algorithms

Abstract: The nonlinear discrete-time optimal control problem with terminal constraints is treated using a new evolutionary approach which combines a genetic search for finding the control sequence with a solution of the initial value problem for the state variables. The main advantage of the method is that it does not require to obtain the solution of the adjoint problem which usually leads to a two-point boundary value problem combined with an optimality condition for finding the control sequence. The method is verified by solving the problem of discrete velocity direction programming with the effects of gravity and thrust and a terminal constraint on the final vertical position. The solution compared favorably with the results of gradient methods.

1 INTRODUCTION

A continuous-time optimal control problem consists of finding the time histories of the controls and the state variables such as to maximize (or minimize) an integral performance index over a finite period of time, subject to dynamical constraints in the form of a system of ordinary differential equations (Bryson, 1975). In a discrete-time optimal control problem, the time period is divided into a finite number of time intervals of equal time duration ΔT. The controls are kept constant over each time interval. This results in a considerable simplification of the continuous time problem, since the ordinary differential equations can be reduced to difference equations and the integral performance index can be reduced to a finite sum over the discrete time counter (Bryson, 1999). In some problems, additional constraints may be prescribed on the final states of the system. In this paper, we concentrate on the discrete-time optimal control problem with terminal constraints.Modern methods for solving the optimal control problem are extensions of the classical methods of the calculus of variations (Fox, 1950).

These methods are known as indirect methods and are based on the maximum principle of Pontryagin, which is astatement of the first order necessary conditions for optimality, and results in a two-point boundary value problem (TPBVP) for the state and adjoint variables (Pontryagin, 1962). It has been known, however, that the TPBVP is much more difficult to solve than the initial value problem (IVP). As a consequence, a second class of solutions, known as the direct method has evolved.

For example, attempts have been made to recast the original dynamic problem as a static optimization problem, also known as a nonlinear programming (NLP) problem.

This can be achieved by parameterisation of the state variables or the controls, or both. In this way, the original dynamical differential equations or difference equations are reduced to algebraic equality constraints. The problems with this approach is that it might result in a large scale NLP problem which has stability and convergence problems and might require excessive computing time. Also, the parameterisation might introduce spurious local minima which are not present in the original problem.

Crispin Y. (2004).
AN EVOLUTIONARY APPROACH TO NONLINEAR DISCRETE-TIME OPTIMAL CONTROL WITH TERMINAL CONSTRAINTS.
In *Proceedings of the First International Conference on Informatics in Control, Automation and Robotics*, pages 49-57
DOI: 10.5220/0001134700490057

Several gradient based methods have been proposed for solving the discrete-time optimal control problem (Mayne, 1966). For example, Murray and Yakowitz (Murray, 1984) and (Yakowitz, 1984) developed a differential dynamic programming and Newton's method for the solution of discrete optimal control problems, see also the book of Jacobson and Mayne (Jacobson, 1970), (Ohno, 1978), (Pantoja, 1988) and (Dunn, 1989). Similar methods have been further developed by Liao and Shoemaker (Liao, 1991). Another method, the trust region method, was proposed by Coleman and Liao (Coleman, 1995) for the solution of unconstrained discrete-time optimal control problems. Although confined to the unconstrained problem, this method works for large scale minimization and is capable of handling the so called hard case problem.

Each method, whether direct or indirect, gradient-based or direct search based, has its own advantages and disadvantages. However, with the advent of computing power and the progress made in methods that are based on optimization analogies from nature, it became possible to achieve a remedy to some of the above mentioned disadvantages through the use of global methods of optimization. These include stochastic methods, such as simulated annealing (Laarhoven, 1989), (Kirkpatrick, 1983) and evolutionary computation methods (Fogel, 1998), (Schwefel, 1995) such as genetic algorithms (GA) (Michalewicz, 1992a), see also (Michalewicz, 1992b) for an interesting treatment of the linear discrete-time problem.

Genetic algorithms provide a powerful mechanism towards a global search for the optimum, but in many cases, the convergence is very slow. However, as will be shown in this paper, if the GA is supplemented by problem specific heuristics, the convergence can be accelerated significantly. It is well known that GAs are based on a guided random search through the genetic operators and evolution by artificial selection. This process is inherently very slow, because the search space is very large and evolution progresses step by step, exploring many regions with solutions of low fitness. What is proposed here, is to guide the search further, by incorporating qualitative knowledge about potential good solutions. In many problems, this might involve simple heuristics, which when combined with the genetic search, provide a powerful tool for finding the optimum very quickly.

The purpose of the present work, then, is to incorporate problem specific heuristic arguments, which when combined with a modified hybrid GA, can solve the discrete-time optimal control problem very easily. There are significant advantages to this approach. First, the need to solve the two-point boundary value problem (TPBVP) is completely avoided. Instead, only initial value problems (IVP) are solved. Second, after finding an optimal solution, we verify that it approximately satisfies the first-order necessary conditions for a stationary solution, so the mathematical soundness of the traditional necessary conditions is retained. Furthermore, after obtaining a solution by direct genetic search, the static and dynamic Lagrange multipliers (the adjoint variables) can be computed and compared with the results from a gradient method. All this is achieved without directly solving the TPBVP. There is a price to be paid, however, since, in the process, we are solving many initial value problems (IVPs). This might present a challenge in advanced difficult problems, where the dynamics are described by a higher order system of ordinary differential equations, or when the equations are difficult to integrate over the required time interval and special methods are required. On the other hand, if the system is described by discrete-time difference equations that are relatively well behaved and easy to iterate, the need to solve the initial value problem many times does not represent a serious problem. For instance, the example problem presented here , the discrete velocity programming problem (DVDP) with the combined effects of gravity, thrust and drag, together with a terminal constraint (Bryson, 1999), runs on a 1.6 GHz pentium 4 processor in less than a minute of CPU time.

In the next section, a mathematical formulation of the optimal control problem is given. The evolutionary approach to the solution is then described. In order to illustrate the method, a specific example, the discrete velocity direction programming (DVDP) is solved and results are presented and compared with the results of an indirect gradient method developed by Bryson (Bryson, 1999).

2 OPTIMAL CONTROL OF NONLINEAR DISCRETE TIME DYNAMICAL SYSTEMS

In this section, we describe a formulation of the nonlinear discrete-time optimal control problem subject to terminal constraints. Consider the nonlinear discrete-time dynamical system described by a system of difference equations with initial conditions

(1) $\boldsymbol{x}(i+1) = \boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i]$

(2) $\boldsymbol{x}(0) = \boldsymbol{x_0}$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of state variables, $\boldsymbol{u} \in \mathbb{R}^p,\ p < n$ is the vector of control variables and $i \in [0, N-1]$ is a discrete time counter. The function $\boldsymbol{f}$ is a nonlinear function of the state vector, the control vector and the discrete time i, i.e., $\boldsymbol{f} : \mathbb{R}^n$ x $\mathbb{R}^p$ x $\mathbb{R} \mapsto \mathbb{R}^n$. Define a performance index

(3)
$$J[\boldsymbol{x}(i), \boldsymbol{u}(i), i, N] = \phi[\boldsymbol{x}(N)] + \sum_{i=0}^{N-1} L[\boldsymbol{x}(i), \boldsymbol{u}(i), i]$$

where
$\phi : \mathbb{R}^n \mapsto \mathbb{R}, \qquad L : \mathbb{R}^n$ x $\mathbb{R}^p$ x $\mathbb{R} \mapsto \mathbb{R}$

Here L is the Lagrangian function and $\phi[\boldsymbol{x}(N)]$ is a function of the terminal value of the state vector $\boldsymbol{x}(N)$. Terminal constraints in the form of additional functions $\boldsymbol{\psi}$ of the state variables are prescribed as

(4) $\boldsymbol{\psi}[\boldsymbol{x}(N)] = 0 \qquad \boldsymbol{\psi} : \mathbb{R}^n \mapsto \mathbb{R}^k \quad k \le n$

The optimal control problem consists of finding the control sequence $\boldsymbol{u}(i)$ such as to maximize (or minimize) the performance index defined by (3), subject to the dynamical equations (1) with initial conditions (2) and terminal constraints (4). This is known as the Bolza problem in the calculus of variations (Bolza, 1904).
In an alternative formulation, due to Mayer, the state vector $x_j,\ j \in (1, n)$ is augmented by an additional variable x_{n+1} which satisfies the following initial value problem:

(5) $x_{n+1}(i+1) = x_{n+1}(i) + L[\boldsymbol{x}(i), \boldsymbol{u}(i), i]$

(6) $x_{n+1}(0) = 0$

The performance index can then be written as

(7)
$$J(N) = \phi[\boldsymbol{x}(N)] + x_{n+1}(N) \equiv \phi_a[\boldsymbol{x_a}(N)]$$

where $\boldsymbol{x_a} = [\boldsymbol{x}\ \ x_{n+1}]^T$ is the augmented state vector and ϕ_a the augmented performance function. In this paper, the Meyer formulation is used.

Define an augmented performance index with adjoint constraints $\boldsymbol{\psi}$ and adjoint dynamical constraints $\boldsymbol{f}(\boldsymbol{x}(i), \boldsymbol{u}(i), i) - \boldsymbol{x}(i+1) = 0$, with static and dynamical Lagrange multipliers $\boldsymbol{\nu}$ and $\boldsymbol{\lambda}$, respectively.

(8) $$J_a = \phi + \boldsymbol{\nu}^T \boldsymbol{\psi} + \boldsymbol{\lambda}^T(0)[\boldsymbol{x_0} - \boldsymbol{x}(0)] + \sum_{i=0}^{N-1} \boldsymbol{\lambda}^T(i+1)\{\boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i] - \boldsymbol{x}(i+1)\}$$

Define the Hamiltonian function as

(9) $H(i) = \boldsymbol{\lambda}^T(i+1)\boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i]$

Rewriting the augmented performance index in terms of the Hamiltonian, we get

(10)
$$J_a = \phi + \boldsymbol{\nu}^T\boldsymbol{\psi} - \boldsymbol{\lambda}^T(N)\boldsymbol{x}(N) + \boldsymbol{\lambda}^T(0)\boldsymbol{x_0} + \sum_{i=0}^{N-1} [H(i) - \boldsymbol{\lambda}^T(i)\boldsymbol{x}(i)]$$

A first order necessary condition for J_a to reach a stationary solution is given by the discrete version of the Euler-Lagrange equations

(11)
$$\boldsymbol{\lambda}^T(i) = H_{\boldsymbol{x}}(i) = \boldsymbol{\lambda}^T(i+1)\boldsymbol{f_x}[\boldsymbol{x}(i), \boldsymbol{u}(i), i]$$

with final conditions

(12) $\boldsymbol{\lambda}^T(N) = \phi_x + \boldsymbol{\nu}^T \boldsymbol{\psi_x}$

and the control sequence $\boldsymbol{u}(i)$ satisfies the optimality condition:
(13) $H_{\boldsymbol{u}}(i) = \boldsymbol{\lambda}^T(i+1)\boldsymbol{f_u}[\boldsymbol{x}(i), \boldsymbol{u}(i), i] = 0$

Define an augmented function Φ as

(14) $\Phi = \phi + \boldsymbol{\nu}^T \boldsymbol{\psi}$

Then, the adjoint equations for the dynamical multipliers are given by

(15)
$$\boldsymbol{\lambda}^T(i) = H_{\boldsymbol{x}}(i) = \boldsymbol{\lambda}^T(i+1)\boldsymbol{f}_{\boldsymbol{x}}[\boldsymbol{x}(i), \boldsymbol{u}(i), i]$$

and the final conditions can be written in terms of the augmented function Φ

(16) $$\boldsymbol{\lambda}^T(N) = \Phi_{\boldsymbol{x}} = \phi_{\boldsymbol{x}} + \boldsymbol{\nu}^T \boldsymbol{\psi}_{\boldsymbol{x}}$$

If the indirect approach to optimal control is used, the state equations (1) with initial conditions (2) need to be solved together with the adjoint equations (15) and the final conditions (16), where the control sequence $u(i)$ is to be determined from the optimality condition (13). This represents a coupled system of nonlinear difference equations with part of the boundary conditions specified at the initial time $i = 0$ and the rest of the boundary conditions specified at the final time $i = N$. This is a nonlinear two-point boundary value problem (TPBVP) in difference equations. Except for special simplified cases, it is usually very difficult to obtain solutions for such nonlinear TPBVPs. Therefore, many numerical methods have been developed to tackle this problem, see the introduction for several references.

In the proposed approach, the optimality condition (13) and the adjoint equations (15) together with their final conditions (16) are not used in order to obtain the optimum solution. Instead, the optimal values of the control sequence $u(i)$ are found by genetic search starting with an initial population of solutions with values of $u(i)$ randomly distributed within a given domain. During the search, approximate, less than optimal values of the solutions $u(i)$ are available for each generation. With these approximate values known, the state equations (1) together with their initial conditions (2) are very easy to solve, by a straightforward iteration of the difference equations from $i = 0$ to $i = N - 1$. At the end of this iterative process, the final values $\boldsymbol{x}(N)$ are obtained, and the fitness function can be determined. The search than seeks to maximize the fitness function F such as to fulfill the goal of the evolution, which is to maximize $J(N)$, as given by the following Eq.(17), subject to the terminal constraints $\boldsymbol{\psi}[\boldsymbol{x}(N)] = 0$, as defined by Eq.(18).

(17) maximize $J(N) = \phi[\boldsymbol{x}(N)]$

subject to the dynamical equality constraints, Eqs. (1-2) and to the terminal constraints (4), which are repeated here for convenience as Eq.(18)

(18) $$\boldsymbol{\psi}[\boldsymbol{x}(N)] = 0 \qquad \boldsymbol{\psi} : \mathbb{R}^n \mapsto \mathbb{R}^k \quad k \le n$$

Since we are using a direct search method, condition (18) can also be stated as a search for a maximum, namely we can set a goal which is equivalent to (18) in the form

(19) maximize $$J_1(N) = -\boldsymbol{\psi}^{\mathbf{T}}[\boldsymbol{x}(N)]\boldsymbol{\psi}[\boldsymbol{x}(N)]$$

The fitness function F can now be defined by

(20) $$F(N) = \alpha J(N) + (1-\alpha)J_1(N) =$$

$$= \alpha\phi[\boldsymbol{x}(N)] - (1-\alpha)\boldsymbol{\psi}^{\mathbf{T}}[\boldsymbol{x}(N)]\boldsymbol{\psi}[\boldsymbol{x}(N)],$$

with $\alpha \in [0,1]$ and $\boldsymbol{x}(N)$ determined from the following dynamical equality constraints:

(21) $$\boldsymbol{x}(i+1) = \boldsymbol{f}[\boldsymbol{x}(i), \boldsymbol{u}(i), i],$$
$$i \in [0, N-1]$$

(22) $$\boldsymbol{x}(0) = \boldsymbol{x}_0$$

3 DISCRETE VELOCITY DIRECTION PROGRAMMING FOR MAXIMUM RANGE WITH GRAVITY AND THRUST

In this section, the above formulation is applied to a specific example, namely, the problem of finding the trajectory of a point mass subjected to gravity and thrust and a terminal constraint such as to achieve maximum horizontal distance, with the direction of the velocity used to control the motion. This problem has its roots in the calculus of variations and is related to the classical Brachistochrone problem, in which the shape of a wire is sought along which a bead slides without friction, under the action of gravity alone, from an initial point (x_0, y_0) to a final point (x_f, y_f) in minimum time t_f. The dual problem to the Brachistochrone problem consists of finding the shape of the wire such as the bead will reach a maximum horizontal distance x_f in a prescribed time t_f. Here, we treat the dual problem with the added effects of thrust and a terminal constraint where the final vertical position y_f is prescribed. The more difficult problem, where the body is moving in a viscous fluid and the effect of drag is taken into account was also solved, but due to lack of space, the results will be presented elsewhere. The reader interested in these problems can find an extensive discussion in Bryson's book (Bryson,1999).

Let point O be the origin of a cartesian system of coordinates in which x is pointing to the right and y is pointing down. A constant thrust force $\boldsymbol{F}$ is acting along the path on a particle of mass m which moves in a medium without friction. A constant gravity field with acceleration g is acting downward in the positive y direction. The thrust acts in the direction of the velocity vector $\boldsymbol{V}$ and its magnitude is $F = amg$, i.e. a times the weight mg. The velocity vector $\boldsymbol{V}$ acts at an angle γ with respect to the positive direction of x. The angle γ, which serves as the control variable, is positive when $\boldsymbol{V}$ points downward from the horizontal. The problem is to find the control sequence $\gamma(t)$ to maximize the horizontal range x_f in a given time t_f, provided the particle ends at the vertical location y_f. In other words, the velocity direction $\gamma(t)$ is to be programmed such as to achieve maximum range and fulfill the terminal constraint y_f. The equations of motion are

(23) $dV/dt = g(a + sin\gamma(t))$

(24) $dx/dt = Vcos\gamma(t)$

(25) $dy/dt = Vsin\gamma(t)$

with initial conditions

(26) $V(0) = 0, x(0) = 0,\ y(0) = 0$

and final constraint

(27) $y(t_f) = y_f.$

We would like to formulate a discrete time version of this problem. The trajectory is divided into a finite number N of straight line segments of fixed time duration $\Delta T = t_f/N$, along which the direction γ is constant. We can increase N such as to approach the solution of the continuous trajectory. The velocity $\boldsymbol{V}$ is increasing under the influence of a constant thrust ag and gravity $gsin\gamma$. The problem is to determine the control sequence $\gamma(i)$ at points i along the trajectory, where $i \in [0, N-1]$, such as to maximize x at time t_f, arriving at the same time at a prescribed elevation y_f. The time at the end of each segment is given by $t(i) = i\Delta T$, so i can be viewed as a time step counter at point i. Integrating the first equation of motion, Eq.(17) from a time $t(i) = i\Delta T$ to $t(i+1) = (i+1)\Delta T$, we get

(28) $V(i+1) = V(i) + g[a + sin\gamma(i)]\Delta T$

Integrating the velocity V over a time interval ΔT, we obtain the length of the segment $\Delta d(i)$ connecting the points i and $i+1$.

(29)
$\Delta d(i) = V(i)\Delta T + \frac{1}{2}g[a + sin\gamma(i)](\Delta T)^2$

Once $\Delta d(i)$ is determined, it is easy to obtain the coordinates x and y as

(30) $x(i+1) = x(i) + \Delta d(i)cos\gamma(i) =$

$= x(i) + V(i)cos\gamma(i)\Delta T +$

$+ \frac{1}{2}g[a + sin\gamma(i)]cos\gamma(i)(\Delta T)^2$

(31) $y(i+1) = y(i) + \Delta d(i)sin\gamma(i) =$

$= y(i) + V(i)sin\gamma(i)\Delta T +$

$+ \frac{1}{2}g[a + sin\gamma(i)]sin\gamma(i)(\Delta T)^2$

We now develop the equations in nondimensional form. Introduce the following nondimensional variables denoted by primes:

(32) $t = t_f t',\ V = gt_f V',\ \ x = gt_f^2 x',$
$y = gt_f^2 y'$

Since $t(i) = it_f/N$, the nondimensional time is $t'(i) = i/N$. The time interval was defined as $\Delta T = t_f/N = t_f(\Delta T)'$, so the nondimensional time interval becomes $(\Delta T)' = 1/N$. Substituting the nondimensional variables in the discrete equations of motion and omitting the prime notation, we obtain the nondimensional state equations:

(33) $V(i+1) = V(i) + [a + sin\gamma(i)]/N$

(34) $x(i+1) = x(i) + V(i)cos\gamma(i)/N +$

$+ \frac{1}{2}[a + sin\gamma(i)]cos\gamma(i)/N^2$

(35) $y(i+1) = y(i) + V(i)sin\gamma(i)/N +$

$+ \frac{1}{2}[a + sin\gamma(i)]sin\gamma(i)/N^2$

with initial conditions

(36) $V(0) = 0, x(0) = 0,\ y(0) = 0$

and terminal constraint

(37) $y(N) = y_f$

The optimal control problem now consists of finding the sequence $\gamma(i)$ such as to maximize the range $x(N)$, subject to the dynamical constraints (33-35), the initial conditions (36) and the terminal constraint (37), where y_f is in units of $g\,t_f^2$ and the final time t_f is given.

4 NECESSARY CONDITIONS FOR AN OPTIMUM

In this section we present the traditional indirect approach to the solution of the optimal control problem, which is based on the first order necessary conditions for an optimum. First, we derive the Hamiltonian function for the above DVDP problem. We then derive the adjoint dynamical equations for the adjoint variables (the Lagrange multipliers) and the optimality condition that needs to be satisfied by the control sequence $\gamma(i)$. Since we have used the symbol x for the horizontal coordinate, we denote the state variables by $\boldsymbol{\xi}$. So the state vector for this problem is

$$\boldsymbol{\xi} = [V\ \ x\ \ y]^T$$

The performance index and the terminal constraint are given by

(38) $J(N) = \phi[\boldsymbol{\xi}(N)] = x(N)$

(39) $\psi(N) = \psi[\boldsymbol{\xi}(N)] = y(N) - y_f = 0$

The Hamiltonian $H(i)$ is defined by

(40)
$H(i) = \lambda_V(i+1)\{V(i) + [a + sin\gamma(i)]/N\}$
$+$

$+ \lambda_x(i+1)\{x(i) + V(i)cos\gamma(i)/N +$

$+ \frac{1}{2}[a + sin\gamma(i)]cos\gamma(i)/N^2\} +$

$+ \lambda_y(i+1)\{y(i) + V(i)sin\gamma(i)/N +$

$+ \frac{1}{2}[a + sin\gamma(i)]sin\gamma(i)/N^2\}$

The augmented performance index is given by

(41) $\Phi = \phi + \boldsymbol{\nu}^T\,\boldsymbol{\psi} = x(N) + \nu[y(N) - y_f]$

The discrete Euler-Lagrange equations are derived from the Hamiltonian function:

(42) $\lambda_V(i) = H_V(i) = \lambda_V(i+1) +$

$+ cos\gamma(i)\lambda_x(i+1)/N + sin\gamma(i)\lambda_y(i+1)/N$

(43) $\lambda_x(i) = H_x(i) = \lambda_x(i+1)$

(44) $\lambda_y(i) = H_y(i) = \lambda_y(i+1)$

It follows from the last two equations that the multipliers $\lambda_x(i)$ and $\lambda_y(i)$ are constant. The final conditions for the multipliers are obtained from the augmented function Φ.

(45)
$\lambda_V(N) = \Phi_V(N) = \phi_V(N) + \nu\,\psi_V(N) = 0$

$\lambda_x(N) = \Phi_x(N) = \phi_x(N) + \nu\,\psi_x(N) = 1$

$\lambda_y(N) = \Phi_y(N) = \phi_y(N) + \nu\,\psi_y(N) = \nu$

Since λ_x and λ_y are constant, they can be set equal to their final values:

(46) $\lambda_x(i) = \lambda_x(N) = 1,$
$\lambda_y(i) = \lambda_y(N) = \nu$

With the values given in (46), the equation for $\lambda_V(i)$ becomes

(47)
$\lambda_V(i) = \lambda_V(i+1) + cos\gamma(i)/N + \nu sin\gamma(i)/N$

with final condition

(48) $\lambda_V(N) = 0$

The required control sequence $\gamma(i)$ is determined from the optimality condition

(49)
$H_\gamma(i) = \lambda_V(i+1)cos\gamma(i)/N - V(i)sin\gamma(i)/N - asin\gamma(i)/(2N^2) + cos(2\gamma(i))/(2N^2) + \nu V(i)cos\gamma(i)/N + \nu acos\gamma(i)/(2N^2) + \nu sin(2\gamma(i))/(2N^2) = 0$

$\lambda_V(i+1)$ is determined by the adjoint equation (47) and the Lagrange multiplier ν is determined from the terminal equality constraint $y(N) = y_f$.

5 AN EVOLUTIONARY APPROACH TO OPTIMAL CONTROL

We now describe the direct approach using genetic search. As was mentioned in Sec. , there is no need to solve the two-point boundary value problem described by the state equations (33-35) and the adjoint equation (47), together with the initial conditions (36), the final condition (48), the terminal constraint (37) together with the optimality condition (49) for the optimal control $\gamma(i)$. Instead, the direct evolutionary method allows us to evolve a population of solutions such as to maximize the objective function or fitness function $F(N)$. The initial population is built by generating a random population of solutions $\gamma(i)$, $i \in [0, N-1]$, uniformly distributed within a domain $\gamma \in [\gamma_{min}, \gamma_{max}]$. Typical values are $\gamma_{max} = \pi/2$ and either $\gamma_{min} = -\pi/2$ or $\gamma_{min} = 0$ depending on the problem. The genetic algorithm evolves this initial population using the operations of selection, mutation and crossover over many generations such as to maximize the fitness function:

(50) $F(N) = \alpha J(N) + (1-\alpha)J_1(N) =$

$= \alpha\phi[\boldsymbol{\xi}(N)] - (1-\alpha)\boldsymbol{\psi}^{\mathbf{T}}[\boldsymbol{\xi}(N)]\boldsymbol{\psi}[\boldsymbol{\xi}(N)],$

with $\alpha \in [0,1]$ and $J(N)$ and $J_1(N)$ given by:

(51) $J(N) = \phi[\boldsymbol{\xi}(N)] = x(N)$

(52) $J_1(N) = \psi^2[\boldsymbol{\xi}(N)] = (y(N) - y_f)^2$

For each member in the population of solutions, the fitness function depends on the final values $x(N)$ and $y(N)$, which are determined by solving the initial value problem defined by the state equations (33-35) together with the initial conditions (36). This process is repeated over many generations. Here, we run the genetic algorithm for a predetermined number of generations and then we check if the terminal constraint (52) is fulfilled. If the constraint is not fulfilled, we can either increase the number of generations or readjust the weight $\alpha \in [0,1]$. After obtaining an optimal solution , we can check the first order necessary conditions by first solving the adjoint equation (47) with its final condition (48). Once the control sequence is known, the solution of (47-48) is obtained by direct iteration backwards in time. We then check to what extent the optimality condition (49) is fulfilled by determining $H_\gamma(i) = e(i)$ for $i \in [0, N-1]$ and plotting the result as an error $e(i)$ measuring the deviation from zero.
The results for the DVDP problem with gravity and thrust, with $a = 0.5$ and the terminal constraint $y_f = -0.1$ are shown in Figs.(1-3). A value of $\alpha = 0.01$ was used. Fig.1 shows the evolution of the solution over 50 generations. The best fitness and the average fitness of the population are given. In all calculations the size of the population was 50 members.

The control sequence $\gamma(i)$, the velocity $V(i)$ and the optimal trajectory are given in Fig.2 where the sign of y is reversed for plotting. The trajectory obtained here was compared to that obtained by Bryson (Bryson, 1999) using a gradient method and the results are similar. In Fig.3 we plot the expression for $dH/d\gamma(i)$ as given by
the right-hand side of Eq.(49). Ideally, this should be equal to zero at every point i. However, since we

are not using (49) to determine the control sequence, we obtain a small deviation from zero in our calculation. Finally, after determining the optimal solution, i.e. after the control and the trajectory are known, the adjoint variable $\lambda_V(i)$ can be estimated by using Eqs.(47-48). The result is shown in Fig.3.

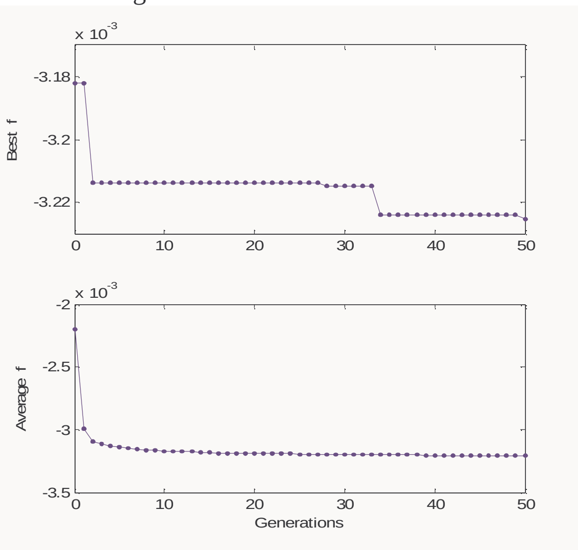

Figure 1: Convergence of the DVDP solution with gravity and thrust, $a = 0.5$ and terminal constraint $y_f = -0.1$

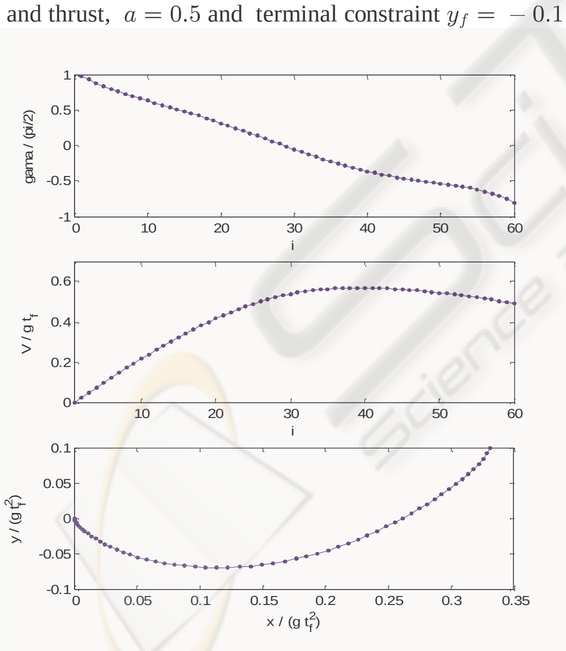

Figure 2: The control sequence $\gamma(i)$, the velocity $V(i)$ and the optimal trajectory for the DVDP problem with gravity g, thrust $a = 0.5$ and terminal constraint $y_f = -0.1$. The sign of y is reversed for plotting

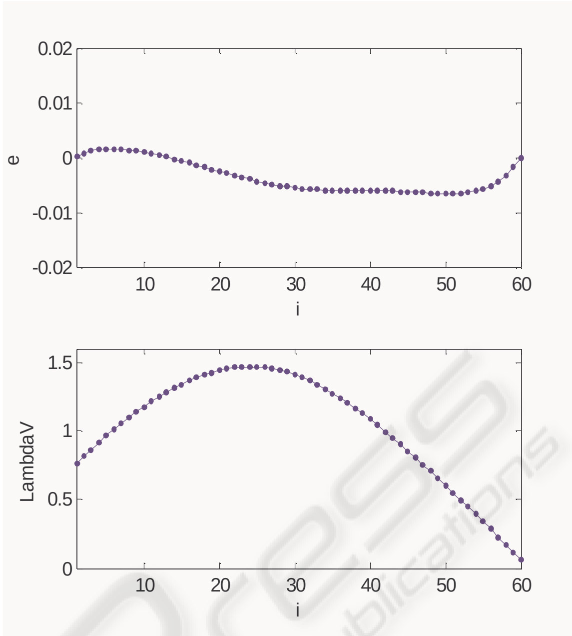

Figure 3: The error $e(i)$ measuring the deviation of the optimality condition $H_\gamma(i) = e(i)$ from zero, and the adjoint variable $\lambda_V(i)$ as a function of the discrete-time counter i. DVDP with gravity and thrust, $a = 0.5$. Here $N = 60$ time steps

6 CONCLUSIONS

A new method for solving the discrete-time optimal control problem with terminal constraints was developed. The method seeks the best control sequence diretly by genetic search and does not make use of the first-order necessary conditions to find the optimum. As a consequence, the need to develop a Hamiltonian formulation and the need to solve a difficult two-point boundary value problem for finding the adjoint variables is completely avoided. This has a significant advantage in more advanced and higher order problems where it is difficult to solve the TPBVP with large systems of differential equations, but when it is still easy to solve the initial value problem (IVP) for the state variables. The method was demonstrated by solving a discrete-time optimal control problem, namely, the DVDP or the discrete velocity direction programming problem that was pioneered by Bryson using both analytical and gradient methods. This problem includes the effects of gravity and thrust and was solved easily using the proposed approach. The results compared favorably with those of

Bryson, who used analytical and gradient techniques.

REFERENCES

Bryson, A. E. and Ho, Y. C., Applied Optimal Control, Hemisphere, Washington, D.C., 1975.

Bryson, A.E., Dynamic Optimization, Addison-Wesley Longman, Menlo Park, CA, 1999.

Coleman, T.F. and Liao, A., An efficient trust region method for unconstrained discrete-time optimal control problems, Computational Optimization and Applications, 4, pp. 47-66, 1995.

Dunn, J., and Bertsekas, D.P., Efficient dynamic programming implementations of Newton's method for unconstrained optimal control problems, J. of Optimization Theory and Applications, 63 (1989), pp. 23-38.

Fogel, D.B., Evolutionary Computation, The Fossil Record, IEEE Press, New York, 1998.

Fox, C., An Introduction to the Calculus of Variations, Oxford University Press, London, 1950.

Jacobson, D. and Mayne, D. , Differential Dynamic Programming, Elsevier Science Publishers, 1970.

Kirkpatrick, and Gelatt, C.D. and Vecchi, Optimization by Simulated Annealing, Science, 220, 671-680, 1983.

Laarhoven, P.J.M. and Aarts, E.H.L., Simulated Annealing: Theory and Applications, Kluwer Academic, 1989.

Liao, L.Z. and Shoemaker, C.A., Convergence in unconstrained discrete-time differential dynamic programming, IEEE Trans. Automat. Contr., 36, pp. 692-706, 1991.

Mayne, D., A second-order gradient method for determining optimal trajectories of non-linear discrete time systems, Intnl. J. Control, 3 (1966), pp. 85-95.

Michalewicz, Z., Genetic Algorithms + Data Structures = Evolution Programs, Springer-Verlag, Berlin, 1992a.

Michalewicz, Z., Janikow, C.Z. and Krawczyk, J.B., A Modified Genetic Algorithm for Optimal Control Problems, Computers Math. Applications, 23(12), pp. 83-94, 1992b.

Murray, D.M. and Yakowitz, S.J., Differential dynamic programming and Newton's method for discrete optimal control problems, J. of Optimization Theory and Applications, 43, pp. 395-414, 1984.

Ohno, K., A new approach of differential dynamic programming for discrete time systems, IEEE Trans. Automat. Contr., 23 (1978), pp. 37-47.

Pantoja, J.F.A. de O. , Differential Dynamic Programming and Newton's Method, International Journal of Control, 53: 1539-1553, 1988.

Pontryagin, L. S. Boltyanskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., The Mathematical Theory of Optimal Processes, Moscow, 1961, translated by Trirogoff, K. N. Neustadt, L. W. (Ed.), Interscience, New York, 1962.

Schwefel H.P., Evolution and Optimum Seeking, Wiley, New York, 1995.

Yakowitz, S.J., and Rutherford, B., Computational aspects of discrete-time optimal control, Appl. Math. Comput., 15 (1984), pp. 29-45.