AN LMI OPTIMIZATION APPROACH FOR GUARANTEED COST CONTROL OF SYSTEMS WITH STATE AND INPUT DELAYS

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Abstract: The robust control problem for linear systems with parameter uncertainties and time-varying delays is examined. By using an appropriate uncertainty description, a linear state feedback control law is found ensuring the closed-loop system's stability and a performance measure, in terms of the *guaranteed cost*. An LMI objective minimization approach allows to determine the "optimal" choice of free parameters in the uncertainty description, leading to the minimal guaranteed cost.

1 INTRODUCTION

Model uncertainties and time-delays are frequently encountered in physical processes and may cause performance degradation and even instability of control systems (Malek-Zavarei and Jamshidi, 1987), (Mahmoud, 2000), (Niculescu, 2001). Hence, stability analysis and robust control problems of uncertain dynamic systems with delayed states have been studied in recent control systems literature; for details and references see e.g. (Niculescu et al., 1998), (Kolmanovskii et al., 1999), (Richard, 2003). Since control input delays are often imposed by process design demands as in the case of transmission lines in hydraulic or electric networks, it is necessary to consider uncertain systems with both, state and input time-delays. Moreover, when the delays are imperfectly known, one has to consider uncertainty on the delay terms, as well. In recent years, LMIs are used to solve complex analysis and control problems for uncertain delay systems (e.g. (Li and de Souza, 1997), (Li et al., 1998), (Tarbouriech and da Silva Jr., 2000), (Bliman, 2001), (Kim, 2001), and related references).
The purpose of the present paper is to design control laws for systems with uncertain parameters and uncertain time-delays affecting the state and the control input. Although the uncertainties are assumed to enter linearly in the system description, it is well known that they may be time-varying and nonlinear in nature, in most physical systems. Consequently, the closed-loop system's stability has to be studied in the Lyapunov-Krasovskii framework; the notion of quadratic stability is then extended to the class of time-delay systems (Barmish, 1985),(Malek-Zavarei and Jamshidi, 1987). On the other hand, it is desirable to ensure some performance measure despite uncertainty and time-delay, in terms of guaranteed upper bounds of the performance index associated with the dynamic system. The latter specification leads to the guaranteed cost control (Chang and Peng, 1972), (Kosmidou and Bertrand, 1987).
In the proposed approach the uncertain parameters affecting the state, input, and delay matrices are allowed to vary into a pre-specified range. They enter into the system description in terms of the so-called *uncertainty matrices* which have a given structure. Different unity rank decompositions of the uncertainty matrices are possible, by means of appropriate scaling. This description is convenient for many physical system representations (Barmish, 1994). An LMI optimization solution (Boyd et al., 1994)is then sought in order to determine the appropriate uncertainty decomposition; the resulting guaranteed cost control law ensures the minimal upper bound. The closed-loop system's quadratic stability follows as a direct consequence.
The paper is organized as follows: The problem formulation and basic notions are given in Section 2. Computation of the solution in the LMI framework is presented in Section 3. Section 4 presents a numerical example. Finally, conclusions are given in Section 5.

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2 PROBLEM STATEMENT AND DEFINITIONS

Consider the uncertain time-delay system described in state-space form,

$$\dot{x}(t) = [A_1 + \Delta A_1(t)]x(t) + [A_2 + \Delta A_2(t)]x(t - d_1(t)) + [B_1 + \Delta B_1(t)]u(t) + [B_2 + \Delta B_2(t)]u(t - d_2(t)) \quad (1)$$

for $t \in [0, \infty)$ and with $x(t) = \phi(t)$ for $t < 0$.

In the above description, $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control vector and A_1, A_2 $\in \mathbf{R}^{n\times n}$, B_1, $B_2 \in \mathbf{R}^{n\times m}$ are constant matrices.

The model uncertainty is introduced in terms of

$$\begin{aligned}
\Delta A_1(t) &= \sum_{i=1}^{k_1} A_{1i} r_{1i}(t), & | r_{1i}(t) | &\leq \bar{r_1} \\
\Delta A_2(t) &= \sum_{i=1}^{k_2} A_{2i} r_{2i}(t), & | r_{2i}(t) | &\leq \bar{r_2} \\
\Delta B_1(t) &= \sum_{i=1}^{l_1} B_{1i} p_{1i}(t), & | p_{1i}(t) | &\leq \bar{p_1} \\
\Delta B_2(t) &= \sum_{i=1}^{l_2} B_{2i} p_{2i}(t), & | p_{2i}(t) | &\leq \bar{p_2} \quad (2)
\end{aligned}$$

where A_{1i}, A_{2i}, B_{1i}, B_{2i} are given matrices with constant elements determining the uncertainty structure in the state, input, and delay terms.
The uncertain parameters r_{1i}, r_{2i}, p_{1i}, p_{2i} are Lebesgue measurable functions, possibly time-varying, that belong into pre-specified bounded ranges (2), where $\bar{r_1}$, $\bar{r_2}$, $\bar{p_1}$, $\bar{p_2}$ are positive scalars; since their values can be taken into account by the respective uncertainty matrices, it is assumed that $\bar{r_1} = \bar{r_2} = \bar{p_1} = \bar{p_2} = 1$, without loss of generality. Moreover, the advantage of the affine type uncertainty description (2) is that it allows the uncertainty matrices to have unity rank and thus to be written in form of vector products of appropriate dimensions,

$$\begin{aligned}
A_{1i} &= d_{1i} e_{1i}^T, & i &= 1, ..., k_1 \\
A_{2i} &= d_{2i} e_{2i}^T, & i &= 1, ..., k_2 \\
B_{1i} &= f_{1i} g_{1i}^T, & i &= 1, ..., l_1 \\
B_{2i} &= f_{2i} g_{2i}^T, & i &= 1, ..., l_2 \quad (3)
\end{aligned}$$

Obviously, the above decomposition is not unique; hence, the designer has several degrees of freedom in choosing the vector products, in order to achieve the design objectives. By using the vectors in (3), define the matrices,

$$\begin{aligned}
D_1 &:= [d_{11}...d_{1k_1}], & E_1 &:= [e_{11}...e_{1k_1}] \\
D_2 &:= [d_{21}...d_{2k_2}], & E_2 &:= [e_{21}...e_{2k_2}] \\
F_1 &:= [f_{11}...f_{1l_1}], & G_1 &:= [g_{11}...g_{1l_1}] \\
F_2 &:= [f_{21}...f_{2l_2}], & G_2 &:= [g_{21}...g_{2l_2}] \quad (4)
\end{aligned}$$

which will be useful in the proposed guaranteed cost approach.

The time delays in (1) are such that,

$$\begin{aligned}
0 \leq d_1(t) \leq \bar{d_1} < \infty, \qquad \dot{d_1}(t) \leq \beta_1 < 1 \\
0 \leq d_2(t) \leq \bar{d_2} < \infty, \qquad \dot{d_2}(t) \leq \beta_2 < 1 \quad (5)
\end{aligned}$$

$\forall t \geq 0$.

Associated with system (1) is the quadratic cost function

$$J(x(t), t) = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \quad (6)$$

with $Q > 0$, $R > 0$, which is to be minimized for a linear constant gain feedback control law of the form

$$u(t) = Kx(t) \quad (7)$$

by assuming (A_1, B_1) stabilizable, $(Q^{1/2}, A_1)$ detectable and the full state vector $x(t)$ available for feedback.

In the absence of uncertainty and time-delay, the above formulation reduces to the optimal quadratic regulator problem (Anderson and Moore, 1990). Since uncertainties and time-delays are to be taken into account, the notions of *quadratic stability* and *guaranteed cost control* have to be considered. The following definitions are given.

Definition 2.1

The uncertain time-delay system (1)-(5) is *quadratically stabilizable independent of delay*, if there exists a static linear feedback control of the form of (7), a constant $\theta > 0$ and positive definite matrices $P \in \mathbf{R}^{n\times n}$, $R_1 \in \mathbf{R}^{n\times n}$ and $R_2 \in \mathbf{R}^{m\times m}$, such that the time derivative of the Lyapunov-Krasovskii functional

$$\mathcal{L}(x(t), t) = x^T(t)Px(t) + \int_{t-d_1(t)}^t x^T(\tau)R_1 x(\tau)d\tau + \int_{t-d_2(t)}^t u^T(\tau)R_2 u(\tau)d\tau \quad (8)$$

satisfies the condition

$$\dot{\mathcal{L}}(x(t), t) = \frac{d\mathcal{L}(x(t), t)}{dt} \leq -\theta \parallel x(t) \parallel^2 \quad (9)$$

along solutions $x(t)$ of (1) with $u(t) = Kx(t)$, for all $x(t)$ and for all admissible uncertainties and time-delays, i.e. consistent with (2), (3) and (5), respectively.
The resulting closed-loop system is called *quadratically stable* and $u(t) = Kx(t)$ is a *quadratically stabilizing control law*.

Definition 2.2

Given the uncertain time-delay system (1)-(5) with quadratic cost (6), a control law of the form of (7) is called a *guaranteed cost control*, if there exists a positive number $\mathcal{V}(x(0), \phi(-\bar{d}_1), \phi(-\bar{d}_2))$, such that

$$J(x(t), t) \leq \mathcal{V}(x(0), \phi(-\bar{d}_1), \phi(-\bar{d}_2)) \tag{10}$$

for all $x(t)$ and for all admissible uncertainties and time-delays. The upper bound $\mathcal{V}(.)$ is then called a *guaranteed cost*.

The following Proposition provides a sufficient condition for quadratic stability and guaranteed cost control.

Proposition 2.3

Consider the uncertain time-delay system (1)-(5) with quadratic cost (6). Let a control law of the form of (7) be such that the derivative of the Lyapunov-Krasovskii functional (8) satisfies the condition

$$\dot{\mathcal{L}}(x(t), t) \leq -x^T(t)[Q + K^T RK]x(t) \tag{11}$$

for all $x(t)$ and for all admissible uncertainties and time-delays. Then, (7) is a guaranteed cost control law and

$$\mathcal{V}(x(0), \phi(-\bar{d}_1), \phi(-\bar{d}_2)) = x^T(0)Px(0) + \int_{-\bar{d}_1}^{0} \phi^T(\tau)R_1\phi(\tau)d\tau + \int_{-\bar{d}_2}^{0} \phi^T(\tau)K^T R_2 K\phi(\tau)d\tau \tag{12}$$

is a guaranteed cost for (6). Moreover, the closed-loop system is quadratically stable.

Proof

By integrating both sides of (11) one obtains

$$\int_0^T \dot{\mathcal{L}}(x(t), t)dt \leq -\int_0^T [x^T(t)Qx(t)+u^T(t)Ru(t)]dt \tag{13}$$

and thus

$$\mathcal{L}(x(T), T) - \mathcal{L}(x(0), 0) = x^T(T)Px(T) + \int_{T-d_1(t)}^{T} x^T(\tau)R_1x(\tau)d\tau + \int_{T-d_2(t)}^{T} x^T(\tau)K^T R_2 Kx(\tau)d\tau - x^T(0)Px(0) - \int_{-d_1(t)}^{0} x^T(\tau)R_1x(\tau)d\tau - \int_{-d_2(t)}^{0} x^T(\tau)K^T R_2 Kx(\tau)d\tau \leq -\int_0^T [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \tag{14}$$

Since (11) is satisfied for $\mathcal{L} > 0$, it follows from Definition 2.1 that (9) is also satisfied for $\theta = \lambda_{min}(Q + K^T RK)$. Consequently $u(t) = Kx(t)$ is a quadratically stabilizing control law and thus $\mathcal{L}(x(t), t) \to 0$ as $T \to \infty$ along solutions $x(.)$ of system (1). Consequently, the above inequality yields

$$\int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \leq x^T(0)Px(0) + \int_{-\bar{d}_1}^{0} \phi^T(\tau)R_1\phi(\tau)d\tau + \int_{-\bar{d}_2}^{0} \phi^T(\tau)K^T R_2 K\phi(\tau)d\tau \tag{15}$$

and thus

$$J(x(t), t) \leq \mathcal{V}(x(0), \phi(-\bar{d}_1), \phi(-\bar{d}_2)) \tag{16}$$

for $\mathcal{V}(x(0), \phi(-\bar{d}_1), \phi(-\bar{d}_2))$ given by (12).

Remark 2.4

The condition resulting from Proposition 2.3 is a *delay-independent* condition. It is well known (Li and de Souza, 1997), (Kolmanovskii et al., 1999)that control laws arising from delay-independent conditions are likely to be conservative. Besides, quadratic stability and guaranteed cost approaches provide conservative solutions, as well. A means to reduce conservatism consists in minimizing the upper bound of the quadratic performance index by finding the appropriate guaranteed cost control law. Such a solution will be sought in the next Section.

3 SOLUTION IN THE LMI FRAMEWORK

In order to determine the unity rank uncertainty decomposition in an "optimal" way such that the guaranteed cost control law minimizes the corresponding guaranteed cost bound, an LMI objective minimization problem will be solved. The uncertainty decomposition (3) can be written as (Fishman et al., 1996),

$$\begin{array}{ll} A_{1i} = (s_{1i}^{1/2} d_{1i})(s_{1i}^{-1/2} e_{1i})^T & i = 1, ..., k_1 \\ A_{2i} = (s_{2i}^{1/2} d_{2i})(s_{2i}^{-1/2} e_{2i})^T & i = 1, ..., k_2 \\ B_{1i} = (t_{1i}^{1/2} f_{1i})(t_{1i}^{-1/2} g_{1i})^T & i = 1, ..., l_1 \\ B_{2i} = (t_{2i}^{1/2} f_{2i})(t_{2i}^{-1/2} g_{2i})^T & i = 1, ..., l_2 \end{array} \quad (17)$$

where s_{ij}, t_{ij} are positive scalars to be determined during the design procedure. For this purpose, diagonal positive definite matrices are defined,

$$\begin{array}{l} S_1 := diag(s_{11}, ..., s_{1k_1}) \\ S_2 := diag(s_{21}, ..., s_{2k_2}) \\ T_1 := diag(t_{11}, ..., t_{1k_1}) \\ T_2 := diag(t_{21}, ..., t_{2k_2}) \end{array} \quad (18)$$

The existence of a solution to the guaranteed cost control problem is obtained by solving an LMI feasibility problem. The following Theorem is presented:

Theorem 3.1

Consider the uncertain time-delay system (1)-(5) with quadratic cost (6). Suppose there exist positive definite matrices S_1, S_2, T_1, T_2, W, $\bar{R}_1$, $\bar{R}_2$, such that the LMI

$$\begin{array}{cccc} \Lambda(.) & WE_1 & WE_2 & B_1R^{-1}G_1 \\ E_1^T W & -S_1 & 0 & 0 \\ E_2^T W & 0 & -S_2 & 0 \\ G_1^T R^{-1} B_1^T & 0 & 0 & -T_1 \\ G_2^T R^{-1} B_1^T & 0 & 0 & 0 \\ R^{-1} B_1^T & 0 & 0 & 0 \\ W & 0 & 0 & 0 \\ W & 0 & 0 & 0 \end{array}$$

$$\begin{array}{cccc} B_1R^{-1}G_2 & B_1R^{-1} & W & W \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -T_2 & 0 & 0 & 0 \\ 0 & -\frac{1}{\beta_2}\bar{R}_2 & 0 & 0 \\ 0 & 0 & -\frac{1}{\beta_1}\bar{R}_1 & 0 \\ 0 & 0 & 0 & -\hat{Q} \end{array} < 0 \quad (19)$$

where

$$\Lambda(.) = A_1W + WA_1^T - B_1R^{-1}B_1^T + A_2A_2^T + B_2B_2^T + D_1S_1D_1^T + D_2S_2D_2^T + F_1T_1F_1^T + F_2T_2F_2^T + B_1R^{-1}R^{-1}B_1^T \quad (20)$$

and

$$\hat{Q} = (I_n + Q)^{-1} \quad (21)$$

$$\bar{R}_1 = R_1^{-1} \quad (22)$$

$$\bar{R}_2 = R_2^{-1} \quad (23)$$

admits a feasible solution (S_1, S_2, T_1, T_2, W, $\bar{R}_1$, $\bar{R}_2$). Then, the control law

$$u(t) = -R^{-1}B_1^T Px(t) \quad (24)$$

with

$$P := W^{-1} \quad (25)$$

is a guaranteed cost control law. The corresponding guaranteed cost is

$$\mathcal{V}(x(0), \phi(-\bar{d}_1), \phi(-\bar{d}_2)) = x^T(0)Px(0) + \int_{-\bar{d}_1}^{0} \phi^T(\tau)R_1\phi(\tau)d\tau + \int_{-\bar{d}_2}^{0} \phi^T(\tau)PB_1R^{-1}R_2R^{-1}B_1^T P\phi(\tau)d\tau \quad (26)$$

Proof

The time derivative of the Lyapunov-Krasovskii functional is

$$\dot{\mathcal{L}}(x(t), t) = 2x^T(t)P\dot{x}(t) + x^T(t)R_1x(t) - (1 - \dot{d}_1(t))x^T(t - d_1(t))R_1x(t - d_1(t)) + x^T(t)K^TR_2Kx(t) - (1 - \dot{d}_2(t)) x^T(t - d_2(t))K^TR_2Kx(t - d_2(t)) \quad (27)$$

By using (1), (5) and (17), one obtains the inequality

$$\dot{\mathcal{L}}(x(t), t) \leq 2x^T(t)P\{[A_1 + \Delta A_1(t)]x(t) + [A_2 + \Delta A_2(t)]x(t - d_1(t)) - [B_1 + \Delta B_1(t)]R^{-1}B_1^T Px(t) - [B_2 + \Delta B_2(t)]R^{-1}B_1^T Px(t - d_2(t))\} + x^T(t)R_1x(t) - (1 - \beta_1)x^T(t - d_1(t))R_1x(t - d_1(t)) + x^T(t)PB_1R^{-1}R_2R^{-1}B_1^T Px(t) - (1 - \beta_2) x^T(t - d_2(t))PB_1R^{-1}R_2R^{-1}B_1^T Px(t - d_2(t)) \quad (28)$$

which is to be verified for all $x(.) \in \mathbf{R}^n$. Furthermore, by using the identity $2 \mid ab \mid \leq a^2 + b^2$, for any a, b, $\in \mathbf{R}^n$, as well as (2)-(4), (17) and (18), the following quadratic upper bounding functions are derived,

$$2x^T(t)P\Delta A_1(t)x(t) \le \mid 2x^T(t)P\Delta A_1(t)x(t) \mid \le \sum_{i=1}^{k_1} \mid 2x^T(t)PA_{1i}r_{1i}(t)x(t) \mid \le \sum_{i=1}^{k_1} \mid 2x^T(t)P(d_{1i}s_{1i}^{1/2})(e_{1i}s_{1i}^{-1/2})^T x(t) \mid \le x^T(t)P\sum_{i=1}^{k_1} s_{1i}d_{1i}d_{1i}^T Px(t) + x^T(t)\sum_{i=1}^{k_1} s_{1i}^{-1}e_{1i}e_{1i}^T x(t) = x^T(t)PD_1S_1D_1^T Px(t) + x^T(t)E_1S_1^{-1}E_1^T x(t) \quad (29)$$

$\forall x(.) \in \mathbf{R}^n$. In a similar way one obtains

$$2x^T(t)PA_2x(t-d_1(t)) \le x^T(t)PA_2A_2^T Px(t) + x^T(t-d_1(t))x(t-d_1(t)) \quad (30)$$

$$2x^T(t)P\Delta A_2(t)x(t-d_1(t)) \le x^T(t)PD_2S_2D_2^T Px(t) + x^T(t-d_1(t))E_2S_2^{-1}E_2^T x(t-d_1(t)) \quad (31)$$

$$-2x^T(t)P\Delta B_1(t)R^{-1}B_1^T Px(t) \le x^T(t)PF_1T_1F_1^T Px(t) + x^T(t)PB_1R^{-1}G_1T_1^{-1}G_1^T R^{-1}B_1^T Px(t) \quad (32)$$

$$-2x^T(t)PB_2R^{-1}B_1^T Px(t-d_2(t)) \le x^T(t)PB_2R^{-1}B_2^T Px(t) + x^T(t-d_2(t))PB_1R^{-1}B_1^T Px(t-d_2(t)) \quad (33)$$

$$-2x^T(t)P\Delta B_2(t)R^{-1}B_1^T Px(t-d_2(t)) \le x^T(t)PF_2T_2F_2^T Px(t) + x^T(t-d_2(t))PB_1R^{-1}G_2T_2^{-1}G_2^T R^{-1}B_1^T P x(t-d_2(t)) \quad (34)$$

The above inequalities are true $\forall x(.) \in \mathbf{R}^n$. By using these quadratic upper bounding functions, it is straightforward to show that the guaranteed cost condition (11) is satisfied if

$$A_1 + A_1^T P - PB_1R^{-1}B_1^T P + Q + P[D_1S_1D_1^T + F_1T_1F_1^T + D_2S_2D_2^T + F_2T_2F_2^T + A_2A_2^T + B_2B_2^T]P + PB_1R^{-1}[G_1T_1^{-1}G_1^T + G_2T_2^{-1}G_2^T + I_m + \beta_2R_2]R^{-1}B_1^T P + I_n + E_1S_1^{-1}E_1^T + E_2S_2^{-1}E_2^T + \beta_1R_1 < 0 \quad (35)$$

By multiplying every term of the above matrix inequality on the left and on the right by $P^{-1} := W$ and by applying Shur complements, the LMI form (19) is obtained.

In the sequel, a minimum value of the guaranteed cost is sought, for an "optimal" choice of the rank-1 uncertainty decomposition. For this purpose, the minimization of the guaranteed cost is obtained by solving an objective minimization LMI problem.

Theorem 3.2

Consider the uncertain time-delay system (1)-(5) with quadratic cost (6). Suppose there exist positive definite matrices S_1, S_2, T_1, T_2, W, $\bar{R}_1$, $\bar{R}_2$, M_1, M_2, M_3, such that the following LMI objective minimization problem

$$min_{\mathcal{X}} \bar{J} = min_{\mathcal{X}}(Tr(M_1) + Tr(M_2) + Tr(M_3)) \quad (36)$$

$\mathcal{X} = (S_1, S_2, T_1, T_2, W, \bar{R}_1, \bar{R}_2, M_1, M_2, M_3)$, with LMI constraints (19) and

$$\begin{bmatrix} -M_1 & I_n \\ I_n & -W \end{bmatrix} < 0 \quad (37)$$

$$\begin{bmatrix} -M_2 & I_n \\ I_n & -\bar{R}_1 \end{bmatrix} < 0 \quad (38)$$

$$\begin{bmatrix} -M_3 & I_m \\ I_m & -\bar{R}_2 \end{bmatrix} < 0 \quad (39)$$

has a solution $\mathcal{X}(.)$. Then, the control law

$$u(t) = -R^{-1}B_1^T Px(t) \quad (40)$$

with

$$P := W^{-1} \quad (41)$$

is a guaranteed cost control law. The corresponding guaranteed cost

$$\mathcal{V}(x(0), \phi(-\bar{d}_1), \phi(-\bar{d}_2)) = x^T(0)Px(0) + \int_{-\bar{d}_1}^{0} \phi^T(\tau)R_1\phi(\tau)d\tau + \int_{-\bar{d}_2}^{0} \phi^T(\tau)PB_1R^{-1}R_2R^{-1}B_1^T P\phi(\tau)d\tau \quad (42)$$

is minimized over all possible solutions.

Proof

According to Theorem (3.1), the guaranteed cost (42) for the uncertain time-delay system (1)-(5) is ensured by any feasible solution $(S_1, S_2, T_1, T_2, W, \bar{R}_1, \bar{R}_2)$ of the convex set defined by (19). Furthermore, by taking the Shur complement of (37) one has $-M_1 + W^{-1} < 0 \Longrightarrow M_1 > W^{-1} = P \Longrightarrow Trace(M_1) >$

$Trace(P)$. Consequently, minimization of the trace of M_1 implies minimization of the trace of P. In a similar way it can be shown that minimization of the traces of M_2 and M_3 implies minimization of the traces of R_1 and R_2, respectively. Thus, minimization of $\bar{J}$ in (36) implies minimization of the guaranteed cost of the uncertain time-delay system (1)-(5) with performance index (6). The optimality of the solution of the optimization problem (36) follows from the convexity of the objective function and of the constraints.

4 EXAMPLE

Consider the second order system of the form of equation (1)with nominal matrices,

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

The uncertainty matrices on the state and control as well as on the delay matrices are according to (2),

$$\Delta A_1 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \Delta A_2 = \begin{bmatrix} 0 & 0 \\ 0.03 & 0.03 \end{bmatrix},$$
$$\Delta B_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \Delta B_2 = \begin{bmatrix} 0 \\ 0.03 \end{bmatrix}$$

and hence, the rank-1 decomposition may be chosen such that,

$$D_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0.03 \end{bmatrix}, E_1 = E_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, F_2 = \begin{bmatrix} 0 \\ 0.03 \end{bmatrix}, G_1 = G_2 = 1$$

The state and control weighting matrices are, respectively, $Q = I_2$, $R = 4$. It is $\beta_1 = 0.3$, $\beta_2 = 0.2$.
By solving the LMI objective minimization problem one obtains the guaranteed cost $\bar{J} = 25.1417$. The optimal rank-1 decomposition is obtained with $s_1 = 0.6664$, $s_2 = 2.2212$, $t_1 = 2.5000$, $t_2 = 8.3333$. Finally, the corresponding control gain is $K = [-0.2470 \; -6.0400]$.

5 CONCLUSIONS

In the previous sections, the problem of guaranteed cost control has been studied for the class of uncertain linear systems with state and input time varying delays. A constant gain linear state feedback control law has been obtained by solving an LMI feasibility problem. The closed-loop system is then quadratically stable and preserves acceptable performance for all parameter uncertainties and time varying delays of a given class. The system performance deviates from the optimal one, in the sense of LQR design of the nominal system, due to uncertainties and time delays. However, the performance deterioration is limited and this is expressed in terms of a performance upper bound, namely the guaranteed cost. In order to make the GC as small as possible, one has to solve an LMI minimization problem. The minimal upper bound corresponds to the ”optimal” rank-1 decomposition of the uncertainty matrices.

REFERENCES

Anderson, B. and Moore, J. (1990). *Optimal Control:Linear Quadratic Methods*. Prentice Hall, Englewood Cliffs, N.J., 2nd edition.

Barmish, B. (1985). Necessary and sufficient conditions for quadratic stabilizability of uncertain linear systems. *Journal of Optimization Theory and Application*, 46:399–408.

Barmish, B. (1994). *New tools for robustness of linear systems*. Macmillan, NY.

Bliman, P.-A. (2001). Solvability of a lyapunov equation for characterization of asymptotic stability of linear delay systems. In *Proceedings of the ECC ’01 European Control Conference,Porto, Portugal*, pages 3006–3011. ICEIS Press.

Boyd, S., Ghaoui, L. E., Feron, E., and Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*. SIAM Studies in Applied Mathematics, Philadelphia.

Chang, S. and Peng, T. (1972). Adaptive guaranteed cost control of systems with uncertain parameters. *I.E.E.E. Trans.on Automatic Control*.

Fishman, A., Dion, J.-M., Dugard, L., and Trofino-Neto, A. (1996). A linear matrix inequality approach for guaranteed cost control. In *Proceedings of the 13th IFAC World Congress, San Francisco,USA*.

Kim, J.-H. (2001). Delay and its time-derivative dependent robust stability of time-delayed linear systems with uncertainty. *I.E.E.E. Trans.on Automatic Control*, 46:789–792.

Kolmanovskii, V., Nikulescu, S., and Richard, J.-P. (1999). On the liapunov-krasovskii functionals for stability analysis oflinear delay systems. *International Journal of Control*, 72:374–384.

Kosmidou, O. and Bertrand, P. (1987). Robust controller design for systems with large parameter variations. *International Journal of Control*, 45:927–938.

Li, H., Niculescu, S.-I., Dugard, L., and Dion, J.-M., editors (1998). *Robust guaranteed cost control for uncertain linear time-delay systems*, pages 283–301. In

Stability and control of time-delay systems. Springer-Verlag, London.

Li, X. and de Souza, C. (1997). Delay-dependent robust stability and stabilization of uncertain linear delay systems: a linear matrix inequality approach. *IEEE Trans. on Automatic Control*, 42:1144–1148.

Mahmoud, M. (2000). *Robust control and filtering for time-delay systems*. Marcel Dekker, NY.

Malek-Zavarei, M. and Jamshidi, M. (1987). *Time-delay systems:analysis, optimization and applications*. North Holland, Amsterdam.

Niculescu, S. (2001). *Delay effects on stability*. Springer-Verlag, London.

Niculescu, S., Verriest, E., Dugard, L., and Dion, J.-M., editors (1998). *Stability and robust stability of time-delay systems: A guided tour*, pages 1–71. In Stability and control of time-delay systems. Springer-Verlag, London.

Richard, J.-P. (2003). Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39:1667–1694.

Tarbouriech, S. and da Silva Jr., J. G. (2000). Synthesis of controllers for continuous-time delay systems with saturating controls via lmi's. *IEEE Trans. on Automatic Control*, 45:105–111.