ZEROS, OUTPUT-NULLING SUBSPACES AND ZERO DYNAMICS IN MIMO LTI SYSTEMS
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Abstract: In standard MIMO LTI continuous-time systems S(A,B,C) the classical notion of the Smith zeros does not characterize fully the output-zeroing problem nor the zero dynamics. The question how this notion can be extended and related to the state-space methods is discussed. Nothing is assumed about the relationship of the number of inputs to the number of outputs nor about the normal rank of the underlying system matrix. The proposed extension treats zeros (called further the invariant zeros) as the triples (complex number, nonzero state-zero direction, input-zero direction). Such treatment is strictly connected with the output zeroing problem and in that spirit the zeros can be easily interpreted even in the degenerate case (i.e., when any complex number is such zero). A simple sufficient and necessary condition of degeneracy is presented. The condition decomposes the class of all systems S(A,B,C) such that $B \neq 0$ and $C \neq 0$ into two disjoint subclasses: of nondegenerate and degenerate systems. In nondegenerate systems the Smith zeros and the invariant zeros are exactly the same objects which are determined as the roots of the so-called zero polynomial. The degree of this polynomial equals the dimension of the maximal (A,B)-invariant subspace contained in Ker C, while the zero dynamics are independent upon control vector. In degenerate systems the zero polynomial determines merely the Smith zeros, while the set of the invariant zeros equals the whole complex plane. The dimension of the maximal (A,B)-invariant subspace contained in Ker C is strictly larger than the degree of the zero polynomial, whereas the zero dynamics essentially depend upon control vector.

1 INTRODUCTION

During the past three decades considerable attention has been paid to the determination and computation of zeros of a LTI MIMO system S(A,B,C). A large number of types of zeros has been defined (MacFarlane,Karcanias, 1976; Schrader, Sain, 1989). The commonly used definitions employ the Smith form (Callier, Desoer, 1982; Chen, 1984; Gantmacher, 1988) of the system matrix $P(s)=\begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix}$. Recall (Callier, Desoer, 1982) that for P(s) there exist unimodular matrices U(s) and V(s) and a polynomial matrix $\Psi(s)$ such that $P(s)=U(s)\Psi(s)V(s)$ and $\Psi(s)$ has the form

$$\Psi(s)=\begin{bmatrix} \psi_1(s) & 0 & & 0 \\ 0 & . & 0 & \\ & & \psi_\nu(s) & 0 \\ 0 & & 0 & 0 \quad 0 \end{bmatrix}.$$

Here $\Psi(s)$ is called the Smith form of P(s) when polynomials $\psi_i(s)$ are monic and $\psi_i(s)$ divides $\psi_{i+1}(s)$ for $i=1,\ldots,\nu-1$, and ν is the normal rank of P(s). The polynomials $\psi_i(s)$ are known as invariant factors of P(s) and their product $\psi(s)=\psi_1(s)\ldots\psi_\nu(s)$ is called the zero polynomial of P(s) (and of S(A,B,C)). The roots of $\psi(s)$ are the Smith zeros of S(A,B,C) (they are commonly known rather as invariant zeros (Basile, Marro, 1992; Marro

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et al., 2002)). The transmission zeros of S(A,B,C) are the Smith zeros of its minimal (controllable and observable) subsystem. The zeros of a transfer-function matrix G(s) can be defined (Misra et al., 1994) as the Smith zeros of the system matrix obtained from any given minimal state-space realization of G(s). The Smith zeros of the pencil $\begin{bmatrix} sI-A & -B \end{bmatrix}$ (i.e., uncontrollable ($\bar{c}$) modes of A) are the input decoupling (i.d.) zeros and the Smith zeros of $\begin{bmatrix} sI-A \\ C \end{bmatrix}$ (i.e., unobservable ($\bar{o}$) modes of A) are the output-decoupling (o.d.) zeros of S(A,B,C). The input-output decoupling (i.o.d.) zeros of S(A,B,C) are those o.d. zeros which disappear when the i.d. zeros are eliminated (Rosenbrock, 1970; 1973). Defined above multivariable zeros are involved in several problems of control theory such as zeroing the system output, tracking the reference output, disturbance decoupling, noninteracting control, model matching and output regulation (Basile, Marro, 1992; Isidori, 1995; Marro, 1996; Sontag, 1990; Wonham, 1979). The Smith zeros were discussed, at various simplifying assumptions concerning the systems considered, by many authors (Schrader, Sain, 1989). As is known (MacFarlane, Karcanias, 1976) the Smith zeros are related (through the corresponding zero directions) with zeroing of the system output. Simple examples (see 5) show however that they do not characterize fully the output-zeroing problem (in particular, the zero dynamics nor the maximal (A,B)-invariant subspace contained in Ker C). In order to remove this disadvantage we consider a set (denoted as Z^I) of complex numbers such that for each its element there exists a zero direction with nonzero state-zero direction. The set Z^S of the Smith zeros, where $Z^S := \{\lambda \in C : \text{rank } P(\lambda) < \text{normal rank } P(s)\}$, is contained in Z^I. To any element of Z^I there corresponds an output-zeroing input which produces nontrivial solution of the state equation. Under typical tansfomations, Z^I has the same invariance propeties as Z^S. For the reasons mentioned above, Z^I is treated as an extension of Z^S and to the elements of Z^I we do not assign a new name; we call them simply the invariant zeros.

The paper is organized as follows. In section 2 we give an overview concerning the basic properties and the algebraic characterization of Z^I (based on singular value decomposition (SVD) of the first nonzero Markov parameter) and explicit formulas for the maximal (A,B)-invariant subspace contained in Ker C. Main results are given in sections 3 and 4.

By R, C we denote the fields of real and complex numbers; $\sigma(A) = \{\lambda \in C : \det(\lambda I - A) = 0\}$ stands for the spectrum of matrix A and the Moore-Penrose pseudoinverse of a matrix M is denoted by M^+.

2 PRELIMINARY RESULTS

2.1 Definition and Basic Properties of Invariant Zeros

Consider system S(A,B,C) of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t), \tag{1}$$

$t \geq 0$, $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^r$, where A and $B \neq 0$, $C \neq 0$ are real matrices of appropriate dimensions. The set U of admissible inputs is assumed to consist of all piecewise continuous functions $u(.):[0,\infty) \to R^m$. The first nonzero Markov parameter of (1) is denoted by CA^kB, where $0 \leq k \leq n-1$, i.e., $CB = \ldots = CA^{k-1}B = 0$ and $CA^kB \neq 0$.

The four-fold canonical decomposition (Kalman, 1982) (2) of (1)

$$A = \begin{bmatrix} A_{c\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{co} & 0 & A_{24} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{c}o} \end{bmatrix}, \quad B = \begin{bmatrix} B_{c\bar{o}} \\ B_{co} \\ 0 \\ 0 \end{bmatrix}, \tag{2}$$

$$C = \begin{bmatrix} 0 & C_{co} & 0 & C_{\bar{c}o} \end{bmatrix}, \quad x = \begin{bmatrix} x_{c\bar{o}} \\ x_{co} \\ x_{\bar{c}\bar{o}} \\ x_{\bar{c}o} \end{bmatrix}$$

is not unique, however in any such form the orders $n_{c\bar{o}}, n_{co}, n_{\bar{c}\bar{o}}$ and $n_{\bar{c}o}$ of the corresponding matrices on the diagonal of the A-matrix are uniquely determined by the order of A (n), the degree of G(s) (n_{co}) and rank defects of the controllability and observability matrices ($n_{\bar{c}}$ and $n_{\bar{o}}$) as $n_{c\bar{o}} = n - n_{co} - n_{\bar{c}}$, $n_{\bar{c}\bar{o}} = n_{co} + n_{\bar{c}} + n_{\bar{o}} - n$, $n_{\bar{c}o} = n - n_{co} - n_{\bar{o}}$. The characteristic polynomials (up to a constant) of these matrices also remain unchanged and the elements of $\sigma(A_{c\bar{o}})$ are known as controllable and unobservable ($c\bar{o}$) modes of (1); analogously, co-modes are eigenvalues of A_{co} (these are poles of G(s)), $\bar{c}\bar{o}$-modes are eigenvalues of $A_{\bar{c}\bar{o}}$ and $\bar{c}o$-modes are eigenvalues of $A_{\bar{c}o}$.

Definition 1 (Tokarzewski, 2000; 2002a,b)
(i) A number $\lambda \in C$ is an invariant zero of (1) if and only if (iff) there exist vectors $0 \neq x^o \in C^n$ (state-zero direction) and $g \in C^m$ (input-zero direction) such that the triple λ, x^o, g satisfies

$$\begin{bmatrix} \lambda I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x^o \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (3)$$

The set of the invariant zeros is denoted by Z_I.
System (1) is called degenerate iff Z_I is infinite (otherwise, (1) is called nondegenerate).
(ii) The transmission zeros of (1) are the invariant zeros of its minimal subsystem.
(iii) The o.d. zeros are the unobservable ($\bar{o}$) modes of (1). The i.d. zeros are the uncontrollable ($\bar{c}$) modes of (1). The i.o.d. zeros are the uncontrollable and unobservable ($\bar{c}\bar{o}$) modes of (1).

The set Z_I is invariant under nonsingular coordinate transformations in the state-space, nonsingular transformations of the inputs or outputs, constant state or output feedback to the inputs, constant output feedback to the integrator input. Any o.d. zero of (1) is in Z_I as well as any transmission zero of (1) is in Z_I.
The Kalman form (2) determines individual kinds of decoupling zeros (including multiplicities) of (1) via the polynomials $\chi_{i.o.d.}(s) = \det(sI_{\bar{c}\bar{o}} - A_{\bar{c}\bar{o}})$ and $\chi_{o.d.}(s) = \det(sI_{c\bar{o}} - A_{c\bar{o}})\det(sI_{\bar{c}\bar{o}} - A_{\bar{c}\bar{o}})$ and $\chi_{i.d.}(s) = \det(sI_{\bar{c}\bar{o}} - A_{\bar{c}\bar{o}})\det(sI_{\bar{c}o} - A_{\bar{c}o})$.

Definition 2 (Tokarzewski, 2000; 2002a) For a transfer-function matrix G(s) a number $\lambda \in C$ is a transmission zero of G(s) iff it is an invariant zero of any given minimal realization of G(s).

2.2 Invariant Zeros and Output-Zeroing Problem

The dynamical interpretation of the elements of Z_I in (1) is based on the following formulation of the output-zeroing problem (Isidori, 1995). Find all pairs $(x^o, u_o(t))$ consisting of an initial state $x^o \in R^n$ and a $u_o(.) \in U$ such that the corresponding system response satisfies $y(t) = 0$ for all $t \geq 0$. Any nontrivial pair of this kind (i.e., such that $x^o \neq 0$ or $u_o(.) \neq 0$) is called an output-zeroing input. The internal dynamics of (1) consistent with the constraint $y(t) = 0$ for all $t \geq 0$ are called the zero dynamics of the system.
The same symbol x^o is used to denote state-zero direction (Definition 1(i)) and initial state in output-zeroing inputs. The state-zero direction must be a nonzero vector (real or complex), whereas the initial state must be a real vector (not necessarily nonzero). If state-zero direction x^o is a complex vector, then it gives two initial states $\mathrm{Re}\, x^o$ and $\mathrm{Im}\, x^o$ (see (Tokarzewski, 2000; 2002a) for an explicit form of output-zeroing inputs and the corresponding solutions of the state equation generated by the elements of Z_I).
The set of all output-zeroing inputs completed by the trivial pair $(x^o = 0, u_o(t) \equiv 0)$ forms a linear space over R. In this space we can distinguish a subspace of all pairs $(x^o = 0, u_o^h(t))$ such that $u_o^h(.) \in U$ and $u_o^h(t) \in \mathrm{Ker}\, B$ for all $t \geq 0$. Any such pair affects (1) in the same way as the trivial pair, i.e., it gives identically zero solution and $y(t) = 0$ for all $t \geq 0$. We do not relate these pairs with invariant zeros (we associate them with the trivial pair).
Recall (Wonham, 1979) that a subspace $X \subseteq R^n$ is (A,B)-invariant if there exists a *m*x*n* real matrix F such that $(A + BF)(X) \subseteq X$ (in (Basile, Marro, 1992) X is called an (A,B)-controlled invariant). The maximal (A,B)-invariant subspace contained in $\mathrm{Ker}\, C$ (denoted as $X^*(A, B, C)$) is an unique (A,B)-invariant subspace contained in $\mathrm{Ker}\, C$ with the property that any (A,B)-invariant subspace X contained in $\mathrm{Ker}\, C$ must satisfy $X \subseteq X^*(A, B, C)$. If $(x^o, u_o(t))$ is an output-zeroing input and $x^o(t)$ is the corresponding solution, then $x^o(t) \in X^*(A, B, C)$ for all $t \geq 0$. Moreover, for any $x^o \in X^*(A, B, C)$ there exists an output-zeroing input such that the corresponding solution passes through x^o (Tokarzewski, 2002a).

2.3 Relationship between Z^S and Z^I

Proposition 1 (Tokarzewski, 2002b) In system (1) the sets Z^S and Z^I are interrelated as follows.
(i) $Z^S \subseteq Z^I$
(ii) System (1) is nondegenerate iff $Z^I = Z^S$
(iii) System (1) is degenerate iff $Z^I = C$.

Thus in (1) the set Z^I may be empty, finite or equal to *C*, and when (1) is nondegenerate, then λ is its invariant zero iff λ is a root of the zero polynomial. If in (1) there exists at least one invariant zero which is not a Smith zero, then (1) is degenerate. On the other hand, if (1) is degenerate, we can have $Z^S \neq \emptyset$ or $Z^S = \emptyset$ (see section 5). Moreover, if λ is a Smith zero of (1), then there exist $0 \neq x^o \in C^n$ and $g \in C^m$ such that λ, x^o, g satisfy (3) (i.e., to any Smith zero there corresponds zero direction with nonzero state-zero direction).

2.4 Invariant Zeros and (A,B)-Invariant Subspaces

Lemma 1 (Tokarzewski, 2002a) If in (1) is $G(s) \equiv 0$ (i.e., all Markov parameters are zero), then $Z^I = C$ and $X^*(A,B,C) = \bigcap_{l=0}^{n-1} \mathrm{Ker}CA^l$ (i.e., $X^*(A,B,C)$ is the unobservable subspace for (1)).

Suppose now that in (1) not all Markov parameters are zero and let the first nonzero Markov parameter CA^kB, $0 \le k \le n-1$, have rank $0 < p \le \min\{m,r\}$. Define the projective matrix (Tokarzewski, 2000)

$$K_k := I - B(CA^kB)^+CA^k \tag{4}$$

The approach presented below (based on SVD of CA^kB) enables us to decide the question of degeneracy/nondegeneracy and to characterize the invariant zeros as well as the subspace $X^*(A,B,C)$. In general, the invariant zeros of (1) will be characterized as invariant zeros (in particular, when (1) is nondegenerate, as output-decoupling zeros) of certain closed-loop system (obtained from (1) via introducing appropriate pre- and postcompensator and state feedback matrix), while $X^*(A,B,C)$ will be characterized as the unobservable subspace for that system.

Let us write SVD (Callier, Desoer, 1982) of CA^kB as

$$CA^kB = U\Lambda V^T, \text{ where } \Lambda = \begin{bmatrix} M_p & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

is *r* x *m*-dimensional, M_p is *p* x *p* nonsingular and diagonal (with positive singular values of CA^kB) and *rxr* U and *mxm* V are orthogonal. Introducing into (1) V and U^T as pre- and postcompensator we associate with (1) a new system $S(A, \overline{B}, \overline{C})$

$$\dot{x}(t) = Ax + \overline{B}\overline{u}(t), \ \overline{y}(t) = \overline{C}x(t), \tag{6}$$

where $\overline{B} = BV$, $\overline{C} = U^TC$ and $\overline{u} = V^Tu$, $\overline{y} = U^Ty$ are decomposed as follows

$$\overline{B} = [\overline{B}_p \quad \overline{B}_{m-p}], \quad \overline{C} = \begin{bmatrix} \overline{C}_p \\ \overline{C}_{r-p} \end{bmatrix}, \quad \overline{u} = \begin{bmatrix} \overline{u}_p \\ \overline{u}_{m-p} \end{bmatrix}$$

$$\overline{y} = \begin{bmatrix} \overline{y}_p \\ \overline{y}_{r-p} \end{bmatrix} \tag{7}$$

and $\overline{B}_p$ consists of first *p* columns of $\overline{B}$ whereas $\overline{C}_p$ consists of first *p* rows of $\overline{C}$. Moreover, $\overline{C}A^k\overline{B} = \Lambda$ is the first Markov parameter for (6) and $M_p = \overline{C}_pA^k\overline{B}_p$. Since (6) is obtained from (1) by nonsingular transformations of inputs and outputs, the sets of the invariant zeros for S(A,B,C) and $S(A, \overline{B}, \overline{C})$ coincide. For the associated with (1) system (6) we form the projective matrix

$$\overline{K}_k := I - \overline{B}(\overline{C}A^k\overline{B})^+\overline{C}A^k \tag{8}$$

which, in view of (5) and (7), may be expressed as

$$\overline{K}_k = I - \overline{B}_pM_p^{-1}\overline{C}_pA^k \tag{9}$$

Lemma 2 The matrix $\overline{K}_k$ in (9) satisfies
(i) $\overline{K}_k^2 = \overline{K}_k$,
(ii) $\overline{\Sigma}_k := \{x : \overline{K}_k x = x\} = \mathrm{Ker}\,\overline{C}_pA^k$,
 $\dim \overline{\Sigma}_k = n - p$,
(iii) $\overline{\Omega}_k := \{x : \overline{K}_k x = 0\} = \mathrm{Im}\,\overline{B}_p$, $\dim \overline{\Omega}_k = p$,
(iv) $C^n(R^n) = \overline{\Sigma}_k \oplus \overline{\Omega}_k$,

(v) $\overline{K}_k\overline{B}_p = 0$, $\overline{K}_k\overline{B}_{m-p} = \overline{B}_{m-p}$,

$\overline{C}_p A^k \overline{K}_k = 0$,

(vi) $\overline{C}_p(\overline{K}_k A)^l = \begin{cases} \overline{C}_p A^l & \text{for } 0 \le l \le k \\ 0 & \text{for } l \ge k+1 \end{cases}$,

(vii) $\overline{C}(\overline{K}_k A)^l = \overline{C}A^l$ for $0 \le l \le k$.

Remark 1 For (4) and (8),(9) is $K_k = \overline{K}_k$.

Proposition 2 In (1) let $\operatorname{rank} CA^kB = p < m$ and in $S(A,\overline{B},\overline{C})$ in (6) let $\overline{B}_{m-p} \ne 0$. Then the sequence of transformations

$S(A,B,C) \to S(A,\overline{B},\overline{C}) \to S(\overline{K}_k A, \overline{B}_{m-p}, \overline{C})$

has the following properties:

(i) it preserves the set of the invariant zeros, i.e., $Z^I_{S(A,B,C)} = Z^I_{S(A,\overline{B},\overline{C})} = Z^I_{S(\overline{K}_k A,\overline{B}_{m-p},\overline{C})}$,

(ii) it preserves the maximal (A,B)-invariant subspace contained in $\operatorname{Ker} C$, i.e.,

$X^*(A,B,C) = X^*(A,\overline{B},\overline{C}) = X^*(\overline{K}_k A, \overline{B}_{m-p}, \overline{C})$,

(iii) it preserves the zero polynomial for S(A,B,C), i.e.,

$\psi_{S(A,B,C)}(s) = \psi_{S(A,\overline{B},\overline{C})}(s) = \psi_{S(\overline{K}_k A,\overline{B}_{m-p},\overline{C})}(s)$,

and consequently, the set of the Smith zeros, i.e., $Z^S_{S(A,B,C)} = Z^S_{S(A,\overline{B},\overline{C})} = Z^S_{S(\overline{K}_k A,\overline{B}_{m-p},\overline{C})}$.

Proposition 3 (Tokarzewski, 2002a) In (1) let $\operatorname{rank} CA^kB = m$. Then (1) is nondegenerate and $\lambda \in C$ is an invariant zero of (1) iff λ is an o.d. zero of $S(\overline{K}_k A, \overline{B}, \overline{C})$. Moreover, $X^*(A,B,C)$ equals the unobservable subspace for $S(\overline{K}_k A, \overline{B}, \overline{C})$, i.e.,

$$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker}\overline{C}(\overline{K}_k A)^l .$$

Proposition 4 (Tokarzewski, 2002a) In (1) let $m > r$ and let CA^kB have full row rank r. Then:

(i) S(A,B,C) is degenerate iff in $S(A,\overline{B},\overline{C})$ in (6) is $\overline{B}_{m-r} \ne 0$. Moreover, $\lambda \in C$ is an invariant zero of (1) iff λ is an invariant zero of the system $S(\overline{K}_k A,\overline{B}_{m-r},\overline{C})$ whose transfer-function matrix equals zero identically. Furthermore, $X^*(A,B,C)$ equals the unobservable subspace for $S(\overline{K}_k A,\overline{B}_{m-r},\overline{C})$, i.e.,

$$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker}\overline{C}(\overline{K}_k A)^l .$$

(ii) S(A,B,C) is nondegenerate iff $\overline{B}_{m-r} = 0$. Moreover, $\lambda \in C$ is an invariant zero of (1) iff λ is an o.d. zero of the system $S(\overline{K}_k A,\overline{B}_r,\overline{C})$. Furthermore, $X^*(A,B,C)$ equals the unobservable subspace for $S(\overline{K}_k A,\overline{B}_r,\overline{C})$, i.e.,

$$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker}\overline{C}(\overline{K}_k A)^l .$$

Proposition 5 (Tokarzewski, 2002a) In (1) let CA^kB have rank $p < \min\{m,r\}$ and in $S(A,\overline{B},\overline{C})$ in (6) let $\overline{C}_{r-p} = 0$. Then:

(i) S(A,B,C) is degenerate iff $\overline{B}_{m-p} \ne 0$. Moreover, $\lambda \in C$ is an invariant zero of (1) iff λ is an invariant zero of the system $S(\overline{K}_k A,\overline{B}_{m-p},\overline{C})$ whose transfer-function matrix equals zero identically. Furthermore, $X^*(A,B,C)$ equals the unobservable subspace for $S(\overline{K}_k A,\overline{B}_{m-p},\overline{C})$, i.e.,

$$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker}\overline{C}(\overline{K}_k A)^l .$$

(ii) S(A,B,C) is nondegenerate iff $\overline{B}_{m-p} = 0$. Moreover, $\lambda \in C$ is an invariant zero of (1) iff λ is an o.d. zero of the system $S(\overline{K}_k A,\overline{B}_p,\overline{C})$. Furthermore, $X^*(A,B,C)$ equals the unobservable subspace for $S(\overline{K}_k A,\overline{B}_p,\overline{C})$, i.e.,

$$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker}\overline{C}(\overline{K}_k A)^l .$$

Proposition 6 (Tokarzewski, 2002a) In (1) let $\operatorname{rank} CA^kB = p < \min\{m,r\}$ and in $S(A,\overline{B},\overline{C})$ in (6) let $\overline{C}_{r-p} \ne 0$ and let $\overline{B}_{m-p} = 0$. Then S(A,B,C) is nondegenerate; moreover $\lambda \in C$ is its invariant zero iff λ is an o.d. zero of the system $S(\overline{K}_k A, \overline{B}_p, \overline{C})$. Furthermore, $X^*(A,B,C)$ equals the unobservable subspace for $S(\overline{K}_k A, \overline{B}_p, \overline{C})$, i.e.,

$$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker}\overline{C}(\overline{K}_k A)^l .$$

Propositions 2-6 and Lemma 1 yield a recursive procedure for the computation of invariant zeros and $X^*(A,B,C)$ for system S(A,B,C) in (1).

Procedure 1 (Tokarzewski, 2002a)

1. CA^kB has full column rank.
Invariant zeros of (1) are o.d. zeros of $S(\overline{K}_k A, \overline{B}, \overline{C})$ and
$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker} \overline{C}(\overline{K}_k A)^l$.

2. CA^kB has full row rank *r* and $m > r$.

2a. $\overline{B}_{m-r} = 0$. Invariant zeros of (1) are o.d. zeros of $S(\overline{K}_k A, \overline{B}_r, \overline{C})$ and
$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker} \overline{C}(\overline{K}_k A)^l$

2b. $\overline{B}_{m-r} \neq 0$. S(A,B,C) is degenerate and
$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker} \overline{C}(\overline{K}_k A)^l$.

3. $\operatorname{rank} CA^kB = p < \min\{m, r\}$.

3a. $\overline{C}_{r-p} = 0$.

3a1. $\overline{C}_{r-p} = 0$ and $\overline{B}_{m-p} = 0$. Invariant zeros of (1) are o.d. zeros of $S(\overline{K}_k A, \overline{B}_p, \overline{C})$ and
$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker} \overline{C}(\overline{K}_k A)^l$.

3a2. $\overline{C}_{r-p} = 0$ and $\overline{B}_{m-p} \neq 0$. S(A,B,C) is degenerate and $X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker} \overline{C}(\overline{K}_k A)^l$.

3b. $\overline{C}_{r-p} \neq 0$.

3b1. $\overline{C}_{r-p} \neq 0$ and $\overline{B}_{m-p} = 0$. Invariant zeros of (1) are o.d. zeros of $S(\overline{K}_k A, \overline{B}_p, \overline{C})$ and
$X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker} \overline{C}(\overline{K}_k A)^l$.

3b2. $\overline{C}_{r-p} \neq 0$ and $\overline{B}_{m-p} \neq 0$. The question is not decided at this step. Start the next step applying Procedure 1 to system $S(\overline{K}_k A, \overline{B}_{m-p}, \overline{C})$.

4. In (1) all Markov parameters are zero. S(A,B,C) is degenerate and $X^*(A,B,C) = \bigcap_{l=0}^{n-1} \operatorname{Ker} CA^l$.

In the case 3b2 we begin the second step applying Procedure 1 to system $S(A', B', C')$ with the matrices $A' = \overline{K}_k A$, $B' = \overline{B}_{m-p}$, $C' = \overline{C}$ and $m' = m - p$ inputs (i.e., we find the first nonzero Markov parameter for $S(A', B', C')$ and its SVD and then we form the associated system $S(A', \overline{B}', \overline{C}')$; in $S(A', B', C')$ the cases 2a and 2b are not possible since its Markov parameters have no full row rank). The process ends after at most *n* steps. At the last step we can meet only two possible situations: we get a system whose transfer-function matrix is identically zero or a system with the first nonzero Markov parameter of full column rank.

Corollary 1

(i) The question of seeking invariant zeros and $X^*(A,B,C)$ for (1) can be decided at the first step (the cases 1, 2a, 2b, 3a1, 3a2, 3b1 or 4 in Procedure 1) or, in case 3b2, applying successively Procedure 1, after at most *n* steps.

(ii) The recursive process generated by point 3b2 preserves $Z^I_{S(A,B,C)}$, $\psi_{S(A,B,C)}(s)$ and $X^*(A,B,C)$ (comp. Proposition 2). Thus, $Z^I_{S(A,B,C)}$ can be found out as the set of the invariant zeros of the system obtained at the last step (similarly for $\psi_{S(A,B,C)}(s)$ and $X^*(A,B,C)$).

(iii) The process ends when we approach a nondegenerate system (the case 1, 3a1 or 3b1) or a degenerate system (the case 3a2 or 4).

3 SMITH ZEROS, INVARIANT ZEROS AND ZERO DYNAMICS

Proposition 7 If (1) is nondegenerate, then its Smith zeros and the invariant zeros are the same objects (including multiplicities). Moreover, the degree of the zero polynomial for (1) equals $\dim X^*(A,B,C)$, while the zero dynamics, in appropriate coordinates, have the form $\dot{\xi}(t) = N\xi(t)$, where the characteristic polynomial of matrix N equals the zero polynomial for (1) and ξ belongs to the subspace $X^*(A,B,C)$ (when taken in the same coordinates).

Proposition 8 If (1) is degenerate, then the dimension of $X^*(A,B,C)$ is larger than the degree of the zero polynomial for (1), i.e.,

$\dim X^*(A,B,C) > \deg \psi_{S(A,B,C)}(s)$. Moreover, the Smith zeros of (1) are i.o.d. zeros of certain system whose transfer-function matrix equals zero identically and the zero dynamics for (1) depend esentially upon control vector.

4 SUFFICIENT AND NECESSARY CONDITION OF DEGENERACY

Proposition 9 System (1) is degenerate iff

$$\text{normal rank } P(s) < n + \text{rank } B. \tag{10}$$

Proposition 10 System (1) is nondegenerate iff

$$\text{normal rank } P(s) = n + \text{rank } B. \tag{11}$$

From Proposition 9 and from the relation $\text{normal rank } P(s) = n + \text{normal rank } G(s)$ we get

Proposition 11 Let G(s) be a *r*x*m* transfer-function matrix and let S(A,B,C) (1) stand for its minimal *n*-dimensional state-space realization. Then $G(s)$ is degenerate (comp. Definition 2) iff

$$\text{normal rank } G(s) < \text{rank } B. \tag{12}$$

Proposition 12 G(s) is nondegenerate iff

$$\text{normal rank } G(s) = \text{rank } B. \tag{13}$$

5 EXAMPLES

Example 1 In (1) let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The system is minimal, asymptotically stable and has no Smith zeros. In SVD of CB we take

$$U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \Lambda = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \quad V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Via Procedure 1 (3b2) we consider system $S(\overline{K}_k A, \overline{B}_{m-p}, \overline{C})$, where

$$\overline{K}_k A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \quad \overline{B}_{m-p} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & 0 \end{bmatrix}.$$

Since all Markov parameters of $S(\overline{K}_k A, \overline{B}_{m-p}, \overline{C})$ are zero, $S(\overline{K}_k A, \overline{B}_{m-p}, \overline{C})$ and consequently, system (1) are degenerate. The zero dynamics for (1) are $\dot{x}_3 = -x_3 + u_1$ and $X^*(A,B,C) = \text{Ker } C$.

Example 2 (Emami-Naeini, Van Dooren, 1982) In (1) let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}.$$

The system has one single Smith zero at 2. Since $G(s) \equiv 0$, we have $Z^I = C$; moreover, $X^*(A,B,C) = \text{Ker } C$.

Example 3 In S(A,B,C) in (1) let $G(s) \equiv 0$ and let $x' = Hx$ denote a change of coordinates which transforms (1) to its Kalman form $S(A',B',C')$

$$A' = \begin{bmatrix} A_{c\bar{o}} & A_{13} & A_{14} \\ 0 & A_{\bar{c}\bar{o}} & A_{34} \\ 0 & 0 & A_{\bar{c}o} \end{bmatrix} \quad B' = \begin{bmatrix} B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}, \tag{14}$$

$$C' = \begin{bmatrix} 0 & 0 & C_{\bar{c}o} \end{bmatrix} \qquad x' = \begin{bmatrix} x_{c\bar{o}} \\ x_{\bar{c}\bar{o}} \\ x_{\bar{c}o} \end{bmatrix}.$$

Since in (1) we have assumed $B \neq 0$ and $C \neq 0$, in (14) is always $n_{c\bar{o}} > 0$ and $n_{\bar{c}o} > 0$, while $n_{\bar{c}\bar{o}} \geq 0$ (i.e., i.o.d. zeros for S(A,B,C) may not exist). The normal rank of the system matrix for (14) is $n_{c\bar{o}} + n_{\bar{c}\bar{o}} + n_{\bar{c}o} = n$ and the zero polynomial for (14) (and consequently, for S(A,B,C)) is $\psi_{S(A,B,C)}(s) = \det(sI_{\bar{c}\bar{o}} - A_{\bar{c}\bar{o}})$. Thus, the Smith

zeros of (1) are i.o.d. zeros of (1). Of course, $\dim X^*(A,B,C) = \dim X^*(A',B',C') = n_{c\bar{o}} + n_{\bar{c}\bar{o}}$.
The zero dynamics in $S(A',B',C')$ are governed by the equations

$$\begin{aligned} \dot{x}_{c\bar{o}}(t) &= A_{c\bar{o}} x_{c\bar{o}}(t) + A_{13} x_{\bar{c}\bar{o}}(t) + B_{c\bar{o}} u(t) \\ \dot{x}_{\bar{c}\bar{o}}(t) &= A_{\bar{c}\bar{o}} x_{\bar{c}\bar{o}}(t) \end{aligned},$$

where $B_{c\bar{o}} \neq 0$, and their solutions remain in $X'_{\bar{o}} = \{x' \in R^n : x_{\bar{c}o} = 0\} = X^*(A',B',C')$. If we constrain initial conditions to the subspace $X'_{c\bar{o}} = \{x' \in R^n : x_{\bar{c}\bar{o}} = 0, x_{\bar{c}o} = 0\}$, then in this part of $X'_{\bar{o}}$ the zero dynamics are governed by

$$\dot{x}_{c\bar{o}}(t) = A_{c\bar{o}} x_{c\bar{o}}(t) + B_{c\bar{o}} u(t).$$

The source of degeneracy of (14) (and consequently, of (1)) lies in this part (i.e., controllable and unobservable) of the system, since for any $\lambda \notin \sigma(A_{c\bar{o}})$ the triple

$$\lambda,\ x^o = \begin{bmatrix} x^o_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}, g,$$

with $x^o_{c\bar{o}} = (\lambda I_{c\bar{o}} - A_{c\bar{o}})^{-1} B_{c\bar{o}} g$ and $g \notin \operatorname{Ker} B_{c\bar{o}}$, satisfies Definition 1(i) for $S(A',B',C')$.

6 CONCLUSION

The purpose of this paper was to discuss certain geometric aspects of multivariable zeros that are not commonly known from the relevant literature. The presented approach can be extended on non strictly proper systems.

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