SPATIAL RANK AND APPROXIMATE SYMMETRIES IN SEQUENTIAL RECONSTRUCTION OF DENSE PACKINGS

Alexander Vinogradov
Dorodnitsyn Computing Center of Russian Academy of Sciences, Moscow, Russia

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Abstract: General rotation group manifold is used as a base structure for representation of k-point configuration clusters in Hough-type parametric space. This yields to introduce efficiently spatial ranks inside k-point trial set and arrange in multiple dimensions Parzen-like windows with properties analogous to the linear ones'. As a result, asymptotically optimal dense packings of clusters are automatically produced for arbitrary spatial shapes via independent sequential trials.

1 INTRODUCTION

Hough transform is widely used as a powerful technique in various IP applications. (Hough P.V.C., 1962). Recently it appears in many forms, generalized versions, and specialized hard and soft implements (Chen L.1 et al, 2004, Daries C.J. and Nixon M.S., 1998, Monari J. et al, 2006). Essential reduction of dimensionality as main advantage is achieved on this way when more or less complicated spatial shapes are transformed to point-wise clusters in appropriate parametric space (Leavers V.F., 1992, Torii A. et al, 2005, Zhang S.-C., Liu Z.-Q., 2005).

Single point is really good representation in many respects, but it implies a sort of indifference to all movements inside the shape and usually ignores initial feature distribution on it, in particular. In this paper we develop an approach to preserve this information and propose a method for its efficient usage in the case when the set of all inside-shape movements forms a group structure. We describe an application of the approach to one of the typical clustering tasks.

2 GROUP STRUCTURES OF HOUGH TRANSFORM

In its simplest authentic form the Hough transform already uses some group properties of spatial data (Figure 1). Really, population of any Hough cluster representing a line L is indifferent with respect to operations of one-parametric translation group of this line $T_1(L)$. Similarly, clusters of double Hough transform, representing concentric rings, are indifferent with respect to operations of one-parametric central rotation group SO_2. In the case of generalized 4-parametric Hough transform an indifference of this type takes place on spheres with respect to SO_3-movements produced by three independent generators, and so on.

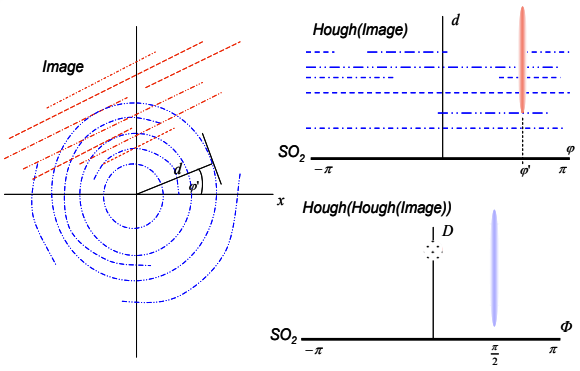

Figure 1: Authentic Hough and double Hough transforms.

An important object appears after first step of double Hough transform in the second example, i.e., a line clusters resulting from rings of initial *Image* plane. Any line cluster of this type preserves empiric density distribution of detected line details of the ring, but the distribution is represented now on the

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angle φ scale. Moreover, since zero rotation angle is defined and thus fixes the unit of Lie group SO_2, this scale turns out as corresponding one-dimensional group manifold itself. The subject matter of the paper is an implementation of the idea that initial density distribution of characteristic features revealed on symmetric spatial shape could be preserved on group manifold in a form of frequencies, with which elements of the group are used in the transform. Below, we describe details of the formal construction along with investigation of some special task of spatial clustering, where representation of such type is essentially needful.

3 DENSE PACKINGS

We consider a class of dense packings with coefficient 1, in which the shapes of elements may change while all the elements have the same volume. Examples in dimensions 2 and 3 are given by the decomposition of asymmetric region into compact domains of equal volume and the arrangement of elastic reservoirs with identical filling in a bounded volume of a space. For different ways of defining admissible shapes of elements (constraints on the linear dimensions and surface area, elasticity, internal potentials; etc.), the analysis of the variants of dense packings turns out to be related to the solution of complex optimization problems. The goal is to find ways to reduce the computational complexity of problems of this type by using the extremal properties of the process of sequential random choice in which a sample $X \subset R^N$ is tested by small sub-samples. Below, we describe the application of a special sequential trial scheme in which the problems of enumeration of close-to-optimal variants of the arrangement of clusters, finding approximate symmetries, as own discrete symmetries of packings as hidden internal symmetries of the domain X, and sequential filtering of optimal solutions and exact symmetries among them are considered from a unified point of view.

Suppose that $X \subset R^N$ is a bounded domain, the space (X,σ,μ) is an a priori distribution in X, and F is a functional that defines the type of a K-cluster packing $O=\{O(x_k),\ k=1,...,K\}$ in X,

$$F(O) \rightarrow max \qquad (1)$$

For nondegenerate distributions (X,σ,μ) with density p_μ, a pair (X,F) defines a certain set of variants of optimal packing, i.e., a certain set of solutions to problem (1) of the form

$$O^* = argmax\ F(O) = \{O^*(x^*_k)\}, \quad \bigcup O(x_i)=X. \qquad (2)$$

If it is uniquely specified how the centers are ranked, then each set (2) defines a point in the space R^{NK} that describes the arrangement of the centers of optimal clusters in R^N

$$\chi = (x^*_{1,...,} x^*_K) \in R^{NK}. \qquad (3)$$

We will refer to the sets $\chi \in R^{NK}$ as *configurations* and the sequential acts of choosing configurations as *trials*.

We will seek a solution to the problem of enumerating various kinds of packing in the class of algorithms requiring constant resources for computation in all trials. The example is the computation of centers as cluster averages, where the means can be permanently refined by the same recurrence formula

$$M\bar{x}_M + x = (M+1)\bar{x}_{M+1}. \qquad (4)$$

Each portion of M_0 elements extracted from X considered as the general population reflects the form of the distribution (X,σ,μ). In one-dimensional case M_0+1 linear blocks known as "Parzen windows" (Parzen E., 1962) are widely used in nonparametric density estimations. Applying certain specialization of F for finite sets, one can translate this approximation into a K-point representation $\chi_0 \in Y_0(M_0)$) whatever the volume of the portion M_0. If we follow the model chosen, then, in order to combine particular solutions of the form χ_0 into the summarized result, we should apply the same procedure in all trials carried out in a fixed space of memory, which includes an array for storing this summarized result. Here, we apply the following standard procedure:

(a) choose a sufficiently large number of initial trials M_1;

(b) construct an appropriate set $Y_1(M_0)$ of particular configurations of the form $\chi_0 \in Y_0(M_0)$;

(c) analyze $Y_1(M_0)$ and single out from R^{NK} a base set of clusters and the corresponding set $Y^*_1(M_0)$ of their central elements $\chi^* \in Y^*_1(M_0)$ by using certain functional F_1 in R^{NK};

(d) fill the base clusters with the results of further trials so that the next configuration χ is related to the nearest center of the base cluster as a template by a certain measure of proximity or the metric $\rho(\chi, \chi')$. To avoid pathological situations, we assume that the proximity is consistent with the usual Euclidean metric: $\rho(\chi, \chi') \rightarrow 0$, as $||\chi - \chi'|| \rightarrow 0$ in R^{NK};

(e) the solution being refined will correspond to the filling levels of clusters as neighborhoods of the central elements χ^*.

There is extremely efficient implementation of outlined scheme in one dimension that is based on good asymptotic behavior of rank statistics. Temporal means for ranks are normalized sums of IID values and thus are consistent estimates for

bounding points of Parzen windows. This estimates can be calculated by the formula (4); their a priory distributions are rapidly normalized in sequential trials; and corresponding variance decreases with the rate $D=1/O(M)$ as $M\to\infty$. So, the sample X composes "by itself" the optimal packing via always the same process and regardless to the forms of the sample X and initial distribution (X,σ,μ).

In multiple dimensions the scheme looses many of these attractive properties. The main reason is the luck of immediate analogues of the ranks in view of the absence of a certain linear ordering in the sample space (Vinogradov A.P., 2001). If elements of the problem posing are insensitive or invariant with respect to the action of some symmetry group of the sample, regardless exact or approximate, it's impossible to establish a doubtless correspondence between ranks in different trials, and thus it's a problem to arrange the addition (4) correctly. In the same time, if symmetry G is exact, i.e. probability space (X,σ,μ) is G-invariant, any action of $g\in G$ results in a metric isomorphism between measurable spaces (X,σ,μ) and $g(X,\sigma,\mu)$, and we can use fiducial equivalents of ranks. So, we can postulate as basic some G-invariant method of ranking configurations in accordance with (1) and thus factorize the entire sequential scheme by the action of G. In the case of approximate symmetry the same is valid for some equivariant conditional distribution on X.

4 GENERALIZED HOUGH PARAMETERIZATION FOR CENTRAL SYMMETRIES

Here we take into consideration only the simplest linear groups of central symmetry. To efficiently look for exact and approximate symmetries we use a special parametric space S of Hough type built of $|Y^*_1(M_0)|$ copies of SO_N manifold, where centers $j\in Y^*_1(M_0)$ of configuration clusters (c) are used as templates of ranking (Figure 2). This type of parameterization is applicable in more general case too, when N exceeds 3. If spatial isomers are treated as separate templates in each case when $N>2$, then the whole sequential scheme is automatically factorized also with respect to the Weyl group of R^N.

It was shown that for any particular symmetry group $G_1\subset SO_N$ the orbit of each base configuration fills a manifold of an adjacent G_1-class (Vinogradov A.P., Voracek J., Zhuravlev Yu.I., 2006). Absolutely asymmetric solutions appear as salient domains in S. If an essential approximate symmetry takes place, the filling of the adjacent class must be continuous; for exact symmetry it must be either uniform. For a ranking template $j\in Y^*_1(M_0)$ and a number m_i of its realizations in trials performed the population of the G_1-orbit represents instant sample distribution that approximates associated fiducial distribution on the scale of parameter $g\in G_1$. In particular, Figure 2 represents the scale space for G_1-orbit sample being refined in trials, where number of realizations m_i (of symmetry G_1 for ranking template j) serves as semi-group parameter

$$U(\Sigma m_i,p(m_1,K_1))=U(m_{max},\dots,U(m_2,p(m_1,K_1)\dots). \quad (5)$$

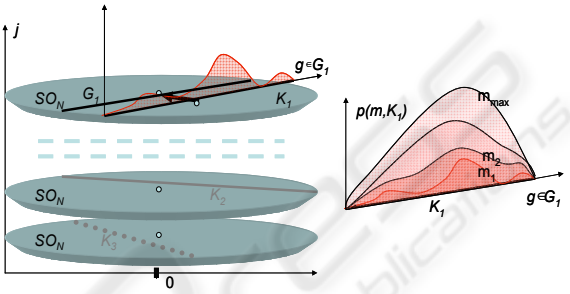

Figure 2: Hough-type parameterization, left. A scale space for samples refined on the G_1-orbit manifold K_1, right. K_3 depicts some dotted manifold of a discrete symmetry. 0 points the SO_N unit.

One can watch the process of filling orbits in S and test their continuity or uniformity. The process of filling corresponding clusters chosen on the step (e) in configuration space R^{NK} still obeys the IID condition mentioned above. It was shown, that configuration clusters in R^{NK} can be processed with basic set-theoretical operations without the lost of this condition.

In the same time, if the size of the portion M_0 is small, then the deviations of particular solutions from the optimal one may be significant; the clusters may be hardly separable; and the set $Y^*_1(M_0)$, which represents the entire set of noticeable variants, may be too large. It is impossible to change the situation by increasing the parameter M_1, although the primary structure of similarity (b), which is strongly determined by the size of the portion M_0, will be displayed with increasing accuracy as $M\to\infty$.

The situation can be changed by increasing the parameter M_0: in more representative samples, foreign configurations will more rarely be chosen as the centers $\chi^*\in Y^*_1(M_0)$. In particular, approximate symmetries and other particular solutions, which represent the local extrema of F, will have less pronounced representatives in the primary structure of similarity in R^{NK}. Nevertheless, the ergodicity of the scheme is preserved; namely, the following assertion holds: whatever the parameter M_1 and the approximate symmetry G with maximal support X^*_G in X, if $\mu(X^*_G)>0$, then, for any finite M_0 and $M\to\infty$, the configuration of the symmetry G fills its own

cluster in R^{NK} with nonzero probability. Moreover, for any approximate (exact) symmetry G the filling of the G-orbit manifold in S becomes strongly continuous (respectively, strongly uniform) as $M \to \infty$, whatever the corresponding configuration template χ^* was chosen on the step (c).

5 CONCLUSIONS

We have considered above only the most important questions related to the application of a sequential scheme to the construction of optimal packings and to the determination of symmetries on the base of some group-theoretical generalization of the Hough transform. As usual, direct solution of such problems requires complex optimization computations. The key point of the approach proposed consists in the application, instead of the laborious direct solution, of simple algorithm, that uses this generalization essentially, during a long enough period of time.

We have introduced a certain parameterization of Hough type that is able to preserve information about feature distribution on spatial shapes in the case of central symmetries, as exact as approximate. On this base an invariant definition was done for spatial ranks, which are used in multi-dimensional sequential scheme in a manner analogous to efficient usage of boundary points of Parzen windows in nonparametric density estimation in one dimension. Some special structure S was constructed, in which exact and approximate symmetries of arbitrary spatial shape are revealed along with sequential reconstruction of optimal K-cluster packings. All symmetries are sought for via the generalized Hough transform adjusted to certain types of adjacent classes on the SO_N group manifold equipped with indices j of ranking templates. It was shown that optimal solutions can be efficiently filtered out due to good asymptotic behavior of the independent sequential choice procedure.

The introduction of additional structures into the sequential scheme that enhance the handling of detected approximate and exact symmetries could serve as a development of the approach. Certain variants of this kind appear due to the presence of natural scale-spaces associated with maximal filters of approximate symmetries in the standard subgroup lattice of SO_N. The construction of a priori estimates that describe the dynamics of filling these scale spaces by the realizations of approximate and exact symmetries seems to be very important. These estimates would enable one to obtain an integral idea about the subordination between symmetries, about the difference between the approximate and exact solutions found, and about the number of trials necessary to reliably filter out exact solutions and exact symmetries from among them.

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