MODEL PREDICTIVE CONTROL FOR HYBRID SYSTEMS UNDER A STATE PARTITION BASED MLD APPROACH (SPMLD)

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Abstract This paper presents the State Partition based Mixed Logical Dynamical (SPMLD) formalism as a new modeling technique for a class of discrete-time hybrid systems, where the system is defined by different modes with continuous and logical control inputs and state variables, each model subject to linear constraints. The reformulation of the predictive strategy for hybrid systems under the SPMLD approach is then developed. This technique enables to considerably reduce the computation time (with respect to the classical MPC approaches for PWA and MLD models), as a positive feature for real time implementation. This strategy is applied in simulation to the control of a three tanks benchmark.

1 INTRODUCTION

Hybrid systems become an attractive field of research for engineers as it appears in many control applications in industry. They include both continuous and discrete variables, discrete variables coming from parts described by logic such as for example on/off switches or valves. Various approaches have been proposed for modeling hybrid systems (Branicky *et al*., 1998), like Automata, Petri nets, Linear Complementary (LC), Piecewise Affine (PWA) (Sontag, 1981), Mixed Logical Dynamical (MLD) models (Bemporad, and Morari, 1999).

This paper examines a class of discrete-time hybrid systems, which consists of several models with different dynamics according to the feasible state space partition. Each model is described with continuous and logical states and control inputs. Consequently, the dynamic of the system depends on the model selected in relation to linear constraints over the states and on the inputs values.

On the other hand, model predictive control (MPC) appears to be an efficient strategy to control hybrid systems. Considering the previous particular class of hybrid systems, implementing MPC leads to a problem including at each prediction step the states and inputs vectors (both continuous and discrete variables), the dynamic equation and linear constraints, for which a quadratic cost function has to be optimized. Two classical approaches exist for solving this optimization problem.

First, all possible logical combinations can be studied at each prediction time, which leads solving a great number of QPs. Each of these QPs is related to a particular scenario of logical inputs and modes. This is the PWA approach. The number of QPs can be reduced by reachability considerations (Pena *et al*., 2003).

The second moves the initial problem through the MLD formalism to a single general model used at each prediction step. This MLD formalism introduces many auxiliary logical and continuous variables and linear constraints. At each prediction step, all the MLD model variables have to be solved (even if some of them are not active). However, the MLD transformation allows utilizing the Branch and Bound (B&B) technique (Fletcher and Leyffer, 1995), reducing the number of QPs solved.

This paper develops a technique which aims at implementing MPC strategy for the considered class of hybrid systems, as a mixed solution of the two classical structures presented before. It is based on a

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new modeling technique, called State Partition based MLD approach (SPMLD) formalism, combining the PWA and MLD models. The complexity of this formalism is compared to that obtained with the usual PWA and MLD forms, which can also model this class of hybrid systems as well.

The paper is organized as follows. Section 2 presents a short description of the PWA and MLD hybrid systems. General consideration about model predictive control (MPC) and its classical application to PWA and MLD systems are summarized in Section 3. Section 4 develops the State Partition based MLD approach (SPMLD) and examines the application of MPC to hybrid systems under this formalism. Section 5 gives an application of this strategy to water level control of a three tanks benchmark. Section 6 gives final conclusions.

2 HYBRID SYSTEMS MODELING

2.1 Mixed Logical Dynamical model

The MLD model appears as a suitable formalism for various classes of hybrid systems, like linear hybrid or constrained linear systems. It describes the systems by linear dynamic equations subject to linear inequalities involving real and integer variables, under the form (Bemporad and Morari, 1999)

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}_1\mathbf{u}_k + \mathbf{B}_2\boldsymbol{\delta}_k + \mathbf{B}_3\mathbf{z}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}_1\mathbf{u}_k + \mathbf{D}_2\boldsymbol{\delta}_k + \mathbf{D}_3\mathbf{z}_k \\ \mathbf{E}_2\boldsymbol{\delta}_k + \mathbf{E}_3\mathbf{z}_k &\le \mathbf{E}_1\mathbf{u}_k + \mathbf{E}_4\mathbf{x}_k + \mathbf{E}_5 \end{aligned} \quad (1)$$

where

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_c \\ \mathbf{x}_l \end{pmatrix} \in \Re^{n_c} \times \{0,1\}^{n_l}, \ \mathbf{u} = \begin{pmatrix} \mathbf{u}_c \\ \mathbf{u}_l \end{pmatrix} \in \Re^{m_c} \times \{0,1\}^{m_l},$$

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_c \\ \mathbf{y}_l \end{pmatrix} \in \Re^{p_c} \times \{0,1\}^{p_l}, \quad \boldsymbol{\delta} \in \{0,1\}^{r_l}, \quad \mathbf{z} \in \Re^{r_c}$$

are respectively the vectors of continuous and binary states of the system, of continuous and binary (on/off) control inputs, of output signals, of auxiliary binary and continuous variables.

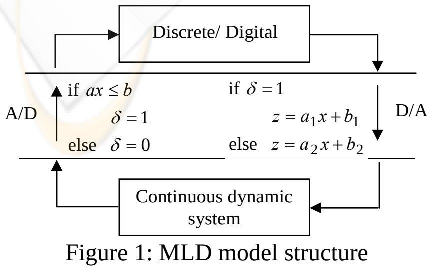

Figure 1: MLD model structure

The auxiliary variables are introduced when translating propositional logic into linear inequalities as described in Figure 1. All matrices appearing in (1) can be obtained through the specification language HYSDEL (Hybrid System Description Language), see (Torrisi *et al.*, 2000).

2.2 Piecewise Affine model

Another framework for discrete time hybrid systems is the PWA model (Sontag, 1981), defined as

$$S^i : \begin{Bmatrix} \mathbf{x}_{k+1} = \mathbf{A}^i\mathbf{x}_k + \mathbf{B}^i\mathbf{u}_k + \mathbf{f}^i \\ \mathbf{y}_k = \mathbf{C}^i\mathbf{x}_k + \mathbf{g}^i \end{Bmatrix}, \text{ for: } \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \in \chi_i \quad (2)$$

where $\{\chi_i\}_{i=1}^{s}$ is the polyhedral partition of the state and input spaces (s being the number of subsystems within the partition). Each χ_i is given by

$$\chi_i = \left\{ \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \middle| \ \mathbf{Q}^i \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \le \mathbf{q}^i \right\} \quad (3)$$

where $\mathbf{x}_k, \mathbf{u}_k, \mathbf{y}_k$ denote the state, input and output vector respectively at instant k. Each subsystem S^i defined by the 7-uple $(\mathbf{A}^i, \mathbf{B}^i, \mathbf{C}^i, \mathbf{f}^i, \mathbf{g}^i, \mathbf{Q}^i, \mathbf{q}^i)$, $i \in (1, 2, \cdots, s)$ is a component of the PWA system (2). $\mathbf{A}^i \in \Re^{n \times n}, \mathbf{B}^i \in \Re^{n \times m}, \mathbf{C}^i \in \Re^{p \times n}, \mathbf{Q}^i \in \Re^{p_i(n+m)}$ and $\mathbf{f}^i, \mathbf{g}^i, \mathbf{q}^i$ are suitable constant vectors or matrices, where n, m, p are respectively the number of states, inputs, outputs, and p_i is the number of hyperplanes defining the i-polyhedral. In this formalism, a logical control input is considered by developing an affine model for each input value (1/0), defining linear inequality constraints linking the model with the relevant input value.

It has been shown in (Bemporad *et al.*, 2000), that MLD and PWA models are equivalent, which enables transformation from one model to the other. A MLD model can be transferred to a PWA model with the number of subsystems inside the polyhedral partition equal to all possible combination of all the integer variables of the MLD model (Bemporad *et al.*, 2000) (a technique for avoiding empty region is presented in (Bemporad, 2003))

$$s = 2^{n_l + m_l + r_l} \quad (4)$$

3 MODEL PREDICTIVE CONTROL

Model predictive control (MPC) has proved to efficiently control a wide range of applications in industry, including systems with long delay times, non-minimum phase, unstable, multivariable and constrained systems.

The main idea of predictive control is the use of a plant model to predict future outputs of the system. Based on this prediction, at each sampling period, a sequence of future control values is elaborated through an on-line optimization process, which maximizes the tracking performance while satisfying constraints. Only the first value of this optimal sequence is applied to the plant according to the 'receding' horizon strategy (Dumur and Boucher, 1998).

Considering the particular class of hybrid systems previously described, implementing MPC leads to a problem including at each prediction step the states vector, the inputs vector (both continuous and discrete), the dynamic equation and linear constraints, for which a quadratic cost function has to be optimized. Two classical approaches exist for solving this optimization problem, the Branch and Bound technique that can be used with the MLD formalism and the enumeration of all possible logical system combinations at each prediction time corresponding to all particular scenarios of logical inputs and modes used with the PWA formalism.

3.1 Model predictive control for the MLD systems

For a MLD system of the form (1), the following model predictive control problem is considered. Let k be the current time, $\mathbf{x}_k$ the current state, $(\mathbf{x}_e, \mathbf{u}_e)$ an equilibrium pair or a reference trajectory value, $k+N$ the final time, find $\mathbf{u}_k^{k+N-1} = \left(\mathbf{u}_k \ \cdots \ \mathbf{u}_{k+N-1}\right)$ the sequence which moves the state from $\mathbf{x}_k$ to $\mathbf{x}_e$ and minimizes

$$\begin{aligned}\min_{\mathbf{u}_k^{k+N-1}} J(\mathbf{u}_k^{k+N-1}, \mathbf{x}_k) = \sum_{i=0}^{N-1} \left\|\mathbf{u}_{k+i} - u_e\right\|_{\mathbf{Q}_1}^2 \ + \\ + \left\|\boldsymbol{\delta}_{k+i/k} - \boldsymbol{\delta}_e\right\|_{\mathbf{Q}_2}^2 + \left\|\mathbf{z}_{k+i/k} - \mathbf{z}_e\right\|_{\mathbf{Q}_3}^2 \ + \\ + \left\|\mathbf{x}_{k+i+1/k} - \mathbf{x}_e\right\|_{\mathbf{Q}_4}^2 + \left\|\mathbf{y}_{k+i/k} - \mathbf{y}_e\right\|_{\mathbf{Q}_5}^2\end{aligned} \tag{5}$$

subject to (1), where N is the prediction horizon, $\boldsymbol{\delta}_e$, $\mathbf{z}_e$ are the auxiliary variables of the equilibrium point or the reference trajectory value, calculated by solving a MILP problem for the inequality equation of (1). $\mathbf{x}_{k+i/k}$ denotes the predicted state vector at time $k+i$, obtained by applying the input sequence $\mathbf{u}_k^{k+N-1}$ to model (1) starting from the current state $\mathbf{x}_k$ (same for the other input and output variables), $\mathbf{Q}_i = \mathbf{Q}_i^{'} > 0$, for $i = 1,4$, and $\mathbf{Q}_i = \mathbf{Q}_i^{'} \geq 0$, for $i = 2,3,5$.

The optimization procedure of (5) leads to MIQP problems with the following optimization vector

$$\begin{aligned}\boldsymbol{\chi} = [\mathbf{u}_k, \cdots, \mathbf{u}_{k+N-1}, \boldsymbol{\delta}_k, \cdots, \boldsymbol{\delta}_{k+N-1}, \\ , \mathbf{z}_k, \cdots, \mathbf{z}_{k+N-1}]^{\mathrm{T}}\end{aligned} \tag{6}$$

The number of binary optimization variables is $L = N\left(m_l + r_l\right)$. In the worst case, the maximum number of solved QP problems is

$$\text{No of QPs} = 2^{L+1} - 1 \tag{7}$$

So the main drawback of this MLD formalism remains the computational burden related to the complexity of the derived Mixed Integer Quadratic Programming (MIQPs) problems. Indeed MIQPs problems are classified as NP-complete, so that in the worst case, the optimization time grows exponentially with the problem size, even if branch and bounds methods (B&B) may reduce this solution time (Fletcher and Leyffer, 1995).

3.2 Model Predictive control for the PWA systems

Considering the PWA system under the form (2), assuming that the current state $\mathbf{x}_k$ is known the model predictive control requires solving at each time step (Pena *et al.*, 2003).

$$\begin{aligned}\min_{\mathbf{u}_k^{k+N-1}} J(\mathbf{u}_k^{k+N-1}, \mathbf{x}_k) = \sum_{i=1}^{N} q_{ii} \left\|\mathbf{y}_{k+i/k} - \mathbf{w}_{k+i}\right\|^2 \\ + \sum_{i=0}^{N-1} r_{ii} \left\|\mathbf{u}_{k+i}\right\|^2 \\ \text{s.t.:} \ \mathbf{u}_{\min} \leq \mathbf{u}_{k+i} \leq \mathbf{u}_{\max}\end{aligned} \tag{8}$$

where N is the prediction horizon, $\mathbf{w}_{k+i}$ is the output reference, and $\mathbf{y}_{k+i/k}$ denotes the predicted output vector at time $k+i$, obtained by applying the input sequence $\mathbf{u}_k^{k+N-1} = \left(\mathbf{u}_k \ \cdots \ \mathbf{u}_{k+N-1}\right)$ to the system starting from the current state $\mathbf{x}_k$. q_{ii}, r_{ii} are the elements of Q, R weighting diagonal matrices.

In order to solve this equation the model applied at each instant has to be determined and all potential sequences of subsystems $I = \left\{I_k, I_{k+1}, \cdots, I_{k+N-1}\right\}$ have to be examined, where I_{k+i} is one sub-region among the s subsystems at prediction time i for $i = 1,2,\cdots,N-1$. As for each model the value of the logical variable is fixed, the MPC problem is solved by a QP for each potential sequence. As the current state $\mathbf{x}_k$ is known, the starting region according to the state partition is identified. But the initial sub-region related to the current input control is not known as it appears in the domain definition (3). Similarly, the next steps subsystems are also unknown, depending on the applied control sequence. In general, all potential sequences of subsystems I have to be examined, which increases the computation burden. If no constraints are considered, the number of possible sequences for a prediction horizon N is $m_p \, s^{N-1}$, where m_p is the number of all possible sub-regions at instant k

according to the input space partition. In order to solve the MPC problem of (8), the number of quadratic programming problems to be solved is

$$\text{No QPs } = m_p \, s^{N-1} \tag{9}$$

4 MPC FOR STATE PARTITION BASED MLD (SPMLD) FORMALISM

4.1 The SPMLD formalism

The SPMLD model is a mixed approach where in each region of the feasible space a simple MLD model is developed. Starting from the MLD model, the auxiliary binary variables are divided into two groups $\boldsymbol{\delta} = [\boldsymbol{\delta}_1 \quad \boldsymbol{\delta}_2]^T$. Where $\boldsymbol{\delta}_2 \in \{0,1\}^{r_{l2}}$ is chosen in order to include the δ variables that are not directly depending on the state variables (the inequalities that define δ variables are not depending on $\mathbf{x}$), and $\boldsymbol{\delta}_1 \in \{0,1\}^{r_{l1}}$ depending on $\mathbf{x}$. This partition will be further justified. The SPMLD model is then developed by giving $\boldsymbol{\delta}_1$ a constant value: for each possible combination of $\boldsymbol{\delta}_1$ a sub-region is defined with the corresponding $\mathbf{Q}^i, \mathbf{q}^i$ constraints as in (3). As some logical combinations may not be feasible, the number of sub-regions of the polyhedral partition is

$$s \le 2^{r_{l1}} \tag{10}$$

Consequently, this model requires a smaller number of sub-regions than the classical PWA model for the same modeled system. Each sub-region has its own dynamic described in the same way as (1) but with a simpler MLD model that represents the system behavior in this sub-region and includes only the active variables in this sub-region. This partition always implies a reduction in the size of $\mathbf{z}$ and/or $\mathbf{u}$. For example, some control variables may not be active in sub-regions and the auxiliary continuous variables $\mathbf{z}$ depending on the $\boldsymbol{\delta}_1$ variables may disappear or become fixed as $\boldsymbol{\delta}_1$ is fixed where:

$$z = \delta f(x) \rightarrow z = \begin{cases} 0 & \text{if } \delta = 0 \\ f(x) & \text{if } \delta = 1 \end{cases} \tag{11}$$

Consequently, simplified sub-regions models can be derived, an example of this simplification is given in the application section.

The system is thus globally modeled as

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}^i \mathbf{x}_k + \mathbf{B}_1^i \mathbf{u}_k + \mathbf{B}_2^i \boldsymbol{\delta}_k + \mathbf{B}_3^i \mathbf{z}_k \\ \mathbf{y}_k &= \mathbf{C}^i \mathbf{x}_k + \mathbf{D}_1^i \mathbf{u}_k + \mathbf{D}_2^i \boldsymbol{\delta}_k + \mathbf{D}_3^i \mathbf{z}_k \\ \mathbf{E}_2^i \boldsymbol{\delta}_k + \mathbf{E}_3^i \mathbf{z}_k &\le \mathbf{E}_1^i \mathbf{u}_k + \mathbf{E}_4^i \mathbf{x}_k + \mathbf{E}_5^i \end{aligned} \tag{12}$$

$\mathbf{A}^i, \mathbf{B}_{1-3}^i, \mathbf{C}^i, \mathbf{D}_{1-3}^i, \mathbf{E}_{1-5}^i$ are the matrices of the i^{th} MLD model defining the dynamics into that sub-region. $\mathbf{Q}^i, \mathbf{q}^i$ constraints has to be included in (12).

4.2 Reformulation of the MPC solution

At this stage, the MPC technique developed for the PWA formalism must be rewritten to fit the new SPMLD model. Consider the initial subsystem I_k

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}^k \mathbf{x}_k + \mathbf{B}_1^k \mathbf{u}_k + \mathbf{B}_2^k \boldsymbol{\delta}_k + \mathbf{B}_3^k \mathbf{z}_k \\ \mathbf{y}_k &= \mathbf{C}^k \mathbf{x}_k + \mathbf{D}_1^k \mathbf{u}_k + \mathbf{D}_2^k \boldsymbol{\delta}_k + \mathbf{D}_3^k \mathbf{z}_k \end{aligned} \tag{13}$$

with $\quad -\mathbf{E}_1^k \mathbf{u}_k + \mathbf{E}_3^k \mathbf{z}_k \le \mathbf{E}_4^k \mathbf{x}_k + \mathbf{E}_5^k - \mathbf{E}_2^k \boldsymbol{\delta}_k$

Where $\mathbf{A}^k$ will now denote for simplification purposes the $\mathbf{A}^i$ matrix of model i at instant k (the same notations are used for $\mathbf{B}_{1-3}^k, \mathbf{C}^k, \mathbf{D}_{1-3}^k, \mathbf{E}_{1-5}^k$). For a given sequence over the prediction horizon N i.e. for $I = \{I_k, I_{k+1}, \cdots, I_{k+N-1}\}$, the system is recursively defined as follows

$$\begin{aligned} \bar{\mathbf{x}} &= \mathbf{F}_x \mathbf{x}_k + \mathbf{H}_x \bar{\mathbf{u}} + \mathbf{P}_x \bar{\boldsymbol{\delta}} + \mathbf{G}_x \bar{\mathbf{z}} \\ \bar{\mathbf{y}} &= \mathbf{F}_y \mathbf{x}_k + \mathbf{H}_y \bar{\mathbf{u}} + \mathbf{P}_y \bar{\boldsymbol{\delta}} + \mathbf{G}_y \bar{\mathbf{z}} \end{aligned} \tag{14}$$

Where $\bar{\mathbf{x}} = [\mathbf{x}_{k+1} \ \ \mathbf{x}_{k+2} \ \cdots \ \mathbf{x}_{k+N}]^{\mathrm{T}}$,
$\bar{\mathbf{u}} = [\mathbf{u}_k \ \cdots \ \mathbf{u}_{k+N-1}]^{\mathrm{T}}, \bar{\mathbf{y}} = [\mathbf{y}_k \ \cdots \ \mathbf{y}_{k+N-1}]^{\mathrm{T}}$
$\bar{\mathbf{z}} = [\mathbf{z}_k \ \cdots \ \mathbf{z}_{k+N-1}]^{\mathrm{T}}, \bar{\boldsymbol{\delta}} = [\boldsymbol{\delta}_k \ \cdots \ \boldsymbol{\delta}_{k+N-1}]^{\mathrm{T}}$

$$\mathbf{F_x} = \begin{bmatrix} \mathbf{A}^k \\ \mathbf{A}^{k+1}\mathbf{A}^k \\ \vdots \\ \mathbf{A}^{k+N-1}\cdots\mathbf{A}^k \end{bmatrix}, \mathbf{F_y} = \begin{bmatrix} \mathbf{C}^k \\ \mathbf{C}^{k+1}\mathbf{A}^k \\ \vdots \\ \mathbf{C}^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^k \end{bmatrix}$$

$$\mathbf{H_x} = \begin{bmatrix} \mathbf{B}_1^k & 0 & \cdots & 0 \\ \mathbf{A}^{k+1}\mathbf{B}_1^k & \mathbf{B}_1^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{k+N-1}\cdots\mathbf{A}^{k+1}\mathbf{B}_1^k & \mathbf{A}^{k+N-1}\cdots\mathbf{A}^{k+2}\mathbf{B}_1^{k+1} & \cdots & \mathbf{B}_1^{k+N-1} \end{bmatrix}$$

$$\mathbf{G_x} = \begin{bmatrix} \mathbf{B}_3^k & 0 & \cdots & 0 \\ \mathbf{A}^{k+1}\mathbf{B}_3^k & \mathbf{B}_3^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{k+N-1}\cdots\mathbf{A}^{k+1}\mathbf{B}_3^k & \mathbf{A}^{k+N-1}\cdots\mathbf{A}^{k+2}\mathbf{B}_3^{k+1} & \cdots & \mathbf{B}_3^{k+N-1} \end{bmatrix}$$

$$\mathbf{P_x} = \begin{bmatrix} \mathbf{B}_2^k & 0 & \cdots & 0 \\ \mathbf{A}^{k+1}\mathbf{B}_2^k & \mathbf{B}_2^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{k+N-1}\cdots\mathbf{A}^{k+1}\mathbf{B}_2^k & \mathbf{A}^{k+N-1}\cdots\mathbf{A}^{k+2}\mathbf{B}_2^{k+1} & \cdots & \mathbf{B}_2^{k+N-1} \end{bmatrix}$$

$$\mathbf{H_y} = \begin{bmatrix} \mathbf{D}_1^k & 0 & \cdots & 0 \\ \mathbf{C}^{k+1}\mathbf{B}_1^k & \mathbf{D}_1^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+1}\mathbf{B}_1^k & \mathbf{C}^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+2}\mathbf{B}_1^{k+1} & \cdots & \mathbf{D}_1^{k+N-1} \end{bmatrix}$$

$$\mathbf{G_y} = \begin{bmatrix} \mathbf{D}_3^k & 0 & \cdots & 0 \\ \mathbf{C}^{k+1}\mathbf{B}_3^k & \mathbf{D}_3^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+1}\mathbf{B}_3^k & \mathbf{C}^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+2}\mathbf{B}_3^{k+1} & \cdots & \mathbf{D}_3^{k+N-1} \end{bmatrix}$$

$$\mathbf{P_y} = \begin{bmatrix} \mathbf{D}_2^k & 0 & \cdots & 0 \\ \mathbf{C}^{k+1}\mathbf{B}_2^k & \mathbf{D}_2^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+1}\mathbf{B}_2^k & \mathbf{C}^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+2}\mathbf{B}_2^{k+1} & \cdots & \mathbf{D}_2^{k+N-1} \end{bmatrix}$$

Then the MPC optimization problem (8) leads to the following cost function

$$\mathbf{J} = \begin{bmatrix} \overline{\mathbf{u}} & \overline{\mathbf{z}} & \overline{\boldsymbol{\delta}} \end{bmatrix} \mathbf{H}_o \begin{bmatrix} \overline{\mathbf{u}} \\ \overline{\mathbf{z}} \\ \overline{\boldsymbol{\delta}} \end{bmatrix} + 2\mathbf{f}_o \begin{bmatrix} \overline{\mathbf{u}} \\ \overline{\mathbf{z}} \\ \overline{\boldsymbol{\delta}} \end{bmatrix} + \mathbf{g}_o \qquad (15)$$

where $\mathbf{H}_o = \begin{bmatrix} \mathbf{H}_y^T\overline{\mathbf{Q}}\mathbf{H}_y + \overline{\mathbf{R}} & \mathbf{H}_y^T\overline{\mathbf{Q}}\mathbf{G}_y & \mathbf{H}_y^T\overline{\mathbf{Q}}\mathbf{P}_y \\ \mathbf{G}_y^T\overline{\mathbf{Q}}\mathbf{H}_y & \mathbf{G}_y^T\overline{\mathbf{Q}}\mathbf{G}_y & \mathbf{G}_y^T\overline{\mathbf{Q}}\mathbf{P}_y \\ \mathbf{P}_y^T\overline{\mathbf{Q}}\mathbf{H}_y & \mathbf{P}_y^T\overline{\mathbf{Q}}\mathbf{G}_y & \mathbf{P}_y^T\overline{\mathbf{Q}}\mathbf{P}_y \end{bmatrix}$

$$\mathbf{f}_o = \begin{bmatrix} \mathbf{x}_k^T\mathbf{F}_y^T\overline{\mathbf{Q}}\mathbf{H}_y - \overline{\mathbf{w}}^T\overline{\mathbf{Q}}\mathbf{H}_y \\ \mathbf{x}_k^T\mathbf{F}_y^T\overline{\mathbf{Q}}\mathbf{G}_y - \overline{\mathbf{w}}^T\overline{\mathbf{Q}}\mathbf{G}_y \\ \mathbf{x}_k^T\mathbf{F}_y^T\overline{\mathbf{Q}}\mathbf{P}_y - \overline{\mathbf{w}}^T\overline{\mathbf{Q}}\mathbf{P}_y \end{bmatrix}^T$$

$$\mathbf{g}_o = [\mathbf{x}_k^T\mathbf{F}_y^T\overline{\mathbf{Q}}\mathbf{F}_y\mathbf{x}_k - 2\overline{\mathbf{w}}^T\overline{\mathbf{Q}}\mathbf{F}_y\mathbf{x}_k + \overline{\mathbf{w}}^T\overline{\mathbf{Q}}\overline{\mathbf{w}}]$$

$$\overline{\mathbf{R}} = \mathrm{diag}[r_{ii}], \quad \left(\overline{\mathbf{R}} = \overline{\mathbf{R}}^T \succ 0\right)$$

$$\overline{\mathbf{Q}} = \mathrm{diag}[q_{ii}], \quad \left(\overline{\mathbf{Q}} = \overline{\mathbf{Q}}^T \succ 0\right)$$

This technique allows choosing different weighting factors for each sub-region according to its priority.

The constraints over the state and input domains for each sub-region are included in the inequality equation of the MLD model of that sub-region using the HYSDEL program. The MPC optimization problem (15) is solved subject to the constraints

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_3 & \mathbf{M}_2 \end{bmatrix} \begin{bmatrix} \overline{\mathbf{u}} \\ \overline{\mathbf{z}} \\ \overline{\boldsymbol{\delta}} \end{bmatrix} \le \mathbf{N} \qquad (16)$$

where $\mathbf{N} = \begin{bmatrix} \mathbf{E}_4^k\mathbf{x}_k + \mathbf{E}_5^k \\ \mathbf{E}_4^{k+1}\mathbf{A}^k\mathbf{x}_k + \mathbf{E}_5^{k+1} \\ \vdots \\ \mathbf{E}_4^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^k\mathbf{x}_k + \mathbf{E}_5^{k+N-1} \end{bmatrix}$

$$\mathbf{M}_1 = \begin{bmatrix} -\mathbf{E}_1^k & 0 & \cdots & 0 \\ -\mathbf{E}_4^{k+1}\mathbf{B}_1^k & -\mathbf{E}_1^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -\mathbf{E}_4^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+1}\mathbf{B}_1^k & -\mathbf{E}_4^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+2}\mathbf{B}_1^{k+1} & \cdots & -\mathbf{E}_1^{k+N-1} \end{bmatrix}$$

$$\mathbf{M}_2 = \begin{bmatrix} \mathbf{E}_2^k & 0 & \cdots & 0 \\ -\mathbf{E}_4^{k+1}\mathbf{B}_2^k & \mathbf{E}_2^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -\mathbf{E}_4^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+1}\mathbf{B}_2^k & \cdots & -\mathbf{E}_4^{k+N-1}\mathbf{B}_2^{k+N-2} & \mathbf{E}_2^{k+N-1} \end{bmatrix}$$

$$\mathbf{M}_3 = \begin{bmatrix} \mathbf{E}_3^k & 0 & \cdots & 0 \\ -\mathbf{E}_4^{k+1}\mathbf{B}_3^k & \mathbf{E}_3^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -\mathbf{E}_4^{k+N-1}\mathbf{A}^{k+N-2}\cdots\mathbf{A}^{k+1}\mathbf{B}_3^k & \cdots & -\mathbf{E}_4^{k+N-1}\mathbf{B}_3^{k+N-2} & \mathbf{E}_3^{k+N-1} \end{bmatrix}$$

The number of binary optimization variables, with $\boldsymbol{\delta}_1$ known and constant, is given by the relation

$$L = \sum_{i=0}^{N-1} m_{lj}^i + r_{l2j}^i, \qquad j \in \{1, 2, \cdots, s\} \qquad (17)$$

Where m_{lj}^i, r_{l2j}^i are the number of modeled logical control and $\boldsymbol{\delta}_2$ elements respectively in the j^{th} sub-region at prediction time i.

Therefore, if the sequence I is fixed, the problem can be solved minimizing (15) subject to the constraints of (16). But, as only I_k is known (where $\mathbf{x}(k)$ is considered as known, and $\boldsymbol{\delta}_1(k)$ depends on $\mathbf{x}(k)$), all possible sequences as in Figure 2 have to be solved. So the number of possible sequences is s^{N-1}. The optimal solution is provided by the resolution of these s^{N-1} MIQPs. In order to find the solution more quickly, these problems are not solved independently and the optimal value of the criterion provided by the solved MIQP is used to bind the result of the others. It can then be used by the B&B algorithm to cut branches.

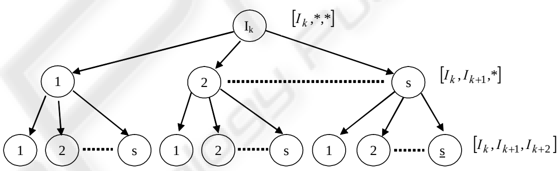

Figure 2: State transition graph of the MPC optimization problem for a system under the SPMLD form ($N = 3$)

4.3 Compared computational burden

The global complexity of the MPC resolution with systems under the SPMLD form is reduced. First the related number of subsystems s is smaller than that with the classical PWA model. Then, the B&B technique considerably decreases the number of solved QPs. The index sequence I imposes the successive values of $\boldsymbol{\delta}_1$ over the prediction horizon and then the succession of region on the state space partition the state has to reach. This leads to non feasible solutions in many sub-problems, effectively reducing the number of solved QPs according to the B&B technique. This is why we partition $\boldsymbol{\delta}$ vector.

First, the SPMLD technique is faster than the classical MLD because for a known sequence of index I, only $2^{r_{l1}(N-1)}$ simple B&B trees with only $\mathbf{u}, \boldsymbol{\delta}_2, \mathbf{z}$ optimization variables have to be solved; i.e. smaller number of optimization binary variables L (17), and simpler MLD models as previously explained. Moreover, as explained, the optimization algorithm just has to look for the control sequence that could force the system to follow the index I and optimize the cost function with respect to all the associated constraints. In many root nodes at level F (Figure 3), this leads to non-feasible solution (more frequent than in classical MLD approach), due to non feasible sequence whatever the value of the control inputs are, thus that MIQP will then quickly be eliminated. Furthermore, if there is a solution for a B&B tree at level F, it is considered as an upper bound for the global optimized solution for all the following B&B trees, which reduces the number of solved QPs according to B&B technique.

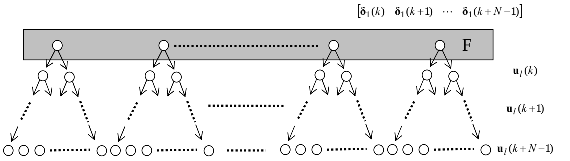

Figure 3: B&B trees for optimization with SPMLD

Then, the SPMLD technique is obviously faster than the classical PWA technique. First the initial index I_k is completely known as the space partition only depends on $\mathbf{x}(k)$ and not on $\mathbf{u}$ (as in the PWA model where m_p possible subsystems at instant k have to be examined). In addition, SPMLD models use the B&B technique, which considerably reduces the number of solved QPs while in classical PWA systems all the QPs must be solved.

4.4 Further improvements of the optimization time

Two different techniques can be considered to reduce the computation load for real time applications. The first one introduces the control horizon N_u, which reduces the number of unknown future control values, i.e. $\mathbf{u}(k+i)$ is constant for $i \geq N_u$. This decreases the number of binary optimization variables (17) and the optimization time

$$L_{N_u} = \sum_{i=0}^{N_u-1} m_{lj}^i + \sum_{i=0}^{N-1} r_{l2j}^i, \quad j \in \{1,2,\cdots,s\} \tag{18}$$

The second one, called the reach set, aims at determining the set of possible regions that can be reached from the actual region in next few sampling times (Kerrigan, 2000). That is, all sequences that cannot be obtained are not considered.

5 APPLICATION

5.1 Description of the benchmark

The proposed control strategy is applied on the three tanks benchmark used in (Bemporad *et al.*, 1999). The simplified physical description of the three tanks system proposed by COSY as a standard benchmark for control and fault detection problems is presented in Figure 4 (Dolanc *et al.*, 1997).

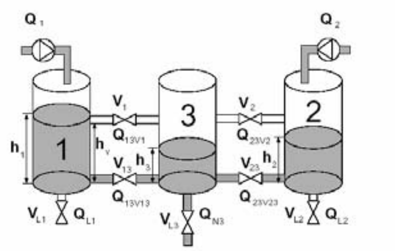

Figure 4: COSY three tanks benchmark system

The system consists of three tanks, filled by two independent pumps acting on tanks 1 and 2, continuously manipulated from 0 up to a maximum flow Q_1 and Q_2 respectively. Four switching valves V_1, V_2, V_{13} and V_{23} control the flow between the tanks, those valves are assumed to be either completely opened or closed ($V_i = 1$ or 0 respectively). The V_{L3} manual valve controls the nominal outflow of the middle tank. It is assumed in further simulations that the V_{L1} and V_{L2} valves are always closed and V_{L3} is open. The liquid levels to be controlled are denoted h_1, h_2 and h_3 for each tank respectively. The conservation of mass in the tanks provides the following differential equations

$$\begin{aligned} \dot{h}_1 &= \frac{1}{A}(Q_1 - Q_{13V1} - Q_{13V13}) \\ \dot{h}_2 &= \frac{1}{A}(Q_2 - Q_{23V2} - Q_{23V23}) \\ \dot{h}_3 &= \frac{1}{A}(Q_{13V1} + Q_{13V13} + Q_{23V2} + Q_{23V23} - Q_N) \end{aligned} \tag{19}$$

where the Qs denote the flows and A is the cross-sectional area of each of the tanks. A MLD model is derived as developed in (Bemporad *et al.*, 1999), introducing the following variables

$$\begin{aligned} \mathbf{x} &= [h_1 \ h_2 \ h_3]' \\ \mathbf{u} &= [Q_1 \ Q_2 \ V_1 \ V_2 \ V_{13} \ V_{23}]' \\ \boldsymbol{\delta} &= [\delta_{01} \ \delta_{02} \ \delta_{03}]' \\ \mathbf{z} &= [z_{01} \ z_{02} \ z_{03} \ z_1 \ z_2 \ z_{13} \ z_{23}]' \end{aligned} \tag{20}$$

where

$$\begin{aligned} &[\delta_{0i}(t) = 1] \leftrightarrow [h_i(t) \geq h_v] \quad i = 1,2,3 \\ &z_{0i}(t) = \delta_{0i}(t)(h_i(t) - h_v) \quad i = 1,2,3 \\ &z_i(t) = V_i(z_{0i}(t) - z_{03}(t)) \quad i = 1,2 \\ &z_{i3}(t) = V_{i3}(h_i(t) - h_3) \quad i = 1,2 \end{aligned}$$

5.2 Application of MPC for the SPMLD formalism

In this system, $\boldsymbol{\delta}_1 = \boldsymbol{\delta}$ since the three introduced auxiliary binary variables depend on the states, thus $r_{l1} = r_l$ and the number of sub-systems is

$$s = 2^{r_{l1}} = 8 \tag{21}$$

Inside each sub-region, a simple MLD model is developed, that takes into account only the system dynamics in this sub-region. In some sub-regions a reduction in the size of **u** and **z** appears; for example in the sub-region where $\boldsymbol{\delta}_1' = [0 \quad 0 \quad 0]$ it clearly appears that the two valves V_1 and V_2 of the input vector are not in progress, as the liquid level in this region is always less than the valves level. Consequently, the continuous auxiliary variables $\{z_{0i}\}_{i=1,2,3}$ and $\{z_i\}_{i=1,2}$ corresponding to the flows that pass through the upper pipes are useless. It results from this a simple model with:

$$\mathbf{x} = [h_1 \ h_2 \ h_3]', \mathbf{u} = [Q_1 \ Q_2 \ V_{13} \ V_{23}]' \qquad (22)$$
$$\mathbf{z}_1 = [\ z_{13} \ z_{23}]'$$

Let us consider now the following specification: starting from zero levels (the three tanks being completely empty), the objective of the control strategy is to reach the liquid levels $h_1 = 0.5\,\text{m}$, $h_2 = 0.5\,\text{m}$ and $h_3 = 0.1\,\text{m}$. The MPC technique for a SPMLD model has been implemented in simulation to reach the level specification with $N = 2$. The results are presented on Figure 5 for the tanks levels and on Figure 6 for the control signals.

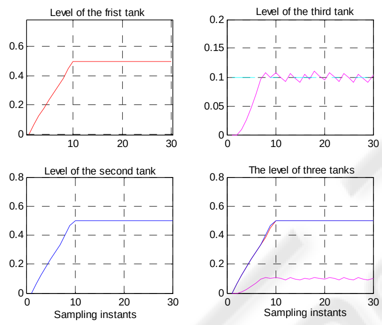

Figure 5: Water levels in the three tanks

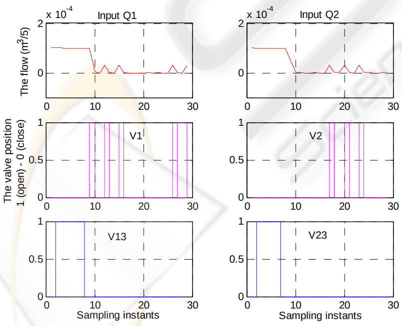

Figure 6: Controlled variables

The level of the third tank oscillates around 0.1 as $h_3 = 0.1$ does not correspond to an equilibrium point. Consequently, the system opens and closes the two valves V_1 and V_2 to maintain the level in the third tank around the desired level of 0.1m.

5.3 Comparison of the approaches

As a comparison purpose between the SPMLD model, the classical MLD model and the classical PWA model strategies, the same previous level specification has been considered with $N = 2$. The MLD model described in (Bemporad *et al.*, 1999) has been used for the three tanks modeled by (20); this MLD model transfers to a PWA model with $s = 128$ subsystems (with 28 empty regions). The classical PWA model has not been developed as it needs 100 sub-models and is in fact not required to compare complexity. For that comparison, looking at the number of QPs that have to be solved during optimization is sufficient.

Table 1 illustrates for $N = 2$ the total time required to reach the specification level, the total number of QPs solved, and the maximum time and maximum QPs to find the optimized solution at each iteration. It can be seen that the difference between the SPMLD technique and the other classical techniques is quite large, the SPMLD model allowing real time implementation and avoiding exponential explosion of the algorithm (the sampling time of the three tanks benchmark is 10 s.). All data given above were obtained using the MIQP Matlab code (Bemporad and Mignone, 2000), on a 1.8 MHz PC with 256 Mo of ram. Same comparisons are presented with $N = 3$ in table 2.

Table 1: Comparison of performances obtained with the SPMLD model, the classical MLD model and the classical PWA model for $N = 2$.

Approach	No of QPs solved	Max. No. QPs / step	Total time	Max. time / step
Classical PWA	8 800	1 600	*	*
Classical MLD	11 130	2 089	822.97 s	138.97 s
SPMLD	832	218	15.28 s	3.90 s

Table 2: Comparison of performances obtained with the SPMLD, MLD and PWA models for $N = 3$

Approach	No of QPs solved	Max. No. QPs / step	Total time	Max. time / step
Classical PWA	880 000	160 000	*	*
Classical MLD	25 606	6 867	5243.6 s	1 147.80 s
SPMLD	3 738	1 054	137.54 s	37.65 s

This table shows that no real time implementation is possible with $N = 3$ for the SPMLD form, although the maximum time per iteration is much smaller in this case. But it must be noticed that the results in table 1 and 2 for the SPMLD model are achieved without applying techniques described in section 4.4. For example using a prediction horizon $N = 3$ and a control horizon $N_u = 1$ leads to the following results enabling real time implementation

No QPs solved = 1224, Max. No QPs/step = 326

Total optimization time = 29.24 s., Max. time/step = 7.92 s.

The technique of MPC for SPMLD systems has been examined also with a simple automata, where automata of Figure 7 have been added to V_1 and V_2 valves of the three tanks benchmark. We assumed for a simplification purpose that $\delta_{03} = 0$ i.e. the level in the third tank is always behind the h_v level.

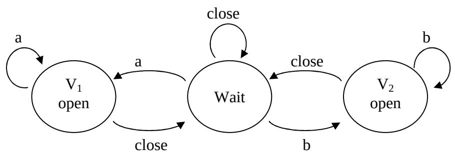

Figure 7: added Automata to the three tanks benchmark.

The automata of Figure 7 can be presented as follows

$$V_{1open} = (wait \,\&\, a) \big| (V_{1open} \,\&\, a)$$
$$V_{2open} = (wait \,\&\, b) \big| (V_{2open} \,\&\, b) \qquad (23)$$
$$Wait = close$$

The SPMLD technique succeeds to reduce the total optimization time to arrive to the specifications, from 5691.4 s for the classical MLD technique to a 173.5 s, solving 3990 QPs instead of 60468 QPs where for each sequence *I*, $\boldsymbol{\delta}$ variables as well as the logical control variables that control the automata are known.

6 CONCLUSION

This paper presents the SPMLD formalism. It is developed by partitioning the feasible region according to the auxiliary binary elements $\boldsymbol{\delta}_1$ of the MLD model that depends on the state variables. A reformulation of the MPC strategy for this formalism has been presented. It is shown that the SPMLD model successfully improves the computational problem of the mixed Logical Dynamical (MLD) model and Piecewise Affine (PWA) model. Moreover, the partition into several sub-regions enables to define particular weighting factors according to the priority of each region. Future work may consider examining $\boldsymbol{\delta}$ variables that depends on the control inputs, by partitioning the feasible region according to those variables also instead of leaving them free included in the optimization vector.

REFERENCES

Bemporad, A., 2003. “A Recursive Algorithm for Converting Mixed Logical Dynamical Systems into an Equivalent Piecewise Affine Form”, *IEEE Trans. Autom. Contr.*

Bemporad, A., Ferrari-Trecate G., and Morari M., 2000. Observability and controllability of piecewise affine and hybrid systems. *IEEE Trans. Automatic Control, 45(10):1864–1876*.

Bemporad, A., and Mignone, D., 2000.: Miqp.m: A Matlab function for solving mixed integer quadratic programs. *Technical Report*.

Bemporad, A., Mignone, D. and Morari, M., 1999. Moving horizon estimation for hybrid systems and fault detection. In *Proceedings of the American Control Conference, San Diego*.

Bemporad, A. and Morari, M., march 1999. Control of systems integrating logical, dynamics, and constraints. *Automatica, 35(3):407-427*.

Branicky, M.S., Borkar, V.S. and Mitter, S.K., January 1998. A unified framework for hybrid control: model and optimal control theory. *IEEE Transaction. on Automatic. Control, 43(1): 31-45*.

Dolanc, G., Juricic D., Rakar A., Petrovcic J. and Vrancic, D., 1998. Three-tank Benchmark Test. *Technical Report Copernicus Project Report* CT94-02337.

Dumur, D. and Boucher, P.,1998. A Review Introduction to Linear GPC and Applications. *Journal A, 39(4), pp. 21-35*.

Fletcher, R. and Leyffer, S., 1995. Numerical experience with lower bounds for MIQP branch and bound. *Technical report, Dept. of Mathematics, University of Dundee, Scotland*.

Kerrigan, E., 2000. Robust Constraint Satisfaction: Invariant sets and predictive control. *PhD thesis University of Cambridge*.

Pena, M.., Camacho, E. F. and Pinon, S., 2003. Hybrid systems for solving model predictive control of piecewise affine system. *IFAC conference, Hybrid system analysis and design Saint Malo, France*.

Sontag, E.D., April 1981: Nonlinear regulation: the piecewise linear approach. *IEEE Transaction. on Automatic. Control, 26(2):346-358*.

Torrisi, F., Bemporad, A. and Mignone, D., 2000. Hysdel – a tool for generating hybrid models. *Technical report, AUT00-03, Automatic control laboratory, ETH Zuerich*.