EXPLICIT PREDICTIVE CONTROL LAWS
On the Geometry of Feasible Domains and the Presence of Nonlinearities

Sorin Olaru, Didier Dumur
Automatic Control Department, Supélec, 3 rue Joliot Curie, Gif-sur-Yvette, France

Simona Dobre
CRAN, Nancy - Université, CNRS UMR 7039, BP 239, F-54506 Vandœuvre-lès-Nancy Cedex, France

Keywords: Predictive control, parameterized polyhedra, explicit control laws.

Abstract: This paper proposes a geometrical analysis of the polyhedral feasible domains for the predictive control laws under constraints. The fact that the system dynamics influence the topology of such polyhedral domains is well known from the studies dedicated to the feasibility of the control laws. Formally the system state acts as a vector of parameters for the optimization problem to be solved on-line and its influence can be fully described by the use of parameterized polyhedra and their dual constraints/generators representation.

Problems like the constraints redundancy or the construction of the associated explicit control laws at least for linear or quadratic cost functions can thus receive fully geometrical solutions. Convex nonlinear constraints can be approximated using a description based on the parameterized vertices. In the case of nonconvex regions the explicit solutions can be obtain by constructing Voronoi partitions based on a collection of points distributed over the borders of the feasible domain.

1 INTRODUCTION

The philosophy behind Model-based Predictive Control (MPC) is to exploit in a "receding horizon" manner the simplicity of the Euler-Lagrange approach for the optimal control. The control action u_t for a given state x_t is obtained from the control sequence $\mathbf{k}_u^* = [u_t^T, \ldots, u_{t+N-1}^T]^T$ as a result of the optimization problem:

$$\begin{array}{ll} \min\limits_{\mathbf{k}_u} & \varphi(x_{t+N}) + \sum\limits_{k=0}^{N-1} l(x_{t+k}, u_{t+k}) \\ \text{subj. to}: & x_{t+1} = f(x_t) + g(x_t)u_t; \\ & h(x_t, \mathbf{k}_u) \leq 0 \end{array} \quad (1)$$

constructed for a finite prediction horizon N, cost per stage $l(.)$, terminal weight $\varphi(.)$, the system dynamics described by $f(.), g(.)$ and the constraints written in a compact form using elementwise inequalities on functions linking the states and the control actions, $h(.)$.

Unfortunately, the control sequence $\mathbf{k}_u^*$ is optimal only for a single initial condition - x_t and produces an open-loop trajectory which contrasts with the need for a feedback control law. This drawback is overcome by solving the local optimization (1) for every encountered (measured) state, thus indirectly producing a state feedback law. The overall methodology is based on computationally tractable optimal control problems for the states found along the current trajectory. However, two important directions are to be studied in order to enlarge the class of systems which can take advantage of the MPC methodology. One is related to the fact that the measurements can be available faster than the optimal control sequence becomes available (as output of the optimization solver) and thus important information can be lost with irreversible consequences on the close-loop performances. Secondly, the lack of a closed form expression for the feedback law notifies about the difficulties that can be encountered when considering properties such as stability, typically established for regions in the state space.

For the optimization problem (1) within MPC, the current state serves as an initial condition and influences both the objective function and the feasible domain. Globally, from the optimization point of view, the system state can be interpreted as a vector of parameters, and the problems to be solved

Olaru S., Dumur D. and Dobre S. (2007).
EXPLICIT PREDICTIVE CONTROL LAWS - On the Geometry of Feasible Domains and the Presence of Nonlinearities.
In *Proceedings of the Fourth International Conference on Informatics in Control, Automation and Robotics*, pages 70-77
DOI: 10.5220/0001633200700077

are part of the multiparametric optimization programming family. From the cost function point of view, the parametrization is somehow easier to deal with and eventually can be entirely translated towards the set of constraints to be satisfied (the MPC literature contains references to schemes based on suboptimality or even to algorithms restraining the demands to feasible solution of the receding horizon optimization (Scokaert et al., 1999)). Unfortunately, similar observation cannot be made about the feasible domain and its adjustment with respect to the parameters evolution.

The optimal solution is often influenced by the limitations, the process being forced to operate at the designed constraints for best performance. The distortion of the feasible domain during the parameters evolution will consequently affect the structure of the optimal solution. Starting from this observation the present paper focuses on the topological analysis of the domains described by the MPC constraints.

The structure of the feasible domain is depending on the model and the set of constraints taken into consideration in (1). If the model is a linear system, the presence of linear constraints on inputs and states can be easily expressed by a system of linear inequalities. In the case of nonlinear systems, these properties are lost and the domains are in general difficult to handle. However, there are several approaches to transform the dynamics to those of a linear system over the operating range as for example by piecewise linear approximation, feedback linearisation or the use of time-varying linear models.

As a consequence, specific attention for the linear constraints and the associated polyhedral feasible domains may be prolific. More than that, the use of polyhedral domains between the convex sets is not hazardous since they offer important advantages, like the closeness over the intersection or the fact that the polyhedral invariant sets (largely used for enforcing stability) are less conservative than the ellipsoidal ones for example. In the current paper, these polyhedral feasible domains will be analyzed with a focus on the parametrization leading to the concept of parameterized polyhedra (Olaru and Dumur,):

$$\begin{array}{ll} \min\limits_{\mathbf{k}_u} & F(x_t, \mathbf{k}_u) \\ \text{subj. to}: & \left\{ \begin{array}{c} A_{in}\mathbf{k}_u \le b_{in} + B_{in}x_t \\ A_{eq}\mathbf{k}_u = b_{eq} + B_{eq}x_t \\ h(x_t, \mathbf{k}_u) \le 0 \end{array} \right. \end{array} \quad (2)$$

where the objective function $F(x_t, \mathbf{k}_u)$ is usually linear or quadratic.

Secondly it will be shown that the optimization problem may take advantage during the real-time implementation either from the possible alleviation of the set of constraints for the on-line optimization routines either from the construction of the explicit solution on geometrical basis when possible. With these two aspects, one can consider that MPC awareness is improved both from the theoretical (insight on the global control law) and practical (computational aspects) point of view.

An important remark concerning the presence of nonlinearities in the constraints is that if the feasible domain remains convex, then an approximation in terms of parameterized polyhedra can lead to an approximate explicit solution in terms of piecewise linear control laws. But, if the feasible domain becomes non-convex due to the presence of nonlinearities, then in order to obtain the explicit solution some assumption have to be relaxed. A special role, in finding the explicit control laws in the nonlinear case, is played by the Voronoi partition.

In the following, Section 2 introduces the basic concepts related to the parameterized polyhedra and interprets the feasible domains of the receding horizon optimization problems (2) in this context. Section 3 presents the use of the feasible domain analysis for the construction of the explicit solution for linear and quadratic objective functions. In Section 4 an extension to nonlinear type of constraints is addressed, simple examples illustrating the construction of explicit solutions.

2 PARAMETRIZATION OF POLYHEDRAL DOMAINS

2.1 Double Representation

A mixed system of linear equalities and inequalities defines a polyhedron (Motzkin and R.M., 1953). In the parameter free case, it is represented by the equivalent dual (Minkowski) formulation:

$$\begin{array}{l} \mathcal{P} = \left\{ \mathbf{k}_u \in \mathbb{R}^p \,\middle|\, A_{eq}\,\mathbf{k}_u = b_{eq}; A_{in}\mathbf{k}_u \le b_{in} \right\} \\ \Longleftrightarrow \mathcal{P} = \underbrace{conv.hull\mathbf{V} + cone\mathbf{R} + lin.space\mathbf{L}}_{generators} \end{array} \quad (3)$$

where $conv.hull\mathbf{V}$ denotes the set of convex combinations of vertices $\mathbf{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_\vartheta\}$, $cone\mathbf{R}$ denotes nonnegative combinations of unidirectional rays in $\mathbf{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_\rho\}$ and $lin.space\mathbf{L} = \{\mathbf{l}_1, \ldots, \mathbf{l}_\lambda\}$ represents a linear combination of bidirectional rays (with ϑ, ρ and λ the cardinals of the related sets). This dual representation (Schrijver, 1986) in terms of generators can be rewritten as:

$$\mathcal{P} = \left\{ \mathbf{k}_u \in \mathbb{R}^p | \mathbf{k}_u = \sum_{i=1}^{\vartheta} \alpha_i \mathbf{v}_i + \sum_{i=1}^{\rho} \beta_i \mathbf{r}_i + \sum_{i=1}^{\lambda} \gamma_i \mathbf{l}_i; \quad 0 \le \alpha_i \le 1,\ \sum_{i=1}^{\vartheta} \alpha_i = 1 \ , \beta_i \ge 0 \ , \forall \gamma_i \right\} \tag{4}$$

with α_i, β_i, γ_i the coefficients describing the convex, non-negative and linear combinations in (3).

Numerical methods like the Chernikova algorithm (Leverge, 1994) are implemented for constructing the double description, either starting from constraints (3) either from the generators (4) representation.

2.2 The Parametrization

A *parameterized polyhedron* (Loechner and Wilde, 1997) is defined in the implicit form by a finite number of inequalities and equalities with the note that the affine part depends linearly on a vector of parameters $x \in \mathbb{R}^n$ for both equalities and inequalities:

$$\begin{aligned} \mathcal{P}(x) &= \{ \mathbf{k}_u(x) \in \mathbb{R}^p \,| A_{eq}\, \mathbf{k}_u = B_{eq} x + b_{eq}; \\ &\qquad A_{in} \mathbf{k}_u \le B_{in} x + b_{in} \} \\ &= \left\{ \mathbf{k}_u(x) |\ \mathbf{k}_u(x) = \sum_{i=1}^{\vartheta} \alpha_i(x) \mathbf{v}_i(x) + \sum_{i=1}^{\rho} \beta_i \mathbf{r}_i + \sum_{i=1}^{\lambda} \gamma_i \mathbf{l}_i \right\} \\ & 0 \le \alpha_i(x) \le 1,\ \sum_{i=1}^{\vartheta} \alpha_i(x) = 1 \ , \beta_i \ge 0 \ , \forall \gamma_i. \end{aligned} \tag{5}$$

This dual representation of the parameterized polyhedral domain reveals the fact that only the vertices are concerned by the parametrization (resulting the so-called *parameterized vertices* - $\mathbf{v}_i(x)$), whereas the rays and the lines do not change with the parameters' variation. In order to effectively use the generators representation in (5), several aspects have to be clarified regarding the parametrization of the vertices (see for exemple (Loechner and Wilde, 1997) and the geometrical toolboxes like POLYLIB (Wilde, 1993)). The basic idea is to identify the parameterized polyhedron with a non-parameterized one in an augmented space:

$$\tilde{\mathcal{P}} = \left\{ \begin{bmatrix} \mathbf{k}_u \\ x \end{bmatrix} \in \mathbb{R}^{p+n} | \left[A_{eq} | - B_{eq} \right] \begin{bmatrix} \mathbf{k}_u \\ x \end{bmatrix} = b_{eq}; \quad [A_{in} | - B_{in}] \begin{bmatrix} \mathbf{k}_u \\ x \end{bmatrix} \le b_{in} \right\} \tag{6}$$

The original polyhedron in (5) can be found for any particular value of the parameters vector x through $P(x) = \mathrm{Proj}_{\mathbf{k}_u} \left(\tilde{P} \cap H(x) \right)$, for any given hyperplane $H(x_0) = \left\{ \begin{pmatrix} \mathbf{k}_u \\ x \end{pmatrix} \in \mathbb{R}^{p+n} | x = x_0 \right\}$ and using $\mathrm{Proj}_{\mathbf{k}_u}(.)$ as the projection from $\mathbb{R}^{p+n}$ to the first p coordinates $\mathbb{R}^p$.

Within the polyhedral domains $\tilde{\mathcal{P}}$, the correspondent of the parameterized vertices in (5) can be found among the faces of dimension n. After enumerating these n-faces: $\left\{ F_1^n(\tilde{\mathcal{P}}), \ldots F_j^n(\tilde{\mathcal{P}}), \ldots, F_\varsigma^n(\tilde{\mathcal{P}}) \right\}$, one can write: $\forall i, \exists j \in \{1, \ldots, \varsigma\}\, s.t. \left[\begin{array}{cc} \mathbf{v}_i(x)^T & x^T \end{array} \right]^T \in F_j^n(\tilde{\mathcal{P}})$ or equivalently:

$$\mathbf{v}_i(x) = \mathrm{Proj}_{\mathbf{k}_u} \left(F_j^n(\tilde{\mathcal{P}}) \cap H(x) \right) \tag{7}$$

From this relation it can be seen that not all the n-faces correspond to parameterized vertices. However it is still easy to identify those which can be ignored in the process of construction of parameterized vertices based on the relation: $\mathrm{Proj}_x \left(F_j^n(\tilde{P}) \right) < n$ with $\mathrm{Proj}_x(.)$ the projection from $\mathbb{R}^{p+n}$ to the last n coordinates $\mathbb{R}^n$ (corresponding to the parameters' space). Indeed the projections are to be computed for all the n-faces, those which are degenerated are to be discarded and all the others are stored as validity domains - $D_{\mathbf{v}_i} \in \mathbb{R}^n$, for the parameterized vertices that they are identifying:

$$D_{\mathbf{v}_i} = \mathrm{Proj}_n \left(F_j^n(\tilde{P}) \right) \tag{8}$$

Once the parameterized vertices identified and their validity domain stored, the dependence on the parameters vector can be found using the supporting hyperplanes for each n-face:

$$\mathbf{v}_i(x) = \begin{bmatrix} A_{eq} \\ \overline{A}_{in_j} \end{bmatrix}^{-1} \begin{bmatrix} B_{eq} \\ \overline{B}_{in_j} \end{bmatrix} x + \begin{bmatrix} b_{eq} \\ \overline{b}_{in_j} \end{bmatrix} \tag{9}$$

where $\overline{A}_{in_j}$, $\overline{B}_{in_j}$, $\overline{b}_{in_j}$ represent the subset of the inequalities, satisfied by saturation for $F_j^n(\tilde{P})$. The inversion is well defined as long as the faces with degenerate projections are discarded.

2.3 The Interpretation from the Predictive Control Point of View

The double representation of the parameterized polyhedra offers a complete description of the feasible domain for the predictive control law as long as this is based on a multiparametric optimization with linear constraints.

Using the generators representation, with simple difference operations on convex sets one can compute the region of the parameters space where no parameterized vertex is defined:

$$\aleph = \mathbb{R}^n \backslash \{ \cup D_{\mathbf{v}_i};\ i = 1 \ldots \vartheta \} \tag{10}$$

representing from the MPC point of view, the set of infeasible states for which no control sequence can be designed due to the fact that the limitations are overly constraining. As a consequence the complete description of the infeasibility is obtained.

Remark: the presence of rays and lines in the set of generators doesn't imply that the infeasibility is avoided. The feasibility is strictly related with the existence of valid parameterized vertices for the given value of the parameter (state) vector.

The vertices of the feasible domain cannot be expressed as convex combinations of other distinct points and, due to the fact that from the MPC point of view, they represent sequences of control actions, one can interpret them in terms of extremal performances of the controlled system (for example in the tracking applications the maximal/minimal admissible setpoint (Olaru and Dumur, 2005)).

3 TOWARDS EXPLICIT SOLUTIONS

In the case of sufficiently large memory resources, construction of the explicit solution for the multiparametric optimization problem can be an interesting alternative to the iterative optimization routines. In this direction recent results where presented at least for the case of linear and quadratic cost functions (see (Seron et al., 2003),(Bemporad et al., 2002),(Goodwin et al., 2004),(Borelli, 2003),(Tondel et al., 2003)). In the following it will be shown that a geometrical approach based on the parameterized polyhedra can bring a useful insight as well.

3.1 Linear Cost Function

The linear cost functions are extensively used in connection with model based predictive control and especially for robust case ((Bemporad et al., 2001), (Kerrigan and Maciejowski, 2004)). In a compact form, the multiparametric optimization problem is:

$$\begin{aligned} &\mathbf{k}_u{}^*(x_t) = \min_{\mathbf{k}_u} f^T \mathbf{k}_u \\ &\text{subject to } A_{in}\mathbf{k}_u \le B_{in}x_t + b_{in} \end{aligned} \tag{11}$$

The problem deals with a polyhedral feasible domain which can be described as previously in a double representation. Further the explicit solution can be constructed based on the relation between the parameterized vertices and the linear cost function (as in (Leverge, 1994)). The next result resumes this idea.

Proposition: The solution for a multiparametric linear problem is characterized as follows:

a) For the subdomain $\aleph \in \mathbb{R}^n$ where the associated parameterized polyhedron has no valid parameterized vertex the problem is infeasible;

b) If there exists a bidirectional ray $\mathbf{l}$ such that $f^T\mathbf{l} \neq 0$ or a unidirectional ray $\mathbf{r}$ such that $f^T\mathbf{r} \le 0$, then the minimum is unbounded;

c) If all bidirectional rays $\mathbf{l}$ are such that $f^T\mathbf{l} = 0$ and all unidirectional rays $\mathbf{r}$ are such that $f^T\mathbf{r} \ge 0$ then there exists a cutting of the parameters in zones where the parameterized polyhedron has a regular shape $\bigcup_{j=1\dots\rho} R_j = \mathbb{R}^n - \aleph$. For each region R_j the minimum is computed with respect to the given linear cost function and for all the valid parameterized vertices:

$$\underline{m}(x) = \min\left\{ f^T\mathbf{v}_i(x) | \mathbf{v}_i(x) \text{ vertex of } \mathcal{P}(x)\right\} \tag{12}$$

The minimum $\underline{m}(x)$ is attained by constant subsets of parameterized vertices of $\mathcal{P}(x)$ over a finite number of polyhedral zones in the parameters space R_{ij} ($\cup R_{ij} = R_j$). The complete optimal solution of the multiparametric optimization is given for each R_{ij} by:

$$\begin{aligned} S_{R_{ij}}(x) = &\ conv.hull\left\{\mathbf{v}_1^*(x), \dots, \mathbf{v}_s^*(x)\right\} + \\ &+ cone\left\{\mathbf{r}_1^*, \dots, \mathbf{r}_r^*\right\} + lin.spaceP(\mathbf{p}) \end{aligned} \tag{13}$$

where $\mathbf{v}_i^*$ are the vertices corresponding to the minimum $\underline{m}(x)$ over R_{ij} and $\mathbf{r}_i^*$ are such that $f^T\mathbf{r}_i^* = 0$□

This result provides *the entire family of solutions* for the linear multiparametric optimization, even for the cases where this family is not finite (for example there are several vertices attaining the minimum).

Remark: For the regions of the parameters space characterized by the case *(a)*, the set of constraints cannot be fulfilled and the feasible domain is empty.

Remark: If the solution of the optimization problem is characterized by the case *(b)*, then the control law based on such an optimization is not well-posed as the optimal control action needs an infinite energy in order to be effectively applied.

Remark: Due to the fact that the parameterized vertices have a linear dependence on the parameter vector, the explicit solution will be piecewise linear. However, the solution is not unique as it can be seen from the case *(c)* and equation (13) and thus for the practical control purposes a continuous piecewise candidate is preferred, eventually by minimizing the number of partitions in the parameters space.

3.2 Quadratic Cost Function

The case of a quadratic const function is one of the most popular at least for the linear MPC. The explicit solution based on the exploration of the parameters space ((Bemporad et al., 2002), (Borelli, 2003), (Tondel et al., 2003)) is extensively studied lately. Alternative methods based on geometrical arguments or dynamical programming ((Goodwin et al., 2004), (Seron et al., 2003)) improved also the awareness of the explicit MPC formulations. The parameterized polyhedra can serve as a base in the construction of such explicit solution (Olaru and Dumur,), for a quadratic multiparametric problem:

$$\begin{aligned} &\mathbf{k}_u{}^*(x_t) = \arg\min_{\mathbf{k}_u} \mathbf{k}_u^T H \mathbf{k}_u + 2\mathbf{k}_u{}^T F x_t \\ &\text{subject to } A_{in}\mathbf{k}_u \leq B_{in}x_t + b_{in} \end{aligned} \tag{14}$$

In this case the main idea is to consider the unconstrained optimum:

$$\mathbf{k}_u^{sc}(x_t) = H^{-1}Fx_t$$

and its position with respect to the feasible domain given by a parameterized polyhedron as in (5).

If a simple transformation is performed:

$$\tilde{\mathbf{k}}_u = H^{1/2}\mathbf{k}_u$$

then the isocost curves of the quadratic function are transformed from ellipsoid into circles centered in $\tilde{\mathbf{k}}_u^{sc}(x_t) = H^{-1/2}Fx_t$. Further one can use the Euclidean projection in order to retrieve the multiparametric quadratic explicit solution.

Indeed if the unconstrained optimum $\tilde{\mathbf{k}}_u^{sc}(x_t)$ is contained in the feasible domain $\tilde{\mathcal{P}}(x_t)$ then it is also the solution of the constrained case, otherwise existence and uniqueness are assured as follows:

Proposition: For any exterior point $\tilde{\mathbf{k}}_u(x_t) \notin \tilde{\mathcal{P}}(x_t)$, there exists an unique point characterized by a minimal distance with respect to $\tilde{\mathbf{k}}_u^{sc}(x_t)$. This point satisfies:

$$(\tilde{\mathbf{k}}_u^{sc}(x_t) - \tilde{\mathbf{k}}_u^*(x_t))^T(\tilde{\mathbf{k}}_u - \tilde{\mathbf{k}}_u^*(x_t)) \leqslant 0, \forall \tilde{\mathbf{k}}_u \in \tilde{\mathcal{P}}(x_t)\square$$

The construction mechanism uses the parameterized vertices in order to split the regions neighboring the feasible domain in zones characterized by the same type of projection.

Remark: The use of these geometrical arguments makes the construction of explicit solution to deal in a natural manner with the so-called *degeneracy* (Bemporad et al., 2002). This phenomenon is identified by the parameters' values where the feasible domain changes its shape (the set of parameterized vertices is modified).

4 GENERALIZATION FOR NONLINEAR PROGRAMS

If the feasible domain is described by a mixed linear/nonlinear set of constraints then the convexity properties are lost and a procedure for the construction of exact explicit solutions do not exist for the general case.

4.1 Nonlinear Constraints Handling

As already mentioned, in dealing with the presence of nonlinearities constraints, a special case is when the associated feasible domain is convex. In this case, the main ideas in finding the explicit control laws are the following:

- considering the augmented space (formed by the extended arguments and parameters space $((x, \mathbf{k}_u))$, find a set of points situated on the borders of the feasible domain (points that will correspond to the parameterized vertices in the associated linear feasible domain);

- using this set of extremal points, construct the dual representation in terms of parameterized polyhedra (as a fact, in the presence of linear constraints, this set of points could represent the input to the linear algorithm, as well as a set of linear constraints);

- build the corresponding explicit solution by reporting the unconstrained optimum to these parameterized vertices and their validity domains.

Remark: The solution will be a continuous piecewise affine function in the state vector, due to the nature (convexity) of the feasible domain, and it is obtained by projecting the unconstrained optimum on the linear subset of constraints (associated with the nonlinear ones). In this nonlinear convex case, the precision of the solution is directly dependent of the linearization of the nonlinear constraints using a finite set of points on the frontier of the feasible domain (knowing that the rest of the algorithm retains the qualities of the linear algorithm).

4.1.1 Simple Example of a Multiparametric Nonlinear Program

Consider the discrete-time linear system:

$$x_{t+1} = \begin{bmatrix} 0.9 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_t \tag{15}$$

and a predictive control law with a prediction horizon of three sampling times and a control horizon of two steps. A nonlinear set of constraints will be also considered:

$$\begin{cases} \sum_{k=0}^{2} u_{t+k}^2 \leq 1 \\ \sum_{k=0}^{2} u_{t+k}^2 \leq \ln(\begin{bmatrix} 0 & 1 \end{bmatrix} x_t + 1) \\ \begin{bmatrix} 0 & 1 \end{bmatrix} x_{t+k} \geq 0; k = 0,1,2 \end{cases} \tag{16}$$

It is obvious that the topology of the feasible domain is changing with the system dynamics, which means that the state vector represents in fact a parameter. More precisely, in our case only the second component of the state, x_t is influencing the shape of the feasible domain and thus one can draw this dependence on the parameter as in figure 1a. Further this parameterized convex shape can be approximated with a set of parameterized linear inequalities and obtain a double description of a parameterized polyhedron as in figure 1b. A precutting in zones with regular shape (figure 1c) can help in the development of explicit solution due to the important degree of redundancy.

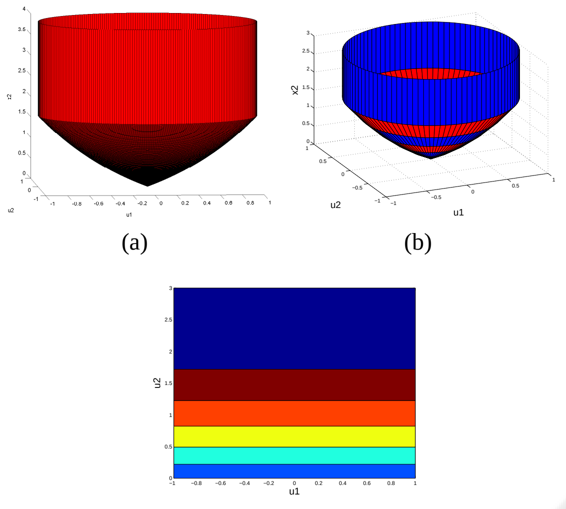

Figure 1: (a) The nonlinear dependence of the feasible domain on the parameters (b) The approximation by a parameterized polyhedron; (c) Regions in the parameters' space corresponding to redundancy-free constraints sets.

Finally the nonlinear MPC law for the system (15) and the constraints (16) can be approximated by the explicit solution found in terms of a piecewise linear control law as in figure 2.

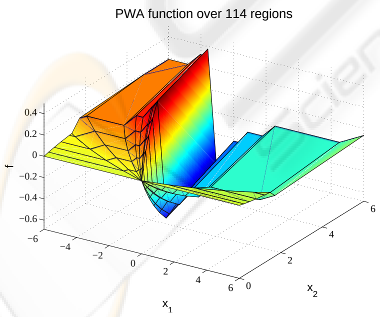

Figure 2: Explicit solution as a piecewise linear function.

4.2 Nonconvex Feasible Domains

In the case when the convexity is lost, one can still construct the convex hull (and an explicit solution) associated with the non-convex domain. From the resulting piecewise affine function one should retain only those regions where the control law is feasible w.r.t. the nonlinear constraints. For the violated constraints a distribution of points will be obtained. Finally the missing regions in the explicit solution will be completed with the Voronoi partition corresponding to these points on the concave border of the feasible domain.

Algorithm:

1. Obtain a set of points ($\mathcal{V}$) on the frontier of the feasible domain D (based on the nonlinear constraints).
2. Considering this set of points, construct the convex $\mathcal{C}_{\mathcal{V}}$ (which will include the non-convex domain D and some other infeasible domains from the MPC point of view).
3. Split the set $\mathcal{V}$ as $\overline{\mathcal{V}}_L \cup \overline{\mathcal{V}}_{NL} \cup \widetilde{\mathcal{V}} \cup \widehat{\mathcal{V}}$
   - $\widehat{\mathcal{V}} \in \mathfrak{Int}(\mathcal{C}_{\mathcal{V}})$
   - $\widetilde{\mathcal{V}} \in \mathfrak{F}(\mathcal{C}_{\mathcal{V}})$ and $\mathcal{C}_{\mathcal{V}} = \mathcal{C}_{\mathcal{V} \setminus \widetilde{\mathcal{V}}}$
   - $\mathcal{V}_L \in \mathfrak{F}(\mathcal{C}_{\mathcal{V}})$, $\mathcal{V}_L \cap \widetilde{\mathcal{V}} = \emptyset$ and $\mathcal{V}_L$ saturate at least one linear constraint in the MPC constraints;
   - $\mathcal{V}_{NL} \in \mathfrak{F}(\mathcal{C}_{\mathcal{V}})$ and $\mathcal{V}_{NL}$ saturate no linear constraint
4. $\mathcal{C}_{\mathcal{V}}$ is described in the dual representation by the intersection of halfspaces $\mathcal{H}$ (which represent the faces of the convex hull $\mathcal{C}_{\mathcal{V}}$). Split this set in $\overline{\mathcal{H}} \cup \widehat{\mathcal{H}}$
   - $\widehat{\mathcal{H}} \subset \mathcal{H}$ such that $\exists x \in \mathcal{C}_{\mathcal{V}}$ with $\mathfrak{Sat}(\widehat{\mathcal{H}}, x) \neq \emptyset$ and $\mathfrak{B}(\mathfrak{R}_{NL}, x) \neq \emptyset$
   - $\overline{\mathcal{H}} = \mathcal{H} \setminus \widehat{\mathcal{H}}$
5. Compute the unconstrained optimum $\mathbf{k}_u^*$
6. Project the unconstrained optimum on $\mathcal{C}_{\mathcal{V}}$:
$$\mathbf{k}_u^* \leftarrow Proj_{\mathcal{C}_{\mathcal{V}}} \{-c\}$$
7. If $\mathbf{k}_u^*$ saturates a subset of constraints $\mathcal{K} \subset \widehat{\mathcal{H}}$
   (a) Retain the set of points:
$$S = \Big\{ v \in \widehat{\mathcal{V}} \,|\, \forall x \in \mathcal{C}_{\mathcal{V}}\ s.t.\ \mathfrak{Sat}(\widehat{\mathcal{H}}, x) = \mathcal{K}; \\ \mathfrak{B}(\mathfrak{R}_{NL}, x) = \mathfrak{Sat}(\mathfrak{R}_{NL}, v) \}$$
   (b) Construct the Voronoi partition for the colection of points in S
   (c) Position $\mathbf{k}_u^*$ w.r.t. this partition and map the suboptimal solution $\mathbf{k}_u^* \leftarrow v$ where v is the vertex corresponding to the active region

8. If the quality of the solution is not satisfactory, improve the distribution of the points $\mathcal{V}$ by augmenting the resolution around $\mathbf{k}_u^*$ and restart from the step 2.

The following notations where used:

$\mathfrak{F}(X)$ The frontier of a compact set X

$\mathfrak{Int}(X)$ The interior of a compact set X

$\mathfrak{R}_L(D)$ The set of linear constraints in the definition of the feasible domain D

$\mathfrak{R}_{NL}(D)$ The set of nonlinear constraints in the definition of the feasible domain D

$\mathfrak{Sat}(\mathfrak{R}_*, x)$ The subset of constraints in $\mathfrak{R}_*$ (either $\mathfrak{R}_L$ either $\mathfrak{R}_{NL}$) saturated by the vector x

$\mathfrak{B}(\mathfrak{R}_*, x)$ The subset of constraints in $\mathfrak{R}_*$. violated by the vector x

4.2.1 Numerical Example

Consider the MPC problem implemented using the first control action of the optimal sequence:

$$k_u^* = \arg\min_{k_u} \sum_{i=0}^{N_y-1} x_{t+k|t}^T Q x_{t+k|t} + u_{t+k|t}^T R u_{t+k|t} + \\ + x_{t+N_y|t}^T P x_{t+N_y|t} \tag{17}$$

with

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}; R = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}; P = \begin{bmatrix} 13.73 & 2.46 \\ 2.46 & 2.99 \end{bmatrix}$$

subject to

$$\begin{cases} x_{t+k+1|t} = \overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}^{A} x_{t+k|t} + \overbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}^{B} u_{t+k|t} \ \ \mathrm{k} \geqslant 0 \\ \begin{bmatrix} -2 \\ -2 \end{bmatrix} \leqslant u_{t+k|t} \leqslant \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ 0 \leqslant \mathrm{k} \leqslant N_y - 1 \\ (u_{t+k|t}^1)^2 + \left(u_{t+k|t}^2 - 2\right)^2 \geqslant \sqrt{3} \ 0 \leqslant \mathrm{k} \leqslant N_y - 1 \\ (u_{t+k|t}^1)^2 + \left(u_{t+k|t}^2 + 2\right)^2 \geqslant \sqrt{3} \ 0 \leqslant \mathrm{k} \leqslant N_y - 1 \\ u_{t+k|t} = \underbrace{\begin{bmatrix} 0.59 & 0.76 \\ -0.42 & -0.16 \end{bmatrix}}_{K_{LQR}} x_{t+k|t} \ \ N_u \leqslant k \leqslant N_y - 1 \end{cases}$$

One can observe the presence of both linear and nonlinear constraints. By following the previous algorithm, in the first stage, the partition of the state space is performed by considering only the linear constraints (figure 3).

Each such region correspond with a specific projection law. By simply verifying the regions where this projection law obey the nonlinear constraints, the exact part of the explicit solution is obtained (fig. 4).

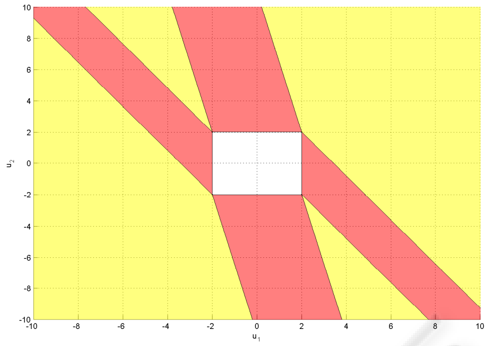

Figure 3: Partition of the arguments space (linear constraints only).

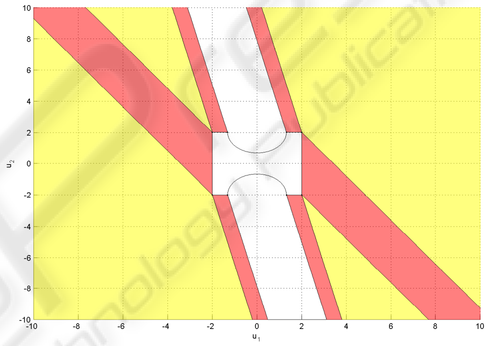

Figure 4: Retention of the regions with feasible linear projections.

Further, a distribution of points on the nonlinear frontier of the feasible domain has to be obtained and based on this distribution of points the associated Voronoi partition constructed. By superposing it to the regions not covered (white zones in the figure 4) at the previous step, one obtain a complete covering of the arguments space. Figure 5 depicts such a complete partition for distribution of 10 points for each nonlinear constraint.

By correspondence, the figure 6 describes the partition of the state space for the explicit solution.

Finally the complete explicit solution is depicted in figure 7. The discontinuities are observable in the regions generated upon the Voronoi partition. In order to give an image of the complexity it can mentioned that the explicit solutions contains 31 regions the computational effort was less than $2s$ mainly spent in the construction of the Voronoi partition.

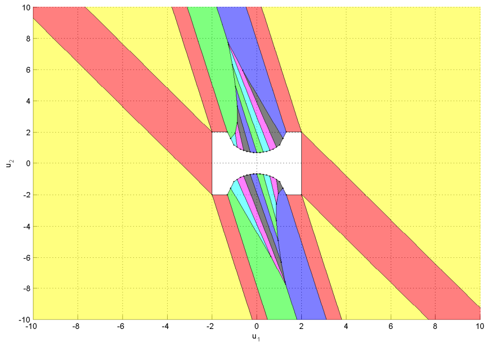

Figure 5: Partition of the arguments space (nonlinear case) - 10 points per nonlinear constraint.

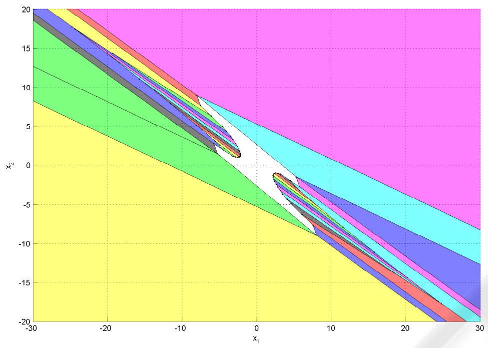

Figure 6: Partition of the state space - 10 points per nonlinear constraint.

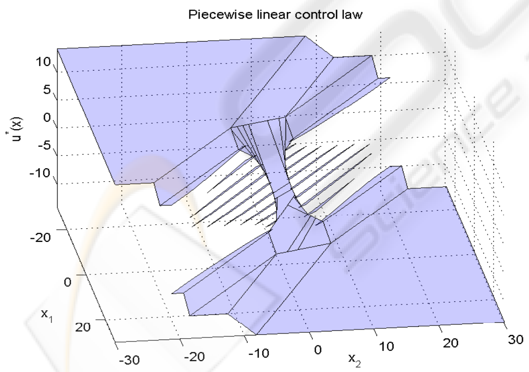

Figure 7: Explicit control law - 10 points per nonlinear constraint.

5 CONCLUSION

The parameterized polyhedra offer a transparent characterization of the MPC degrees of freedom. Once the complete description of the feasible domain as a parameterized polyhedron is obtained explicit MPC laws can be constructed using the projection of the unconstrained optimum. The topology of the feasible domain can lead to explicit solution even if nonlinear constraints are taken into consideration. The price to be paid is found in the degree of suboptimality.

REFERENCES

Bemporad, A., Borrelli, F., and Morari, M. (2001). Robust Model Predictive Control: Piecewise Linear Explicit Solution. In *European Control Conference*, pages 939–944, Porto, Portugal.

Bemporad, A., Morari, M., Dua, V., and Pistikopoulos, E. (2002). The Explicit Linear Quadratic Regulator for Constrained Systems. *Automatica*, 38(1):3–20.

Borelli, F. (2003). *Constrained Optimal Control of Linear and Hybrid Systems*. Springer-Verlag, Berlin.

Goodwin, G., Seron, M., and Dona, J. D. (2004). *Constrained Control and Estimation*. Springer, Berlin.

Kerrigan, E. and Maciejowski, J. (2004). Feedback min-max model predictive control using a single linear program: Robust stability and the explicit solution. *International Journal of Robust and Nonlinear Control*, 14(4):395–413.

Leverge, H. (1994). A note on chernikova's algorithm. In *Technical Report 635*. IRISA, France.

Loechner, V. and Wilde, D. K. (1997). Parameterized polyhedra and their vertices. *International Journal of Parallel Programming*, V25(6):525–549.

Motzkin, T.S., R. H. T. G. and R.M., T. (1953). *The Double Description Method, republished in* Theodore S. Motzkin: Selected Papers, *(1983)*. Birkhauser.

Olaru, S. and Dumur, D. (2005). Compact explicit mpc with guarantee of feasibility for tracking. In *44th IEEE Conference on Decision and Control, and European Control Conference. CDC-ECC '05.*, pages 969–974.

Olaru, S. B. and Dumur, D. A parameterized polyhedra approach for explicit constrained predictive control. In *43rd IEEE Conference on Decision and Control, 2004. CDC.*, pages 1580–1585 Vol.2.

Schrijver, A. (1986). *Theory of Linear and Integer Programming*. John Wiley and Sons, NY.

Scokaert, P. O., Mayne, D. Q., and Rawlings, J. B. (1999). Suboptimal model predictive control (feasibility implies stability). In *IEEE Transactions on Automatic Control*, volume 44, pages 648–654.

Seron, M., Goodwin, G., and Dona, J. D. (2003). Characterisation of receding horizon control for constrained linear systems. In *Asian Journal of Control*, volume 5, pages 271–286.

Tondel, P., Johansen, T., and Bemporad, A. (2003). Evaluation of piecewise affine control via binary search tree. *Automatica*, 39:945–950.

Wilde, D. (1993). A library for doing polyhedral operations. In *Technical report 785*. IRISA, France.