DETECTION OF FACES IN WIRE-FRAME POLYHEDRA
Interactive Modelling of Uniform Polyhedra

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Keywords: Uniform polyhedron, polyhedral graph, simulated elasticity, interactive computing, recreational mathematics.

Abstract: This paper presents an interactive modelling system of uniform polyhedra including regular polyhedra, semi-regular polyhedra, and intersected concave polyhedra. In our system, user can virtually "make" and "handle" them interactively. The coordinate of vertices are computed without the knowledge of faces, solids, or metric information, but only with the isomorphic graph structure. After forming a wire-frame polyhedron, the faces are detected semi-automatically through user-computer interaction. This system can be applied to recreational mathematics, computer assisted education of the graph theory, and so on.

1 INTRODUCTION

This paper presents an interactive modelling system of uniform polyhedra using simulated elasticity. Uniform polyhedra include five regular polyhedra (Platonic solids), thirteen semi-regular polyhedra (Archimedean solids), and four regular concave polyhedra (Kepler-Poinsot solids). Alan Holden is describing in his writing, "*The best way to learn about these objects is to make them, next best to handle them* (Holden, 1971)." Traditionally, these objects are made based on the shapes of faces or solids. Development figures and a set of regular polygons cut from card boards can be used to assemble them. Kepler-Poinsot solids can be formed by stellation of faces of "core" polyhedra. "PolyFormes" is an application program for dialog-based declarative modelling of polyhedra (Martin, 1999). These methods are based on faces. On the other hand, some semi-regular polyhedra can be formed by truncation of other solids. Kepler-Poinsot solids can be also formed by faceting of solids of "case" polyhedra. These methods are based on solids (Coxeter, 1973).

In our system, user can virtually "make" and "handle" all of the uniform polyhedra without the knowledge of faces, solids, or metric information, but only with the isomorphic graph structure. After forming a wire-frame polyhedron with the vertices and the edges, the faces are detected semi-automatically through user-computer interaction.

2 UNIFORM POLYHEDRA

2.1 Platonic Solids

Five Platonic solids are listed in Table 1. The symbol P_{m^n} indicates that the number of faces gathering around a vertex is n, and each face is m-sided regular polygon.

Table 1: The list of Platonic solids.

Symbol	Polyhedron	Vertices	Edges	Faces
P_{3^3}	Tetrahedron	4	6	4
P_{4^3}	Cube	8	12	6
P_{3^4}	Octahedron	6	12	8
P_{5^3}	Dodecahedron	20	30	12
P_{3^5}	Icosahedron	12	30	20

Platonic solids, or regular solids, are convex polyhedra with faces that are regular and congruent polygons, while their vertices lie on the circum-sphere, their vertex figures are also regular and congruent.

2.2 Archimedean Solids

Thirteen Archimedean solids are shown in Figure 1. Archimedean solids, or semi-regular solids, are surrounded by several sorts of regular polygons, and

Nonaka H. (2008).
DETECTION OF FACES IN WIRE-FRAME POLYHEDRA - Interactive Modelling of Uniform Polyhedra.
In *Proceedings of the Third International Conference on Computer Graphics Theory and Applications*, pages 134-137
DOI: 10.5220/0001095301340137

their vertex figures are not regular but congruent polygons. Their vertices lie on the circum-sphere. Some of Archimedean solids can be obtained by truncation of other polyhedra.

2.3 Kepler-Poinsot Solids

Four Kepler-Poinsot solids are shown in Figure 2. $K_{(5/2)^3}$ (great stellated dodecahedron) and $K_{(5/2)^5}$ (small stellated dodecahedron) are regular concave polyhedra with pentagrams (5/2) as faces.

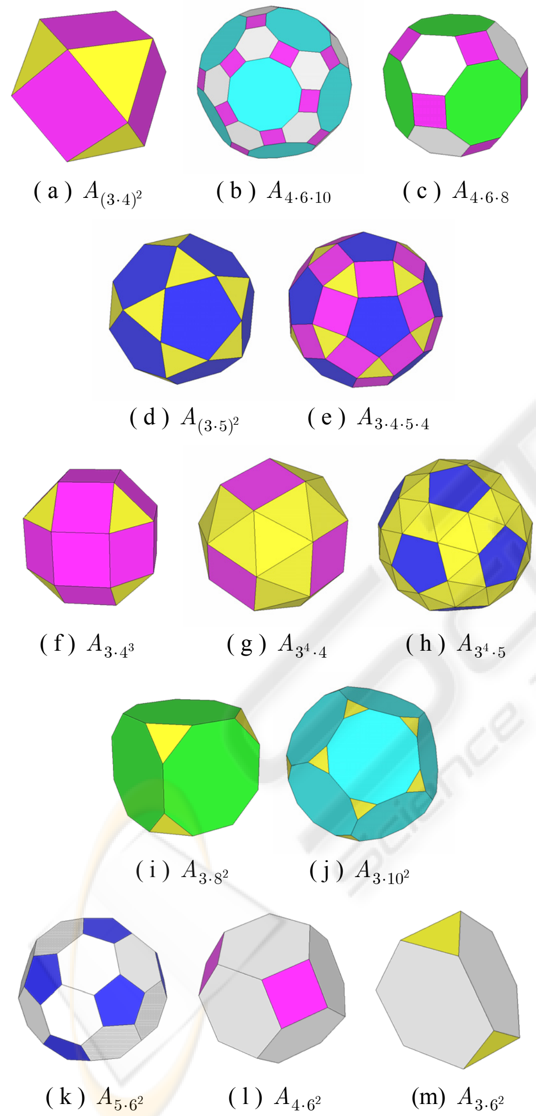


Figure 1: Thirteen Archimedean solids generated by the system.

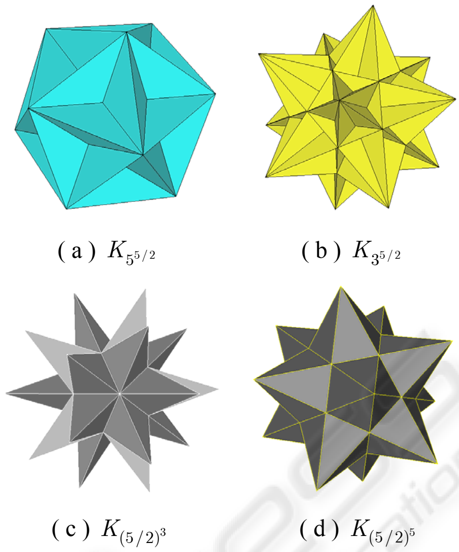


Figure 2: Four Kepler-Poinsot solids generated by the system.

3 POLYHEDRAL GRAPH

3.1 Polyhedral Graph

Drawing graph is the first step of polyhedron modelling in the system. Polyhedral graphs isomorphic to Archimedean solids are illustrated in Figure 3-4. Kepler-Poinsot solids are isomorphic to icosahedron or dodecahedron.

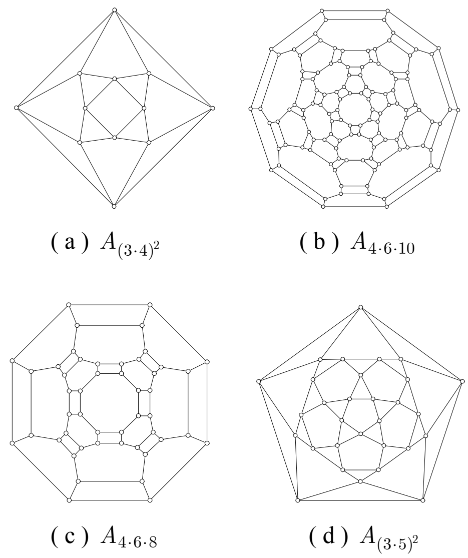


Figure 3: Polyhedral graphs isomorphic to Archimedean solids (1). (to be continued).

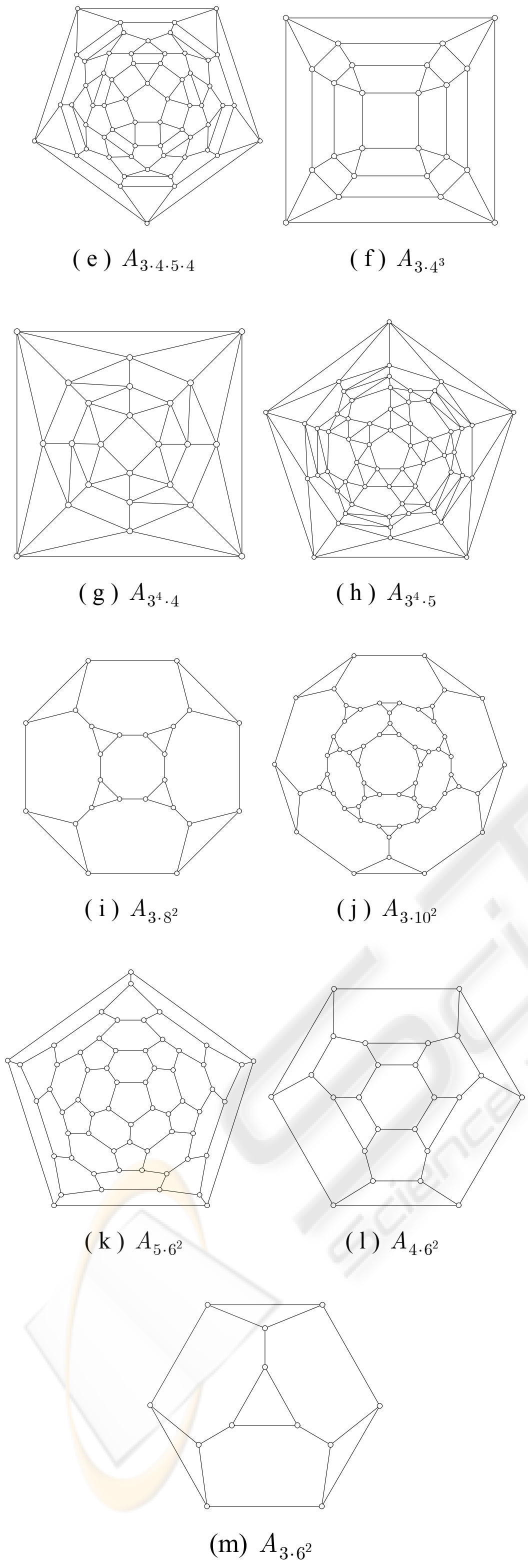


Figure 4: Polyhedral graphs isomorphic to Archimedean solids (2).

3.2 Simulated Elasticity

We define three binary relations between two vertices:

The relation *adjacent* corresponds to the length of an edge in a 3 dimensional space. The relation *neighbour* means that the length of path between two vertices is 2, and two vertices are neighbourhood of another vertex, and it corresponds to the shape of vertex figure in a 3 dimensional space. The relation *diameter* corresponds to the circum-sphere.

Virtual elastic forces are assumed between vertices according to Hooke's law and three relations defined in the previous section. Let L_a, L_n, L_d be natural length of virtual spring, and each lower suffix indicates "adjacent", "neighbour", or "diameter". Let k_a, k_n, k_d be spring constant, which varies from 0 to 1. Bold-faced $\boldsymbol{v}_i,\ i = 0,\cdots p-1$ stands for the 3 dimensional coordinate of vertex $v_i \in V$. Then the total elastic potential E_t is given as follows:

$$E_t = E_a + E_n + E_d \tag{1}$$

$$\begin{cases} E_a = \dfrac{k_a}{2} \displaystyle\sum_{i<j \wedge adjacent(v_i,v_j)} \left(L_a - \left\| \boldsymbol{v}_i - \boldsymbol{v}_j \right\|\right)^2 \\ E_n = \dfrac{k_n}{2} \displaystyle\sum_{i<j \wedge neighbour(v_i,v_j)} \left(L_n - \left\| \boldsymbol{v}_i - \boldsymbol{v}_j \right\|\right)^2 \\ E_d = \dfrac{k_d}{2} \displaystyle\sum_{i<j \wedge diameter(v_i,v_j)} \left(L_d - \left\| \boldsymbol{v}_i - \boldsymbol{v}_j \right\|\right)^2 \end{cases}$$

4 DETECTION OF FACES

After constructing polyhedral graph, and arranging vertices in 3-dimensional space using elastic potential, the next step is detecting and selecting faces. In the case of Platonic solids, Archimedean solids, prisms, and anti-prisms, common routine is available.

The faces of Kepler-Poinsot solid are detected by separate routine. Selecting triangles from great icosahedron is common with selecting triangles from icosahedron. Selecting pentagrams from great stellated dodecahedron is common with selecting pentagons from dodecahedron. Lastly, selecting pentagon from great dodecahedron is common with selecting pentagram from small stellated dodecahedron.

At this stage, the modelling of polyhedron is completed. The final step is rendering the wire-frame polyhedron with detected faces. In the case of Kepler-Poinsot solids, faces are intersecting each other. Then the process of hidden surface removal is required. In the case of presented system, the target machine is low cost PC with single CPU without graphics accelerators. We tried to calculate the geometric interference, and detected the exposed fragments. As an example, such fragments in a triangle of great icosahedron are obtained as is shown in Figure 5, where shaded region is outside of great icosahedron.

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \cdots, \boldsymbol{r}$ be the position vector of each point in the figure: $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \cdots, \overrightarrow{OR}$, then they are expressed by $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows,

$$
\begin{aligned}
\boldsymbol{c} &= \frac{5-\sqrt{5}}{5}\boldsymbol{a} + \frac{3\sqrt{5}-5}{10}\boldsymbol{b}\,,\ \boldsymbol{d} = \frac{\sqrt{5}-1}{2}\boldsymbol{a}\,,\\
\boldsymbol{e} &= \frac{\sqrt{5}}{5}\boldsymbol{a} + \frac{5-2\sqrt{5}}{5}\boldsymbol{b}\,,\ \boldsymbol{f} = \frac{3-\sqrt{5}}{2}\boldsymbol{a}\,,\\
\boldsymbol{g} &= \frac{5-\sqrt{5}}{10}\boldsymbol{a} + \frac{3\sqrt{5}-5}{10}\boldsymbol{b}\,. \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{p} &= \frac{3+\sqrt{5}}{8}\boldsymbol{a} + \frac{3-\sqrt{5}}{8}\boldsymbol{b}\,,\ \boldsymbol{q} = \frac{1}{2}\boldsymbol{a}\\
\boldsymbol{r} &= \frac{1}{4}\boldsymbol{a} + \frac{3-\sqrt{5}}{8}\boldsymbol{b} \qquad (6)
\end{aligned}
$$

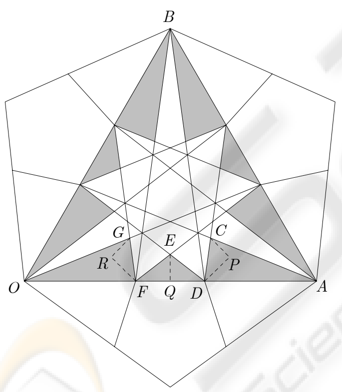
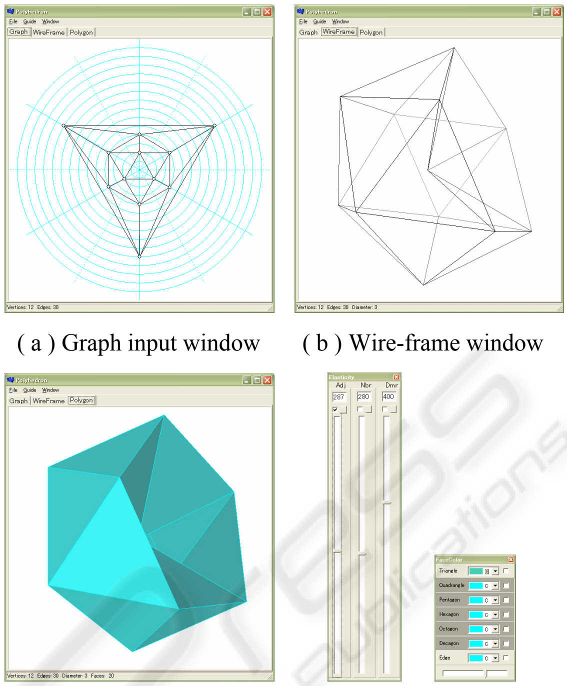


Figure 5: Nine exposed fragments of triangle ΔOAB surrounding the great icosahedron.

5 SYSTEM OVERVIEW

Figure 6 shows snapshots of GUI. In this example, depressed icosahedron is obtained. It means the elastic potential remains at a local minimum. It can be recovered interactively by pulling a proper vertex, or increasing the natural length of diameter, and so on.



(a) Graph input window (b) Wire-frame window

(c) Polygon window (d) Control tools

Figure 6: Snapshots of GUI of the system.

6 CONCLUSIONS

This paper proposed an interactive modelling system of uniform polyhedra. Process of modelling is composed of following three steps. Firstly, a polyhedral graph is constructed by editing graph with several graph operations. Secondly, wire-frame polyhedron is formed by simulated elasticity with the relation of *adjacent*, *neighbour*, and *diameter*. Lastly, proper faces are detected semi-automatically through user-computer interaction. This system can be applied to recreational mathematics, computer assisted education of the graph theory, and so on.

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