STATE-DEPENDENT DELAY SYSTEM MODEL FOR CONGESTION CONTROL

Konstantin E. Avrachenkov
INRIA Sophia Antipolis, 2004 route des Lucioles, B.P. 93
06902, Sophia Antipolis Cedex, France

Wojciech Paszke
Institute of Control and Computation Engineering
University of Zielona Góra, Poland

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Abstract: This paper considers the problem of the stability for a control system with a state-dependent delay. Systems with state-dependent delays arise naturally in the data network congestion control schemes. Simple analytical studies of stability are provided for a particular set of initial conditions. Then, to verify these results and to extend the stability analysis, the Simulink-based simulator is presented and described. The developed simulator allows us to extend stability analysis for a variety of sets of initial conditions.

1 INTRODUCTION

In data networks such as Internet or ATM one needs to control the sending rate of data injected into the network. For instance, in the Internet this task is performed by Transmission Control Protocol (TCP) (Stevens, 1994). The control of the data sending rate is essentially based on the delayed information which is frequently source of instability.

In several recent paper (Deb and Srikant, 2003; Johari and Tan, 2000; Kelly, 2001; Massoulie, 2000; Vinnicombe, 2002) the researchers have analysed the sending rate control models for data networks with fixed delays. However, it is known (see e.g., (Altman et al., 2001)) that the value of the delay actually depends on the sending rate. In the present paper, we make the first attempt to analyse a rate control system with a state-dependent delay. In the next section we introduce our model. In Section 3, we provide an analytic study of the stability for some particular set of the initial conditions. Then, in Section 4 we described a `Simulink` based model and in Section 5 we verify and extend the analytic conditions using that `Simulink` model. Finally, in Section 6 we provide conclusions together with some directions for future research.

2 MODEL FORMULATION

Here we consider a single bottleneck network model. We represent the data sent into the network by the fluid which injection rate evolves according to the following equation

$$\dot{y}(t) = \alpha - \beta y(t - \mu^{-1}x(t)), \tag{1}$$

where $x(t)$ is the amount of the data stored at the bottleneck router. We can think of the above equation as an approximation of the total rate evolution of multiplexed TCP sources passing through the same bottleneck router. The term α corresponds to the additive increase and the term $-\beta y(t - \mu^{-1}x(t))$ corresponds to the multiplicative decrease of TCP in the Congestion Avoidance phase. Since in the present Internet state the main component of the information delay corresponds to data queueing, in our model we neglect the propagation delay and model the queueing delay by $\mu^{-1}x(t)$, where μ is a capacity of the bottleneck router.

By using the fluid approach, the following evolution of data queued at the bottleneck router is considered

$$\dot{x}(t) = \begin{cases} y(t) - \mu, & \text{if} \quad x(t) > 0, \\ (y(t) - \mu)_+, & \text{if} \quad x(t) = 0, \end{cases} \tag{2}$$

where $(z)_+ = \max\{z, 0\}$ and $(z)_- = \min\{z, 0\}$. In general, this expression shows that the derivative of the queue content is equal to the incoming rate minus the drain rate and it cannot become negative. As a result the sending rate can tend to infinity and in fact a

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whole system becomes unstable. To avoid this problem, the extension of (2) is considered

$$\dot{x}(t) = \begin{cases} (y(t)-\mu)_+, & \text{if} \quad x(t)=0, \\ y(t)-\mu, & \text{if} \quad 0 < x(t) < M, \\ (y(t)-\mu)_-, & \text{if} \quad x(t)=M, \end{cases} \qquad (3)$$

where M denotes the maximum queue size in bottleneck router.

To reduce the number of parameters, let us introduce the following change of variables

$$\hat{x} := \mu^{-1}x, \;\; \hat{y} := \mu^{-1}y,$$

$$\hat{\alpha} := \mu^{-1}\alpha, \;\; \hat{\beta} := \mu^{-1}\beta,$$

and to rewrite (1) and (2) as follows

$$\dot{\hat{y}}(t) \;\; = \;\; \hat{\alpha} - \hat{\beta}\hat{y}(t - \hat{x}(t)) \qquad (4)$$

$$\dot{\hat{x}}(t) \;\; = \;\; \hat{y}(t) - 1. \qquad (5)$$

for $0 < \hat{x}(t) < M$.

3 ANALYTICAL STUDY OF STABILITY

It turns out that it is easy to perform the analytical study of the system stability with (2) for a particular set of initial conditions. Namely, we have the following result.

Lemma 1 *Let* $\hat{x}(0) = 0$ *and* $\hat{y}(0) \in [0,1]$ *and*

$$\frac{\hat{\alpha}}{\hat{\beta}} < 1, \qquad (6)$$

then the system of equations (4) *and* (5) *is asymptotically stable.*

Proof 1 *We note that if the system starts from the initial conditions* $\hat{x}(0) = 0$ *and* $\hat{y}(0) \in [0,1]$*, and in addition* $\hat{\alpha}/\hat{\beta} < 1$*, the data backlog* $\hat{x}(t)$ *remains zero and one can view the equation (4) as a linear time-invariant differential equation.* ■

4 SIMULINK-BASED STUDY OF STABILITY

To verify the stability condition (6) and to study system behavior under nonzero initial condition and different evolutions of $\dot{x}(t)$, simulation tools, such as Simulink, can be used.

Since the system is represented by (4) and (5) then the stability region can be considered in two-dimensional space.

In the last few years, Simulink has become the most widely used software package for modeling and simulating dynamic systems (The MathWorks,). It allows us to describe, to simulate, to evaluate, and to refine a system's behavior through standard and custom block libraries. Simulink integrates seamlessly with MATLAB, which provide an immediate access to an extensive range of analysis and design tools. These benefits make Simulink the tool of choice for control system design, signal processing system design, communications system design, and other simulation applications. Moreover, this software package gives us ability to simulate systems that would not be possible to analyze in analytical way.

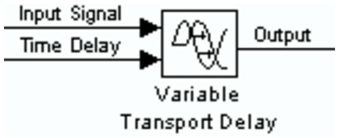

Figure 1: The variable transport delay block

Faced with the above facts, the Simulink package has been chosen as a basis to build the simulator for the system represented by (4) and (5). The main part of this simulator is the variable transport delay block, which is used to simulate a time delay depended on the signal $\hat{x}(t)$ in (4). The block accepts two inputs: the first input is the signal that passes through the block (in our case $\hat{y}(t)$) and the second input is the time delay, as shown in Figure 1. The other parts of the simulator are rather standard and the block diagram of the simulator is presented in Figure 2.

5 SIMULATION RESULTS

In this section we present the results of simulation studies on the system represented by (4) and (5) prepared with described above simulator.

5.1 An example of unstable system

The data for the first example are as follows.

$$\hat{\alpha} = 0.4; \;\; \hat{\beta} = 0.1;$$

Figures 3 and 4 show the $\hat{x}$ signal and the response of this system ($\hat{y}$) respectively. It is clear that the size of delay increases due to $\hat{x}(t) \to \infty$ and as the result we conclude that the system represented by (4) and (5) is unstable.

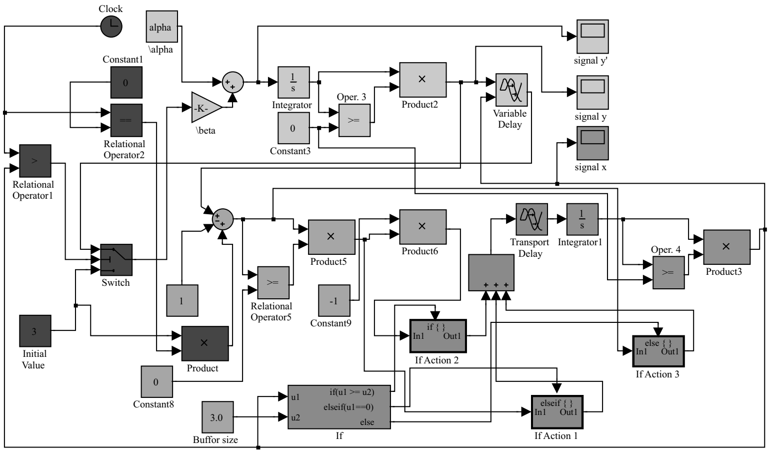

Figure 2: The simulator

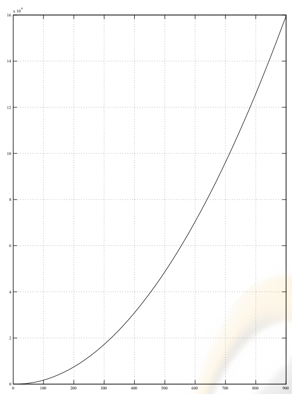

Figure 3: Signal $\hat{x}(t)$ of unstable system

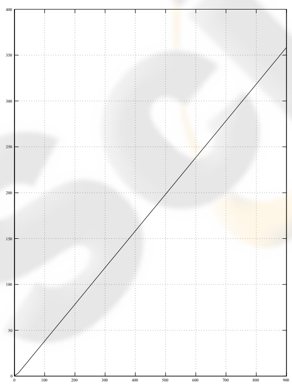

Figure 4: Signal $\hat{y}(t)$ of unstable system

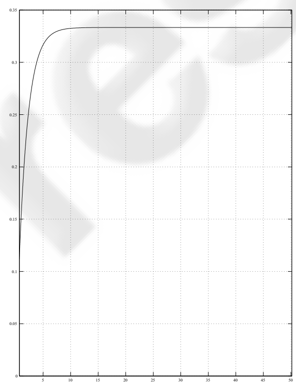

Figure 5: Signal $\hat{y}(t)$ of the stable system

5.2 An example of stable system

As the second example, let us consider the following system parameters

$$\hat{\alpha} = 0.2; \quad \hat{\beta} = 0.6;$$

Figure 5 shows the response of this system ($\hat{y}$). In this case the delay $\hat{x} = 0$ (it is not presented here) and the resulting system represented by (4) and (5) is stable.

5.3 Stability region

Previously presented simulation results do not give us chance to conclude for what conditions (parameters variables) the system is stable. Therefore, thousands of the simulations have to be done to provide the stability region in two-dimensional space of parameters $\hat{\alpha}$ and $\hat{\beta}$.

In order to do a such large number of simulations, a typical `MATLAB` M-*file* is used to run them. This script generates a large amount of the systems (4) and (5) with various values of parameters $\hat{a}$ and $\hat{b}$. The stability of the system is analyzed with the following simple condition

$$\begin{cases} |\hat{y}(\tau) - \text{const}| < \epsilon \Rightarrow \text{ system is stable} \\ |\hat{y}(\tau) - \text{const}| > \epsilon \Rightarrow \text{ system is unstable} \end{cases} \tag{7}$$

for $\tau \in [t, T]$ and for some given t and T. Moreover, the M-*file* is implementation an iterative algorithm of the form

(1) Set $\hat{\alpha}^0 = 0$, $\hat{\beta}^0 = 0$ and zero initial conditions

(2) Run a simulation in `Simulink` with provided parameters

(3) Write the simulation parameters into memory

(4) Check the stability condition (7). If system is stable, then set $k_n = 0$ otherwise set $k_n = 1$ (system is stable)

(5) Increase variables $\hat{a}$ and $\hat{b}$ with the formula $\hat{\alpha}^{n+1} = \hat{\alpha}^n + \Delta\hat{\alpha}$ and $\hat{\beta}^{n+1} = \hat{\beta}^n + \Delta\hat{\beta}$

(6) If a specified number of iteration is performed then exit otherwise set $n = n + 1$ and return to Step 2

Using the above algorithm, the result of the simulation is presented on Figure 6. Note, that rhe resulting value 0 denotes stable system. It is important to note that the simulation result can be put in the `MATLAB` workspace for postprocessing and visualization.
It is straightforward to see that the simulation result shown on Figure 6 confirms the analytical result provided by Proposition 1 (6).

Note that all computations and simulations have been performed with `MATLAB` 6.5 and the `Simulink` 5.0.

5.4 A system with a bottleneck router with finite buffer

It was shown in previous section that system (4) under delay evaluation (2) could be unstable. To overcome this drawback we proceed to apply the evaluation (3) for data stored in bottleneck router. The effectiveness of the proposed control scenario is confirmed by the simulations performed for the data

$$\hat{\alpha} = 0.79; \quad \hat{\beta} = 0.7, \quad y(0) = 3$$

and following maximal buffer sizes

1. $M = 0.6$ (The simulations results are : Figures 7, 8).

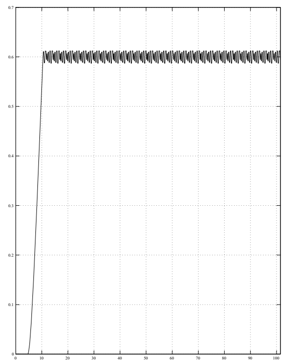

Figure 7: The signal $\hat{x}(t)$

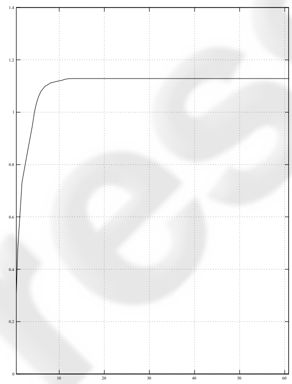

Figure 8: The signal $\hat{y}(t)$

2. $M = 2.3$ (The simulations results are : Figures 9, 10).
3. $M = 2.6$ (The simulations results are : Figures 11, 12).
4. $M = 3.0$ (The simulations results are : Figures 13, 14).

It is important to note that in some cases $\hat{x}(t)$ exceeds the level of M. It is caused by some small delay added to avoid problems with simulations under Simulink (high frequency oscillations on the level of M).

6 CONCLUSION

This paper studies stability of the rate control system with a state-dependent delay for data networks. The analytical condition for stability of the system starting from some particular set of initial conditions is provided. Then, `Simulink` based simulations are used to verify the stability conditions. Clearly, the

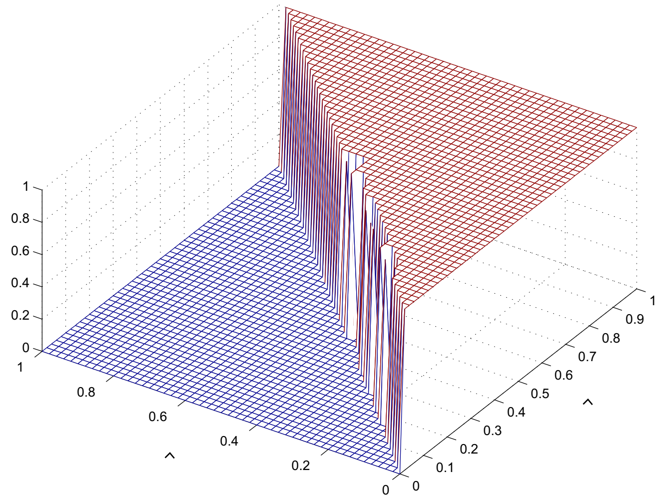

Figure 6: Stability region

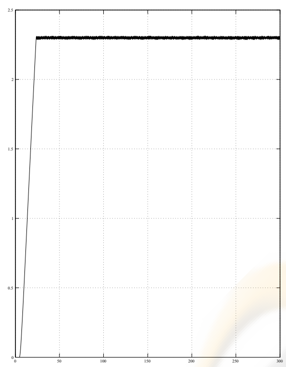

Figure 9: The signal $\dot{\hat{x}}(t)$

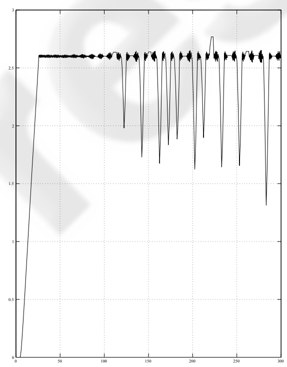

Figure 11: The signal $\hat{x}(t)$

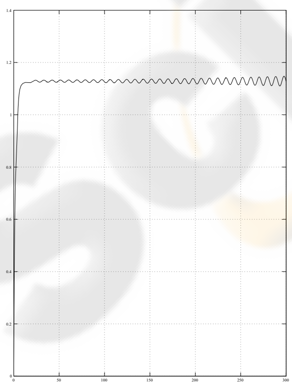

Figure 10: The signal $\hat{y}(t)$

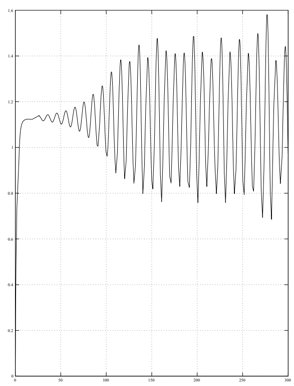

Figure 12: The signal $\dot{\hat{y}}(t)$

presented model provides opportunities for future analytical and simulation based research.

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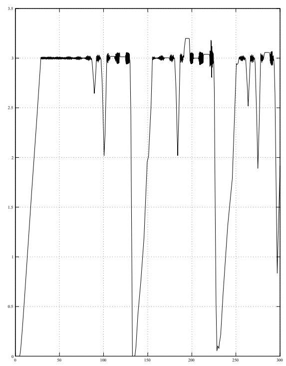

Figure 13: The signal $\hat{x}(t)$

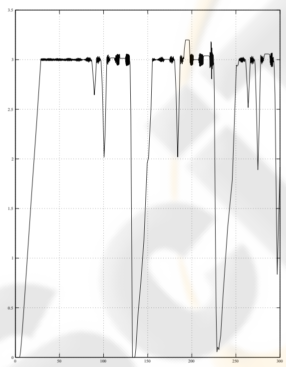

Figure 14: The signal $\dot{y}(t)$