INCLUSION OF ELLIPSOIDS

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Abstract: We present, in this paper, a ready-to-use inclusion detection for ellipsoids. Ellipsoids are used to represent configuration uncertainty of a mobile robile. This kind of test is used in path planning to find the optimal path according to a safety criteria.

1 INTRODUCTION

Nowadays, path planning for mobile robots has taken a new dimension. Due to many failures when experimenting the following of geometrical paths, determined by the first generation of planners (Latombe, 1991), with real robots, searchers concluded that those too simple paths were no longer enough. Planning method must now guarantee that the paths proposed are safe, i.e. that the robot will be able to follow a path without any risk of failure, or at least in warranting a high success rate. To achieve this goal, some parameters must be considered : uncertainties of the model used (not-so-perfect mapping, inaccuracy of the sensors, slipping of the robot on the floor, etc.).

Collision detection is very important in mobile robotic and furthermore when trying to find a safe path in an uncertain-configuration space. Thus, searchers tend to integrate evolved collision detection in their planners. Thus, after having used circular disk to approximate the shape taken by the mobile robot, more and more searchers use elliptic disks as they offer a better accuracy.

Used in the context of *safe* path planning (Pierre and Lambert, 2006; Lozano-Pérez and Wesley, 1979; Gonzalez and Stentz, 2005; Pepy and Lambert, 2006), ellipsoids allow to approximate the shape of the set of positions where the mobile *could be* (ellipsoids thus take the mobile robot's geometry and the uncertainties on its position into account). The the Safe A* with Towers of Uncertainty (SATU*) planner (Pierre and Lambert, 2006) is one of those safe path planner that use ellipsoid to approximate the shape of the set of positions where the mobile could be. Ellipsoids are used in the SATU* to perform collision detection between the mobile robot and its environment. However, the authors of the SATU* have also proposed a new mean of organising the performing of the planner so that the ellipsoids can be used to detect very early beginning of useless paths. In order to achieve this goal, inclusion detection must be performed between two ellipsoids (that correspond to two different ways to come to the same position). In this paper, we are going to present an algebraic method using the resultant of Sylvester (Lang, 1984) to solve this problem. The SATU* algorithm (Alg. 1) has already been presented in (Pierre and Lambert, 2006) and three tests of inclusion of uncertainties (lines 7, 26 and 36) are used. However the authors did not explain how they implemented those tests nor give the algorithms used. As the model of uncertainties used in the SATU* corresponds to an ellipsoid in 3 dimensions, this test of inclusion of uncertainties can be seen as a test of inclusion of ellipsoids.

In the present paper, we are going to propose an algorithm of test of inclusion of ellipsoids. In a first part, the uncertain configuration space will be described. Then, Sylvester's resultant will be used to defined a ready for use inclusion detection test.

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Algorithm 1 (SATU*).

```
CLOSE ← ∅
OPEN ← NodStart
while OPEN≠ ∅ do
  Nod ← Shortest_f*_Path(OPEN)
  CLOSE ← CLOSE + Nod
  OPEN ← OPEN - Nod
  if Base(Nod) = Base(NodGoal) and uncertainty(Nod)
  ⊆ uncertainty(NodGoal) then
    RETURN Success
  end if
  NEWNODES ← Successors(Nod)
  for all NewNod of NEWNODES do
    if NewNod ∉ OPEN,CLOSE then
      g(NewNod) ← g(Nod)+cost(Nod,NewNod)
      f*(NewNod) ←
      g(NewNod)+h(NewNod,NodGoal)
      build(NEWTOWER,base(NewNod))
      AddLevel(NEWTOWER,NewNod)
      OPEN ← OPEN+NewNod
      parent(NewNod) ← Nod
    else
      TOWER ← ExtractTower(base(NewNod))
      level ← -1
      repeat
        AddLevel ← false
        level ← level+1
        LevelNod ← ExtractNode(TOWER,level)
        if (g(NewNod) ≥ g(LevelNod) and uncer-
        tainty(LevelNod) ⊈ uncertainty(NewNod)) or
        g(NewNod) < g(LevelNod) then
          AddLevel ← true
        end if
      until level ≠ TopLevel(TOWER) and Ad-
      dLevel=true
      if AddLevel=true then
        level ← insert(NewNod,TOWER)
        OPEN ← OPEN + NewNod
        parent(NewNod) ← Nod
        UpperNods ←
        nodes(UpperLevels(TOWER,level))
        for all uppernod of UpperNods do
          if uncertainty(NewNod) ⊆ uncer-
          tainty(uppernod) then
            remove(TOWER,uppernod)
            OPEN ← OPEN - uppernod
          end if
        end for
      end if
    end if
  end for
end while
RETURN NoSolution
```

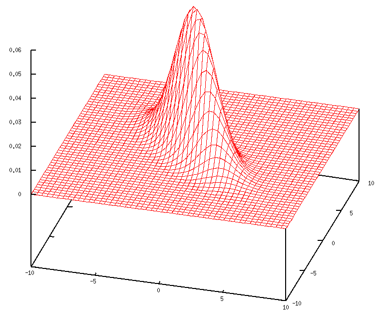

Figure 1: Example of $p_{(x,y)}.\ (x,y)$.

2 SHAPE OF UNCERTAINTIES

SATU* uses the Kalman filter (Kalman, 1960) to determine the uncertainties associated with the estimated positions of the mobile robot. (Smith and Cheeseman, 1986) gave a method to interpret the results of this filter. Indeed, thanks to this filter, we know at each time the covariance matrix $\mathbf{C}$ representing the uncertainties on the position $\mathbf{x}$ of the mobile robot. As we are only interested in the position of the mobile robot (in x and y), our random vector $\mathbf{x}$ varies in $\mathbb{R}^2$. As the Kalman filter uses only gaussian random vectors, the density of probability $p_{\mathbf{x}}(\mathbf{x})$ of the vector $\mathbf{x}$ is

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det \mathbf{C}}} e^{-\frac{1}{2}(\mathbf{x}-\hat{\mathbf{x}})^T \mathbf{C}^{-1}(\mathbf{x}-\hat{\mathbf{x}})},$$

where $\hat{\mathbf{x}}$ represents the estimated state vector.

We know the covariance matrix $\mathbf{C}$, which is given by

$$\mathbf{C} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_y\sigma_x \\ \rho\sigma_y\sigma_x & \sigma_y^2 \end{bmatrix}, \quad (1)$$

where σ_x^2 represents the variance of x, σ_y^2 the variance of y and ρ is the correlation coefficient of x and y.

In translating the coordinate frame to the known point $\hat{\mathbf{x}}$, the probability density function is

$$p_{x,y}(x,y) = \frac{1}{2\pi\sqrt{\det \mathbf{C}}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x^2} + \frac{2\rho xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right]}. \quad (2)$$

An example of drawing of this probability density function is given figure 1.

Our goal is to determine the set of positions where the mobile robot can be for a given confidence threshold p. This corresponds in finding an areo of isodensity contours such that

$$(\mathbf{x}-\hat{\mathbf{x}})^T \mathbf{C}^{-1}(\mathbf{x}-\hat{\mathbf{x}}) = k^2,$$

where k is a constant. The relation between k and p is given by $k = \sqrt{-2\ln(1-p)}$.

Thus, we can determine the equation of the ellipsoid (centered on the known point $\hat{\mathbf{x}}$) where the robot mobile may be with the given probability p:

$$Ax^2 + Bxy + Cy^2 = k^2, \tag{3}$$

with

$$\begin{cases} A &= \frac{1}{(1-\rho^2)\sigma_x^2}, \\ B &= \frac{2\rho}{(1-\rho^2)\sigma_x\sigma_y}, \\ C &= \frac{1}{(1-\rho^2)\sigma_y^2}. \end{cases} \tag{4}$$

This equation describes an ellipsoid. A classical result allows us to write this equation in function of the parameters a, b and φ of the ellipsoid. Considering that a is the half major axis of the ellipsoid, b its half minor axis and φ its orientation, we have

$$\begin{cases} \varphi &= \frac{1}{2}\arctan\left(\frac{B}{A-C}\right), \\ a &= \sqrt{\frac{2k^2}{A+C-\sqrt{A^2+C^2-2AC+B^2}}}, \\ b &= \sqrt{\frac{2k^2}{A+C+\sqrt{A^2+C^2-2AC+B^2}}}. \end{cases} \tag{5}$$

Using (4) and (5), we can determine the parameters of the ellipsoid in function of the covariance matrix given by the EKF.

We are going to work with this ellipsoid in order to find a test of inclusion of ellipsoids. Then, a simple test on σ_θ^2 will allow us to complete the test of inclusion of 3-D ellipsoids.

3 DEFINITIONS AND NOTATIONS

The equation of an ellipsoid E_i with a half major axis a_i and a half minor axis $b_i < a_i$, centered on the origin of the frame of reference and with no orientation is given by

$$\frac{x'^2}{a_i^2} + \frac{y'^2}{b_i^2} = 1, \tag{6}$$

considering x' and y' the coordinates of the ellipsoid.

To determine the equation of the centered ellipsoid of orientation φ_i, the use of a simple rotation matrix of angle $-\varphi_i$ is enough. The new coordinates of the ellipsoid are named x and y. Equation (6) then becomes

$$A_i x^2 + B_i y^2 + C_i xy - 1 = 0 \tag{7}$$

where

$$\begin{cases} A_i &= \frac{(b_i\cos\varphi_i)^2 + (a_i\sin\varphi_i)^2}{(a_i b_i)^2} \\ B_i &= \frac{(b_i\sin\varphi_i)^2 + (a_i\cos\varphi_i)^2}{(a_i b_i)^2} \\ C_i &= \frac{\left(b_i^2 - a_i^2\right)\sin(2\varphi_i)}{(a_i b_i)^2}. \end{cases} \tag{8}$$

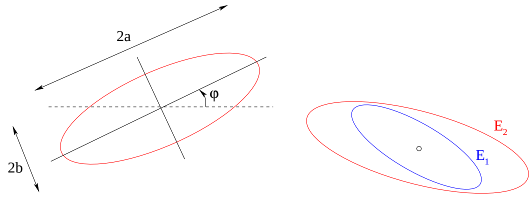

Figure 2: Ellipsoid's parameters.

It represents the equation of a centered ellipsoid of half major axis a_i, half minor axis b_i, and of orientation φ_i (figure 2). We will call this ellipsoid $E_i(a_i, b_i, \varphi_i)$.

In this paper, we will use two ellipsoids $E_1(a_1, b_1, \varphi_1)$ and $E_2(a_2, b_2, \varphi_2)$.

4 INCLUSION TEST

To ensure an ellipsoid E_1 is included in an ellipsoid E_2(see figure 2), we just need to perform two tests. We must check there is no intersection between E_1 and E_2 and then that E_1 lies within E_2.

4.1 Sylvester's Resultant

Checking if two ellipsoids intersect is the same as finding the tuples (x, y), which are solutions of both E_1 and E_2. The use of the resultant allows to find those tuples.

Definition Let $P = u_0 X^p + \cdots + u_p$ and $Q = v_0 X^q + \cdots + v_q$ be two polynoms of degrees $p > 0$ and $q > 0$ respectively. The *resultant* of two polynomials P and Q is the determinant of the *Sylvester's matrix* of P and Q, which is the square matrix of size $p + q$:

$$\begin{pmatrix} u_0 & u_1 & \cdots & \cdots & \cdots & u_p & 0 & \cdots & 0 \\ 0 & u_0 & u_1 & \cdots & \cdots & \cdots & u_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_0 & u_1 & \cdots & \cdots & \cdots & u_p \\ v_0 & v_1 & \cdots & \cdots & v_q & 0 & \cdots & \cdots & 0 \\ 0 & v_0 & v_1 & \cdots & \cdots & v_q & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & v_0 & v_1 & \cdots & \cdots & v_q \end{pmatrix}.$$

Theorem Let $P = u_0 X^p + \cdots + u_p$ and $Q = v_0 X^q + \cdots + v_q$ be two polynomials with their coefficient in a field $\mathbb{K}$, so that $u_0 \neq 0$ and $v_0 \neq 0$. Let $\overline{\mathbb{K}}$ be an algebraically closed field[1] containing $\mathbb{K}$. The two following properties are equivalent:

1. The polynomials P and Q have a common root in $\overline{\mathbb{K}}$.
2. P and Q's resultant is null.

The demonstration is given in (Lang, 1984).

4.2 Intersection Test

Theorem 4.1 indicates that two polynomials have at least one solution in the algebraic closure of the field in which the coefficients are defined if the resultant of those polynomials is nil. As the equations of the ellipsoids are polynomials (bilinear), we can use the resultant to determine the intersection point of the ellipsoids. The coefficients of the polynomials (ellipsoids) are defined on $\mathbb{R}$, the use of the Sylvester's resultant gives the common roots of the polynomial in an algebraically closed field containing $\mathbb{R}$, for example $\mathbb{C}$. We will then need to check if the roots are in $\mathbb{R}$, as we just need *real* intersections. To define the Sylvester's matrix, we need to rewrite the equation of the ellipsoid in function of only one parameter. Let's choose (randomly) x.

$$(7) \Leftrightarrow Ax^2 + (Cy)x + By^2 - 1 = 0,$$

which gives, for the equations of E_1 and E_2:

$$\begin{cases} A_1x^2 + (C_1y)x + B_1y^2 - 1 &= 0 \\ A_2x^2 + (C_2y)x + B_2y^2 - 1 &= 0. \end{cases} \quad (9)$$

Sylvester's matrix of the polynomials is then:

$$S = \begin{pmatrix} A_1 & C_1y & B_1y^2-1 & 0 \\ 0 & A_1 & C_1y & B_1y^2-1 \\ A_2 & C_2y & B_2y^2-1 & 0 \\ 0 & A_2 & C_2y & B_2y^2-1 \end{pmatrix}.$$

The determinant of S is

$$|S| = py^4 + qy^2 + r, \quad (10)$$

with

$$\begin{aligned} p &= -2A_1A_2B_1B_2 + (A_1B_2 - A_2B_1)C_1C_2 \\ &\quad + A_1B_1C_2^2 + A_2B_2C_1^2 + A_1^2B_2^2 + A_2^2B_1^2 \\ q &= 2A_1A_2(B_1+B_2) + (A_2 - A_1)C_1C_2 \\ &\quad - 2A_1^2B_2 - 2A_2^2B_1 - A_1C_2^2 - A_2C_1^2 \\ r &= (A_2 - A_1)^2. \end{aligned}$$

The change of variables $Y = y^2$ allows us to rewrite the determinant:

$$|S| = pY^2 + qY + r.$$

Algorithm 2 ComputeEquation(a,b,φ).

1: $A \leftarrow ((b\cos\varphi)^2 + (a\sin\varphi)^2)/(ab)^2$
2: $B \leftarrow ((b\sin\varphi)^2 + (a\cos\varphi)^2)/(ab)^2$
3: $C \leftarrow ((b^2 - a^2)\sin(2\varphi))/(ab)^2$
4: RETURN (A,B,C)

Algorithm 3 Determinant$(A_1,B_1,C_1,A_2,B_2,C_2)$.

1: $p \leftarrow -2A_1A_2B_1B_2 + (A_1B_2 - A_2B_1)C_1C_2 + A_1B_1C_2^2 + A_2B_2C_1^2 + A_1^2B_2^2 + A_2^2B_1^2$
2: $q \leftarrow 2A_1A_2(B_1+B_2) + (A_2 - A_1)C_1C_2 - 2A_1^2B_2 - 2A_2^2B_1 - A_1C_2^2 - A_2C_1^2$
3: $r \leftarrow (A_2 - A_1)^2$
4: RETURN (p,q,r)

We are looking for the roots of $|S|$ in $\mathbb{R}$. If the discriminant of the resultant (which is $q^2 - 4rp$) is strictly negative, there is no roots in $\mathbb{R}$. If r is nil, we can directly conclude that there is at least one intersection point because there exists at least one real root ($y = 0$).

If the discriminant of $|S|$ is positive or nil, there is at least one root in $\mathbb{R}$. We conclude that the ellipsoids have at least one intersection point in $\mathbb{C}$ (y may be real, x could be complex). We then need to calculate the values of y that gives an intersection point, then check that the corresponding values of x are ni $\mathbb{R}$ too. Roots of 10 are given by:

$$\begin{cases} y_{1,2} &= \pm\frac{-q-\sqrt{\Delta}}{2p} \\ y_{3,4} &= \pm\frac{-q+\sqrt{\Delta}}{2p} \end{cases} \quad (11)$$

Using 11, (9) becomes:

$$(A_1 - A_2)x^2 + y(C_1 - C_2)x + y^2(B_1 - B_2) = 0, \quad (12)$$

which discriminant is

$$\Delta = y^2(C_1 - C_2)^2 - 4y^2(A_1 - A_2)(B_1 - B_2). \quad (13)$$

If $\Delta \geq 0$, then equation (12) has real roots. We conclude that the global system as a tuple (x,y) of real solutions. Thus, ellipsoids have at least one intersection point. On the contrary, if equation (12) has no real root in x, then the ellipsoids do not intersect.

4.3 Test of Length

If the ellipsoids do not intersect, then it means that one of them is fully included in the other (as they have the same center). To test if it is the first ellipsoid that is included in the second, we just need to check that the half major axis of the first ellipsoid, a_1, is smaller

[1]A field $\overline{\mathbb{K}}$ is algebraically closed if every polynomial of $\overline{\mathbb{K}}[X]$ is split on $\overline{\mathbb{K}}$, i.e. product of polynomials of degree 1.

Algorithm 4 ComputeRoots(p,q,Δ).

$y1 \leftarrow (-q-\sqrt{\Delta})/(2p)$
$y3 \leftarrow (-q+\sqrt{\Delta})/(2p)$
RETURN $(y_1,-y_1,y_3,-y_3)$

Algorithm 5 InclusionTest$(a_1,b_1,\varphi_1,a_2,b_2,\varphi_2)$.

$(A_1,B_1,C_1) \leftarrow$ ComputeEquation(a_1,b_1,φ_1)
$(A_2,B_2,C_2) \leftarrow$ ComputeEquation(a_2,b_2,φ_2)
$(p,q,r) \leftarrow$ Determinant$(A_1,B_1,C_1,A_2,B_2,C_2)$
$\Delta \leftarrow q^2-4rp$
if $\Delta \geq 0$ **then**
  $(y_1,y_2,y_3,y_4) \leftarrow$ ComputeRoots(p,q,Δ)
  for all $y \in \{y_1,y_2,y_3,y_4\}$ **do**
    $\Delta_y \leftarrow y^2\,(C_1-C_2)^2-4y^2\,(A_1-A_2)\,(B_1-B_2)$
    if $\Delta_y \geq 0$ **then**
      RETURN NO INCLUSION
    end if
  end for
end if
if $(a_1 \geq a_2)$ or $(b_1 \geq b_2)$ **then**
  RETURN NO INCLUSION
end if
RETURN INCLUSION

than the half major axis of the second ellipsoid, a_2. We can, following the same scheme, check that the half minor axis of the first ellipsoid b_1 is smaller than the half minor axis of the second ellipsoid, b_2.

4.4 Algorithm

The complete algorithm of inclusion of two ellipsoids which have the same center is given alg. 5. Subfunctions ComputeEquation (Alg. 2), Determinant (Alg. 3) and ComputeRoots (Alg. 4) refer respectively to the equations (8), (10) and (11).

5 CONCLUSION

In this article, we presented a method of test of inclusion of ellipsoids when those ellipsoids have the same center. This test of inclusion is necessary in the use of some path planners such as the SATU* planner (Pierre and Lambert, 2006). The proposed method is very easily implementable on computer and have a constant and low complexity.

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