CYCLE TIME OF P-TIME EVENT GRAPHS

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Abstract: The dater equalities constitutes an appropriate tool which allows a linear description of Timed Event Graphs in the field of (max, +) algebra. This paper proposes an equivalent model in the usual algebra which can describe Timed and P-time Event Graphs. Considering 1-periodic behavior, the application of a variant of Farkas' lemma allows the determination of upper and lower bounds of the production rate and necessary conditions of consistency.

1 INTRODUCTION

Event Graphs are a subclass of Petri nets which can be used to model discrete event dynamic systems subject to saturation and synchronization phenomena, typically, transportation networks, multiprocessor systems and manufacturing systems. P-time Event Graphs are convenient tools to model systems whose operation times are included between a minimum and a maximum duration. Therefore, P-time Event Graphs can function at a maximal or a minimal speed and, average cycle time is one of the most important criteria which characterizes the system. An important result about Timed Event Graphs is that a Timed Event Graph reaches a periodic regime after a transient period (G. Cohen and Viot, 1983) (Chrétienne, 1985) in the earliest functioning mode (*i.e.*, transitions fire as soon as they are enabled). In this case, the trajectory is said K-periodic. More precisely, if $x(k)$ represents the date of firings of the transition x at the number of event k, then there is a constant λ (called the cycle time which is the inverse of the periodic throughput) and two integers k_0 in $\mathbb{N}$ and c in $\mathbb{N}^*$(called the cyclicity) such that

$$x(k+c) = x(k) + c \times \lambda \; for \; k \geq k_0$$

and

$$\lambda = \lim_{k\to\infty} \frac{x(k)}{k}$$

(Gaubert, 1995).

However, the periodical behavior is reached only after a transient that can be extremely long, moreover presence of perturbations (faults, maintenance operations,...) can limit the possibility of reaching a periodical behavior. The representativeness of the production rate can be reduced as the effectiveness of the approaches as resources optimization or control using transfert functions.

A possible approach is to generate periodic behaviors without transient period as 1-periodic behavior which is defined by

$$x(k+1) = x(k) + \lambda.$$

This technique assumes that each transition is structurally controllable (F. Baccelli, 1992).

Considering an 1-periodic behavior, the objective of the paper is the calculation of the average cycle time of P-time Event Graphs. The proposed approach introduces a new model based on "daters" in the Section 2. Defined by an inequality, the model completely describes in the usual algebra the trajectories of different Event Graphs as Timed Event Graphs or P-time Event Graphs.

Using a well-known Farkas'lemma of the linear programming (Schijver, 1987), the Sections 3 and 4 presents results about cycle time. Two examples are given in the Section 5 to illustrate the proposed method.

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2 MODEL

Definition 1 *A **Petri net** is a pair (G, M_0), where $G = (R, V)$ is a bipartite graph with a finite number of nodes (the set V) which are partitioned into the disjoint sets of places P and transitions T; R consists of pairs of the form (p_i,q_i) and (q_i,p_i) with $p_i \in P$ and $q_i \in T$. The initial marking M_0 is a vector of dimension $\mid P \mid$ whose elements denote the number of initial tokens in the respective places.*

Definition 2 *For a Petri Net with $\mid P \mid$ places and $\mid T \mid$ transitions, the **incidence matrix** $W = [W_{ij}]$ is an $\mid P \mid \times \mid T \mid$ matrix of integers and its typical entry is given by $W_{ij} = W_{ij}^+ - W_{ij}^-$ where W_{ij}^+ is the weight of the arc from transition j to its output place i and W_{ij}^- is the weight of the arc to transition j from its input place i.*

In a Petri net, from a marking M, a firing sequence implies a string of successive markings. The characteristic vector s of a firing sequence S is a vector for which each component is an integer corresponding to the number of firings of the corresponding transition. Then a marking M reached from M_0 by firing of a sequence S can be deduced using the fundamental relation:

$$M = M_0 + W \times s$$

where M_0 is the initial marking and W is the incidence matrix.

Definition 3 *A Petri net is called an **Event Graph** if each place has exactly one upstream and one downstream transition.*

P-time Petri nets allow the modeling of discrete event dynamic systems with sojourn time constraints of the tokens inside the places. Consistently with the dioid $\overline{\mathbb{R}}_{max}$ (see ((F. Baccelli, 1992))), we associate a temporal interval defined in $\mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$ for each place.

Definition 4 *A **P-time Event Graph** is a pair $< R, IS >$ where R is an Event Graph and the mapping IS: from P to $\mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$ is defined by $p_i \rightarrow [a_i, b_i]$ with $0 \leq a_i \leq b_i$.*

The interval $[a_i, b_i]$ is the static interval of duration time of a token in the place p_i belonging to the set of places P. The token must stay in the place p_i during the minimum residence duration a_i. Before this duration, the token is in a state of unavailability to fire the transition t_j. The value b_i is a maximum residence duration after which the token must leave the place p_i (and can contribute to the enabling of the downstream transitions). If not, the system falls into a token-dead state. So, the token is available to fire the transition t_j in the time interval $[a_i, b_i]$.

2.1 Preliminary Inequalities

For Event Graphs, let us express the firing interval for each transition of the system guaranteing the absence of token-dead states. The set ${}^\bullet p$ is the set of input transitions of p and $p^\bullet$ is the set of output transitions of p. The set ${}^\bullet t_i$ (respectively, $t_i^\bullet$) is the set of the input (respectively, output) places of the transition t_i. The set of upstream (respectively, downstream) transitions of t_i is denoted ${}^\leftarrow t_i = {}^\bullet\ ({}^\bullet t_i\ )$ (respectively, $t_i^\rightarrow = (\ t_i^\bullet)^\bullet$). The following assumption alleviates the notations. We suppose that for each pair of transitions (i, j), there is at the most a unique place denoted p_{ij} between the upstream transition $t_j \in^\bullet p$ and the downstream transition $t_i \in p^\bullet$. Each place p_{ij} is associated with an interval $[a_{ij}, b_{ij}]$, where a_{ij} is the lower bound and b_{ij} the upper bound .

We consider the "dater" type well-known in the (max, +) algebra: each variable $x_i(k)$ represents the date of the k^{th} firing of transition x_i. If we assume a FIFO functioning of the places which guarantees that the tokens do not overtake one another, a correct numbering of the events can be carried out. In this paper, we do not take the assumption of earliest (respectively, latest) functioning which will be the subject of other studies.

Therefore, the evolution can be described by the following inequalities expressing relations between the firing dates of transitions. An Event Graph can be considered as a set of subgraphs made up of a place p_{ij} linking the upstream transition j and the downstream transition i. We denote m_{ij} the corresponding initial marking or initial number of tokens.

For the lower bounds a_{ij} of the upstream place of transition i, we can write:

$\forall x_j \in^\leftarrow x_i,\ a_{ij} + x_j(k - m_{ij}) \leq x_i(k)$,

or equivalently,

$x_j(k - m_{ij}) - x_i(k) \leq -a_{ij}$.

The weight 1 of $x_j(k - m_{ij})$ (respectively, -1 of $x_i(k)$) is the weight of the entering arc of the place p_{ij}, from t_j to place p_{ij} (respectively, the outgoing arc of the place p_{ij}, from place p_{ij} to transition t_i) which is equal to W_{lj}^+ (respectively, $-W_{lj}^-$) if $p_l = p_{ij}$.

Respectively, for the upper bounds b_{ij} of the upstream place of transition i, we have:

$\forall x_j \in^\leftarrow x_i,\ x_i(k) \leq b_{ij} + x_j(k - m_{ij})$,

or equivalently,

$x_i(k) - x_j(k - m_{ij}) \leq b_{ij}$.

The weight 1 of $x_i(k)$ (respectively, -1 of $x_j(k - m_{ij})$) is the weight of the entering arc of the place p_{ij}, from t_j to place p_{ij} (respectively, the outgoing arc of the place p_{ij}, from place p_{ij} to transition t_i) which is equal to W_{lj}^+ (respectively, $-W_{li}^-$) if $p_l = p_{ij}$.

2.2 Matrix Expression

Let m be the maximum number of initial tokens, the set of the previous inequalities can be expressed as follows:

$$H = [H_m H_{m-1} H_{m-2} .... H_1 H_0]\times$$

$$\begin{pmatrix} x(k-m) \\ x(k-m+1) \\ .... \\ x(k-1) \\ x(k) \end{pmatrix} \leq \begin{pmatrix} -A \\ B \end{pmatrix}. \qquad (1)$$

The matrix H contains the weights of the arcs entering and outgoing of the places defined above. Each place p_l linking the upstream transition j and the downstream transition i corresponds to two rows of H and particularly, $-A$ and B are vector of temporizations where $A_l = a_{ij}$ and $B_l = b_{ij}$.

Now, we consider the matrix representation in different cases: the initial marking of all places is equal to zero; the initial marking of all places is equal to one; the general case. The two last cases will be considered in the following sections.

a) The initial marking of all places is null

The evolution can be described by the following inequalities expressing relations between the firing dates of transitions:

$$\begin{cases} x_j(k) - x_i(k) \leq -a_{ij} \\ -x_j(k) + x_i(k) \leq b_{ij} \end{cases}.$$

As $x(k)$ corresponds to firing sequence S, we can deduce from the above description on the weight of the arcs that there is a direct correspondance with the incidence matrix W. Therefore, one can write the system as follows:

$$H_0 \times x(k) \leq \begin{pmatrix} -A \\ B \end{pmatrix} \qquad (2)$$

where $H_0 = \begin{pmatrix} W \\ -W \end{pmatrix}$ and $W = W^+ - W^-$.

b) The initial marking of all places is equal to one

In this case, each place initially contains only one token. One can write:

$$\begin{cases} x_j(k-1) - x_i(k) \leq -a_{ij} \\ -x_j(k-1) + x_i(k) \leq b_{ij} \end{cases}.$$

As $x(k-1)$ and $x(k)$ respectively corresponds to firing sequence S, we can deduce from the above description on the weight of the arcs that respectively, there is a direct correspondance with the incidence matrices W^+ and $-W^-$. Therefore, one can write the system as follows:

$$\begin{pmatrix} H_1 & H_0 \end{pmatrix} \times \begin{pmatrix} x(k-1) \\ x(k) \end{pmatrix} \leq \begin{pmatrix} -A \\ B \end{pmatrix}$$

with $H_1 = \begin{pmatrix} W^+ \\ -W^+ \end{pmatrix}$ and $H_0 = \begin{pmatrix} -W^- \\ W^- \end{pmatrix}$.

c) General case

Now let us give the explicit form of the system (1) or in other words, the objective is to build an equivalent model such that each place of the new Event Graph contains only zero or one token. This new form will simplify the calculations of the cycle time.

As a place contains a maximum number of m tokens, the general idea is to split each place containing m tokens into m places, where each place contains only one token.

Let us introduce the variables $\alpha^{(m-j-1)}$ for $j = 0$ to $m-1$ in the inequations, we have:

$$\begin{pmatrix} x(k-m) \\ x(k-m+1) \\ .... \\ x(k-3) \\ x(k-2) \\ x(k-1) \\ x(k) \end{pmatrix} = \begin{pmatrix} \alpha^{(m-1)}(k-1) \\ \alpha^{(m-2)}(k-1) \\ .... \\ \alpha^{(2)}(k-1) \\ \alpha^{(1)}(k-1) \\ \alpha^{(0)}(k-1) \\ x(k) \end{pmatrix}$$

with

$$\begin{cases} \alpha^{(m-1)}(k) = x(k-m+1) = \alpha^{(m-2)}(k-1) \\ \alpha^{(m-2)}(k) = x(k-m+2) = \alpha^{(m-3)}(k-1) \\ \vdots \\ \alpha^{(2)}(k) = x(k-2) = \alpha^{(1)}(k-1) \\ \alpha^{(1)}(k) = x(k-1) = \alpha^{(0)}(k-1) \\ \alpha^{(0)}(k) = x(k) \end{cases}.$$

Or equivalently,

$$\begin{cases} \alpha^{(m-j-1)}(k) = x(k-m+j+1) = \alpha^{(m-j-2)}(k-1) \\ \text{for } j = 0 \text{ to } m-2 \\ \alpha^{(0)}(k) = x(k) \end{cases}.$$

The new state vector is:

$$X = (\alpha^{(m-1)}, \alpha^{(m-2)}, \alpha^{(m-3)}, ..., \alpha^{(2)}, \alpha^{(1)}, \alpha^{(0)})^t$$

and (1) becomes

$$H' \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq \begin{pmatrix} -A \\ B \end{pmatrix}$$

where H' contains H with the addition of null columns.

The system must be completed with $2(m-1)\times \mid T \mid$ relations in the worst case: for $j = 0$ to $m-2$,

$$\begin{cases} \alpha^{(m-j-2)}(k-1) - \alpha^{(m-j-1)}(k) \leq 0 \\ -\alpha^{(m-j-2)}(k-1) + \alpha^{(m-j-1)}(k) \leq 0 \end{cases}.$$

Therefore, one can write the system as follows:

$$\begin{pmatrix} G_1 & G_0 \end{pmatrix} \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $G_1 = \begin{pmatrix} G_{11} \\ -G_{11} \end{pmatrix}$ and $G_0 = \begin{pmatrix} -G_{21} \\ G_{21} \end{pmatrix}$.

The matrix G_{11} of dimension $((m-1)\times \mid T \mid \times m)$ as G_{21}, is an subdiagonal of identity matrices immediately above the main diagonal, while the matrix G_{21} is a diagonal of identity matrices.

Finally, we can write the algebraic form:

$$\begin{pmatrix} G \\ H' \end{pmatrix} \times \begin{pmatrix} X(k-1) \\ X(k) \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ -A \\ B \end{pmatrix}.$$

3 CYCLE TIME

The aim of this part is the determination of the existence of 1-periodic trajectory in P-time Event Graphs. Let us consider an Event Graph such that $m_{ij} = 0$ or 1.

$$H \times \begin{pmatrix} x(k) \\ x(k+1) \end{pmatrix} \leq \begin{pmatrix} -A \\ B \end{pmatrix} with H = \begin{pmatrix} H_{11} & H_{10} \\ H_{21} & H_{20} \end{pmatrix} \quad (3)$$

The 1-periodic behavior can be defined by $x(k+1) = \lambda \times u + x(k)$ with $u = (1, 1, ..., 1)^t$ and the average cycle time λ.

The following result will be useful.

Corollary 1 *Farkas' lemma (variant) Corollary 7.1.e in (Schijver, 1987) (Hennet, 1989).*

Let A be a matrix and let b a vector. Then the system $A \times x \leq b$ of linear inequalities has a solution x, if and only if $y \times b \geq 0$ for each row vector $y \geq 0$ with $y \times A = 0$

Theorem 1 *The system (3) can follow a 1-periodic behavior for a given cycle time λ, if and only if, for each row vector $y \geq 0$ with*

$$y \times \begin{pmatrix} H_{11} + H_{10} \\ H_{21} + H_{20} \end{pmatrix} = 0, \quad (4)$$

we have:

$$\frac{y \times \begin{pmatrix} -A \\ B \end{pmatrix}}{y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times u} \geq \lambda \quad (5)$$

if $y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times u > 0,$

$$\frac{y \times \begin{pmatrix} -A \\ B \end{pmatrix}}{y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times u} \leq \lambda \quad (6)$$

if $y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times u < 0,$

$$y \times \begin{pmatrix} -A \\ B \end{pmatrix} \geq 0 \quad (7)$$

if $y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times u = 0.$

Proof: We have

$$\begin{pmatrix} H_{11} & H_{10} \\ H_{21} & H_{20} \end{pmatrix} \times \begin{pmatrix} x(k) \\ \lambda \times u + x(k) \end{pmatrix} \leq \begin{pmatrix} -A \\ B \end{pmatrix}$$

i.e.,

$$\begin{cases} H_{11} \times x(k) + H_{10} \times (\lambda \times u + x(k)) \leq -A \\ H_{21} \times x(k) + H_{20} \times (\lambda \times u + x(k)) \leq B \end{cases}$$

i.e.,

$$\begin{cases} (H_{11} + H_{10}) \times x(k) \leq -A - H_{10} \times (\lambda \times u) \\ (H_{21} + H_{20}) \times x(k) \leq B - H_{20} \times (\lambda \times u) \end{cases}$$

or equivalently,

$$\begin{pmatrix} H_{11} + H_{10} \\ H_{21} + H_{20} \end{pmatrix} \times x(k) \leq \begin{pmatrix} -A \\ B \end{pmatrix} - \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times \lambda \times u. \quad (8)$$

From Farkas' lemma, we can deduce that the system (8) of linear inequalities has a solution x, if and only if $y \times (\begin{pmatrix} -A \\ B \end{pmatrix} - \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times \lambda \times u) \geq 0$ for each row vector $y \geq 0$ with $y \times \begin{pmatrix} H_{11} + H_{10} \\ H_{21} + H_{20} \end{pmatrix} = 0$.

So, $y \times \begin{pmatrix} -A \\ B \end{pmatrix} - y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times (\lambda \times u) \geq 0$

$y \times \begin{pmatrix} -A \\ B \end{pmatrix} \geq y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times (\lambda \times u) = \lambda \times y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times u.$

In this relation, the product by u gives the addition of all columns of $\begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix}$. From the sign of $y \times \begin{pmatrix} H_{10} \\ H_{20} \end{pmatrix} \times u$, the two cases (6)(5) and the relevant necessary and sufficient conditions of existence of x (7) for the system (8) can be deduced. ■

Let us note that the existence of a solution depends on λ in the two first relations contrary to the last one.

4 LINKS WITH OTHER RESULTS

We assume that $m_{ij} = 1$, which simplifies the presentation of the connections with notions of incidence matrix and P-semi flows. So, $H_{11} = W^+$, $H_{10} = -W^-$, $H_{21} = -H_{11}$ and $H_{20} = -H_{10}$. The previous theorem is now applied.

To summarize, for each row vector $y \geq 0$ with

$$y \times \begin{pmatrix} W \\ -W \end{pmatrix} = 0 \qquad (9)$$

- if $y \times \begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u > 0$ then

$$\frac{y \times \begin{pmatrix} -A \\ B \end{pmatrix}}{y \times \begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u} \geq \lambda, \qquad (10)$$

- if $y \times \begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u < 0$ then

$$\frac{y \times \begin{pmatrix} -A \\ B \end{pmatrix}}{y \times \begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u} \leq \lambda, \qquad (11)$$

- if $y \times \begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u = 0$ then

$$y \times \begin{pmatrix} -A \\ B \end{pmatrix} \geq 0. \qquad (12)$$

Moreover, we consider particular vectors y: The row-vector y can highlights the lower bounds of the temporizations A which correspond to a Timed Event Graph; The row-vector y can also highlight the upper bounds of the temporizations B. However, they give a rough estimate of λ which must be improved by considering the space of the orthogonal vectors y. Now, we successively consider the upper bounds B and the lower bounds A.

Upper bounds B

Let us consider a row-vector y such that the m first entries are null. It can be defined by the vector $y = (y_1, y_2)$ with $y_1 = 0$. From (9), we deduce that $y_2 \times W = 0$. So, $y_2 \times W^- \times u \geq 0$, then $\frac{y_2 \times B}{y_2 \times W^- \times u} \geq \lambda$.

Lower bounds A

Let us consider a row-vector y such that the m last entries are null. It can be defined by the vector $y = (y_1, y_2)$ with $y_2 = 0$. From (9), we deduce that $y_1 \times W = 0$. As $W^- \geq 0$, $y_1 \times (-W^-) \times u \leq 0$, then

$$\frac{y_1 \times (-A)}{y_1 \times (-W^-) \times u} = \frac{y_1 \times A}{y_1 \times W^- \times u} \leq \lambda. \qquad (13)$$

Calculation of the state

Considering any non-negative row vector y, the set of the relations defined by (11) (respectively, (10)) gives the lower bound (respectively, upper bound) of λ_1. Given an arbitrary cycle time λ_1 satisfying (11) and (10), the objective is the calculation of the date of firing of the transitions for a given k.

As $H_{11} = W^+$, $H_{10} = -W^-$, $H_{21} = -H_{11}$ and $H_{20} = -H_{10}$, from (8), $x(k)$ must satisfy
$\begin{pmatrix} W \\ -W \end{pmatrix} \times x(k) \leq \begin{pmatrix} -A \\ B \end{pmatrix} - \begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times \lambda_1 \times u$.

This inequality follows the general form $A \times x \leq B$ which can be solved by the Fourier-Motzkin algorithm.

4.1 Link with Karp's Theorem

The following well-known result is based on circuits (Gaubert, 1995).

Theorem 2 *(Karp's theorem)*
In a strongly connected system, the minimal cycle time can be defined by the maximum of the ratio of the sum of the delays to the sum of tokens, for each elementary circuit C_k, i.e.,

$$minimal\ cycle\ time\ = \max_k \left(\frac{\textit{sum of delays in } C_k}{\textit{sum of tokens in } C_k}\right).$$

Let us now consider (13). Its numerator $y_1 \times A$ is a sum of durations as $y_1 > 0$ which is the total delay in C_k.

Consider the denominator of (13): $y_1 \times W^- \times u$.

As each row of W^- contains a unique entrie different from zero which can be associated with the unique token of the relevant place, the right-multiplication by u generates a column-vector $v = (1, 1, ..., 1)^t$ whose dimension is m and which is the initial marking M_0. So, the denominator $y_1 \times W^- \times u$ is equal to $y_1 \times M_0$ which is the number of tokens in C_k at M_0. Therefore, there is a correspondance between (13) and the expression of the theorem of Karp.

Strictly speaking, the Karp's theorem can be apply even if the behavior of the graph is not 1-periodic as we suppose here.

4.2 Link with (Murata, 1989)

Another result can be found in ((Murata, 1989)). If we model a Timed Petri Net which is consistent (*i.e.,* $\exists\ x > 0$, $W.x = 0$) by assigning delay d_i to each place p_i, then it can be shown that the minimal cycle time is given by:

$\underset{k}{max}(\frac{y_k.D.W^+.x}{y_k.M_0})$

where y_k is the P-semi flow k and D is the diagonal matrix of d_i,$i = 1, 2, .., m$.

So, $W^+.x = v$ and $y_k.D.W^+.x = y_k.A$ which is the numerator of (13).

5 EXAMPLES

5.1 First Example

Let us consider a simple example based on two elementary strongly connected subgraphs.

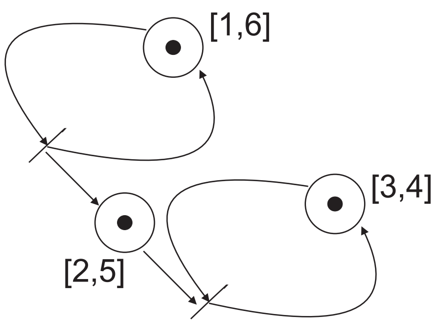

Figure 1: A simple P-time Event Graph.

$$\begin{pmatrix} W^+ & -W^- \\ -W^+ & W^- \end{pmatrix} . \begin{pmatrix} x(k-1) \\ x(k) \end{pmatrix} \leq \begin{pmatrix} -A \\ B \end{pmatrix}$$

with $x(k) = \begin{pmatrix} x_1(k) & x_2(k) & x_3(k) \end{pmatrix}^t$, $W^+ = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, $-W^- = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$, $-A = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$ and $B = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$. We have

$$W = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

A possible integer matrix $Y \geq 0$ such that $Y.\begin{pmatrix} W \\ -W \end{pmatrix} = 0$ is as follows. $Y = \begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 7 & 0 & 0 & 7 & 0 \end{pmatrix}$

$\begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u = \begin{pmatrix} -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}^t$

$Y.\begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u = \begin{pmatrix} -7 & -7 & +7 & +7 & 0 \end{pmatrix}^t$

$Y \times \begin{pmatrix} -A \\ B \end{pmatrix} = \begin{pmatrix} -7 & -21 & +42 & +28 & +21 \end{pmatrix}^t$.

The two first terms lead to lower bounds ($\frac{-7}{-7} = 1$, $\frac{-21}{-7} = 3$), the two successive terms gives the upper bounds ($\frac{+42}{+7} = 6$, $\frac{+28}{+7} = 4$) and the last one is a condition of consistency ($+21 \geq 0$).

Therefore, the 1-periodic trajectory exists with $\max(1, 3) = 3 \leq \lambda \leq \min(6, 4) = 4$.

For $\lambda = 3$, a possible trajectory is $\binom{1}{0} \to \binom{4}{3} \to \binom{7}{6} \to ...$

For $\lambda = 3.5$, a possible trajectory is $\binom{1.5}{0} \to \binom{5}{3.5} \to \binom{8.5}{7} \to ...$

For $\lambda = 4$, a possible trajectory is $\binom{2}{0} \to \binom{6}{4} \to \binom{10}{8} \to ...$

5.2 Second Example

Now, we consider a P-time Event Graph without directed circuit.

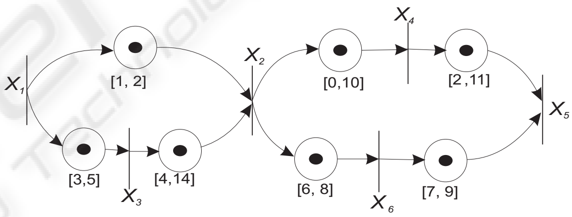

Figure 2: A P-time Event Graph.

$$\begin{pmatrix} W^+ & -W^- \\ -W^+ & W^- \end{pmatrix} . \begin{pmatrix} x(k-1) \\ x(k) \end{pmatrix} \leq \begin{pmatrix} -A \\ B \end{pmatrix}$$

with $x(k)$ $\begin{pmatrix} x_1(k) & x_2(k) & x_3(k) & x_4(k) & x_5(k) & x_6(k) \end{pmatrix}^t$,

$W^+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $W^- = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, $-A = \begin{pmatrix} -1 \\ -3 \\ -4 \\ 0 \\ -6 \\ -2 \\ -7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ 5 \\ 14 \\ 10 \\ 8 \\ 11 \\ 9 \end{pmatrix}$

$$W = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

A possible integer matrix $Y \geq 0$ such that $Y.\begin{pmatrix} W \\ -W \end{pmatrix} = 0$ is as follows. $Y =$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}$$

$\begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u =$
$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$

$Y.\begin{pmatrix} -W^- \\ W^- \end{pmatrix} \times u =$
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & 0 & 0 \end{pmatrix}^t$

$Y \times \begin{pmatrix} -A \\ B \end{pmatrix} =$
$\begin{pmatrix} +1 & +2 & +10 & +10 & +2 & +9 & +2 & +18 & -5 & +15 & +10 \end{pmatrix}^t$

The 9^{nd} term leads to the lower bound ($\frac{-5}{-1} = 5$), the 8^{nd} term gives the upper bound ($\frac{+18}{+1} = 18$) and the last one are conditions of consistency which are satisfied.

Therefore, the 1-periodic trajectory exists with $5 \leq \lambda \leq 18$

For $\lambda = 5$, a possible trajectory is $\begin{pmatrix} 3 & 0 & 1 & 5 & 4 & 1 \end{pmatrix}^t \rightarrow \begin{pmatrix} 8 & 5 & 6 & 10 & 9 & 6 \end{pmatrix}^t \rightarrow \begin{pmatrix} 13 & 10 & 11 & 15 & 14 & 11 \end{pmatrix}^t \rightarrow \ldots$

6 CONCLUSION

Using a new incidence matrix, the model we propose allows the counting of the events in Timed and P-time Event Graphs. The connections with usual incidence matrix has been realized. Considering 1-periodic behavior, the application of a variant of Farkas' lemma leads to the introduction of a generalization of the P-semi flow vectors for Timed and P-time Event Graphs, and allows the determination of upper and lower bounds of the possible cycle time. Each limit is respectively a complex function of lower and upper bounds of the temporizations. Moreover, even if cycle time λ belongs to this interval, the system must also satisfy conditions of consistency such that the finite initial dates of firing exist. With the restriction that a 1-periodic behavior has been considered, the proposed lower bound of the cycle time includes the Karp's relation.

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