A HIERARCHICAL SPATIAL INDEX FOR TRIANGULATED SURFACES

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Abstract: We present the *PM2-Triangle quadtree* (PM2T-quadtree), a new hierarchical spatial index for triangle meshes which has been designed for performing spatial queries on triangle-based terrain models. The PM2T-quadtree is based on a recursive space decomposition into square blocks. Here, we propose a highly compact data structure encoding a PM2T-quadtree, which decouples the spatial indexing structure from the combinatorial description of the mesh. We compare the PM2T-quadtree against other spatial indexes by considering the structure of the underlying domain subdivision, the storage costs of their data structures and the performance in geometric queries.

1 INTRODUCTION

A terrain is a scalar field $z = f(x,y)$ defined on a domain D. Function f is known only at a finite set of data points. When the data points are irregularly spaced in the x-y plane, the terrain is approximated through a triangulated surface, called a *Triangular Irregular Network* (TIN). A TIN consists of a decomposition of the domain in the x-y plane as a *triangle mesh* connecting the data points, and of a piecewise-linear interpolant defined on the triangles of the mesh.

The triangle mesh underlying a TIN is usually encoded in a topological data structure, such as, for instance, the indexed data structure with adjacencies (De Floriani and Hui, 2007). This is well suited for navigation in the mesh through adjacencies, but it becomes quite inefficient for spatial queries, such as point location or window query, which involve focusing on a portion of the domain. Since current models have a large number of triangles, of the order of several millions, it is necessary to superimpose a spatial index on the triangle mesh to expedite the search. An approach for point location without a spatial index (Mücke et al., 1999) is only possible for Delaunay triangulations and convex domains.

There is a vast literature on both object-based or space-based hierarchical spatial indexes, see (Samet, 2006) for a survey. Object-based indexes, such as R-trees or their variants, are hierarchies of bounding boxes (or sometimes spheres), and thus are not designed for encoding connectivity of a set of entities, like a polygonal map or a triangle mesh. The most common space-based spatial indexes are quadtrees and kD trees. Some of them have been defined for sets of points or for polygonal maps (e.g., the PM-quadtrees), but the subdivision is based on the edges of the map, and the topological structure of the map is not encoded.

We propose here a new hierarchical spatial index based on triangles, that we call a *PM2-Triangle quadtree* (or *PM2T-quadtree* for short), and we present a compact data structure for it. Such data structure separately encodes the combinatorial structure of the triangle mesh, which is the basis for mesh navigation, and the structure of the spatial index, that enhances the performances of spatial queries by reducing the searching space.

The remainder of this paper is organized as follows. In Section 2 we present related works. In Sections 3 we define the PM2T-quadtree and its data structure. In Section 4 we show that the PM2T-quadtree outperforms a number of alternative spatial indexes. Finally, Section 5 contains some concluding remarks.

2 RELATED WORK

In this Section, we briefly review space-based hierarchical spatial indexes related to the new indexes presented here.

Quadtrees and kD-trees are spatial indexes for representing a set of points. We describe here only a

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quadtree for representing points in the plane, the PR-quadtree (Orenstein, 1982). A PR-quadtree is generated by the recursive subdivision of the square domain, where the points are distributed, into nested square blocks split through the middle point of the block. The subdivision is described through a quaternary tree, where every node is either a leaf, or an internal node with four children corresponding to the four sub-quadrants of its parent's square. The shape of a PR-quadtree is independent of the order in which the points are inserted, since the subdivision is based on the domain and the points are only in the leaves.

Several quadtrees have been proposed to store collection of edges, see (Samet, 2006) for a detailed treatment. The class of PM-quadtrees extend the PR-quadtree to represent polygonal maps, by using subdivision rules based on vertices and edges. The PMR quadtree handles a collection of edges in the plane, not necessarily forming a polygonal map, and uses a probabilistic splitting rule. All such indexes maintain a list of edges in the leaf blocks, but they do not encode the topological structure of the map. Thus, they are good for searching in space but not for topological navigation. The PM2-Triangle quadtree defined here is a triangle-based extension of an index in the class of PM-quadtrees, namely the PM2-quadtree.

The K-structure (Kirkpatrick, 1983) is a spatial index for triangle meshes, developed in computational geometry with the purpose of performing point location in a triangle mesh in worst-case logarithmic time in the number of vertices. Unlike the PR- and PM-quadtrees, the K-structure organizes the data to be stored and not the embedding space. It consists of a hierarchy of triangle meshes of logarithmic height described as a directed acyclic graph. The hierarchy is computed by starting from the given mesh and repeatedly eliminating an independent set of vertices of low degree. Compared with the PM-quadtree, the K-structure has a better worst-case behavior, but it is more complex to implement and update (Samet, 2006).

3 THE PM2-TRIANGLE QUADTREE

3.1 Space Subdivision

A *PM2-Triangle quadtree* (or *PM2T-quadtree*) is a hierarchical spatial index for triangle meshes. As all quadtrees, it recursively divides a square domain containing the data into four blocks of the same size, and the resulting nested subdivision is described as a quaternary tree. The PM2-Triangle quadtree has a triangle-based subdivision rule. The blocks of the final subdivision, i.e., the leaves of the tree, must satisfy the following four validity conditions:

1. A leaf block may contain at most one vertex.
2. If a leaf block contains one vertex v, it intersects only the triangles incident in v.
3. If a leaf block b does not contain any vertex, the triangles intersecting b are all incident in an external common vertex.
4. Leaf blocks are maximal (i.e., no four sibling leaves can be merged into a valid leaf).

Figure 1 shows a PM2T-quadtree for a mesh of eight triangles.

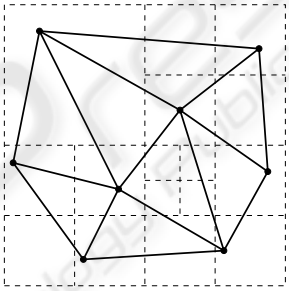


Figure 1: An example of a PM2T-quadtree.

Each validity condition provides an upper bound in the evaluation of the depth of the tree. Condition 1 gives an upper bound $D_1 = 1 + \log_2 \frac{\sqrt{2}}{d_1}$, where d_1 is minimum distance between any two vertices. Conditions 2 and 3 imply an upper bound that depends on the shape of the triangles, being equal to $D_2 = 1 + \log_2 \frac{\sqrt{2}}{d_2}$ where d_2 is the minimum height of the triangles in the mesh. Thus, the global upper bound to the depth of the PM2T-quadtree is $\max(D_1, D_2)$. Details can be found in (Facinoli et al., 2007).

3.2 Data Structure

We present a compact data structure for encoding the PM2T-quadtree, which represents the combinatorial structure of the triangle mesh and the structure of the spatial index separately.

The combinatorial structure of the triangle mesh is encoded through a *triangle-based data structure*, that is the indexed data structure with adjacencies (De Floriani and Hui, 2007). We encode the vertices and the triangles of the mesh into a vertex array and a triangle array, respectively. The index of the position in the array is used to identify the triangle or the vertex. The only geometrical information are the vertex

coordinates and the vertex elevations. For each triangle t, we encode the indexes of the three vertices of t (*Triangle-Vertex (TV)* relation), and the indexes of the three triangles adjacent to t along an edge (*Triangle-Triangle (TT)* relation), ordered consistently with the information in TV relation. For each vertex v, we encode the index in the triangle array of one of its incident triangles (partial *Vertex-Triangle (VT)* relation). This is sufficient to retrieve all the triangles incident in a vertex v, in time linear in their number.

For encoding the structure of the spatial index, we do not explicitly represent the quaternary tree that describes the PM2T-quadtree through all nodes and their parent-child pointers, but we encode only the leaf nodes and their location codes. The *location code* (Gargantini, 1982) of a leaf n describes the position of n in the tree by means of the root-to-leaf path traversed to reach it. Thus, for every leaf node, we store its location code and an integer number, which allows distinguishing among different types of leaves, as well as retrieving the information (vertex and triangles) possibly associated with such leaf. We have the following types of leaves:

- An empty leaf is identified by the value -1.
- For a leaf block intersected by three or more triangles, sharing a vertex v (that can be internal or external to the block), we store the index of v. Triangles are not explicitly stored since they are retrieved by using topological relations.
- For a leaf containing one triangle t, we store a negative even integer $content = -2(t+1)$, from which we can retrieve the index of t as $-(content/2)-1$.
- For a leaf containing two triangles t_1 and t_2, we store a negative odd integer $content = -2(t_1+1)+1$, if this quantity is different from -1 (already denoting an empty leaf); otherwise, it is equal to the result of the same expression with t_2 instead of t_1. The two triangles can be retrieved one as $((1-content)/2)-1$, and the other one as its adjacent triangle which intersects the leaf.

We denote with N the number of vertices of the triangle mesh, with T the number of its triangles and with L_N the number of leaf nodes in the PM2T-quadtree. We consider every base type element (integers and pointers) to have a unit cost. The cost of encoding the vertex coordinates and elevation values, and the partial VT relation is equal to $4N$, while the cost of encoding the triangle array, i.e., the TV and TT relations is equal to $6T$. Finally the cost of encoding the tree structure is equal to $2L_N$. This leads to $4N+6T+2L_N$ stored information items.

Note that $4N+6T$ is the space required for representing just the triangle mesh in a triangle-based data structure, while $2L_N$ expresses the overhead of the spatial index. From Euler formula, $T \simeq 2N$. From our experiments, L_N is about $6N$ on average for synthetic data, and about $5N$ for real data (see Section 4). Thus, the additional memory needed by the spatial index is from 63% to 75% the memory required for storing the triangle mesh itself.

4 ANALYSIS

We compare the PM2T-quadtree with other two spatial indexes: the PM2-quadtree (Samet, 2006) for polygonal maps, which can be used for a triangle mesh as well, and an extension of the PR-quadtree to triangle meshes.

We evaluate the space subdivision generated by the different spatial indexes, the storage costs of the data structures, and the performance on spatial queries. The main requirement for a spatial index is to perform spatial queries efficiently, while a low overhead of the data structure is desirable as an additional requirement. Experiments show that the PM2T-quadtree outperforms the other considered spatial indexes in queries, and the proposed data stucture makes it compact enough.

We present results on synthetic data (Delaunay triangle meshes built on randomly generated point sets) and on triangulated models from the AIM@SHAPE shape repository (`http://shapes.aim-at-shape.net/`): City of Rome (957,456 vertices, 1,914,867 triangles), and Dolomites mountains (810,000 vertices, 1,619,963 triangles).

4.1 Other Spatial Indexes

The *PM2-quadtree* belongs to the class of PM-quadtrees (Samet, 2006) developed for polygonal maps. In a PM2-quadtree, a leaf block contains at most one vertex. If a leaf block contains a vertex, it may intersect only edges incident in such vertex. If a leaf block contains no vertex, it may intersect only edges that meet a common vertex exterior to the block. Each leaf has a list of edges intersecting it.

A PM2-quadtree, applied to a triangle mesh, represents the edges and not the triangles. The data structure we have implemented for the PM2-quadtree is slightly different from the one described in (Samet, 2006). The vertices and edges of the map are stored in a vertex array and an edge array, respectively. For each vertex, we store its coordinates and elevation value, and, for each edge, we store the indexes of its

two endpoints. This latter is the only topological information encoded in the PM2-quadtree. Each leaf block stores its location code, and the list of the edges intersecting the leaf block. The storage cost of a PM2-quadtree is equal to $3N + 2E + 2L_N + 2E_L$, where E denotes the number of edges and E_L the total length of all edge lists. Note that the number of leaves L_N is greater here than in the PM2T-quadtree (as shown in Section 4.2), moreover $E_L \geq E$ and, from Euler formula, $E \simeq 3N$ in the case of a triangle mesh.

We have developed in (Facinoli et al., 2007) the *PM3-Triangle quadtree*, or *PM3T-quadtree*, as an extension of the PR-quadtree for triangle meshes. It is basically a PR-quadtree built on the mesh vertices, where, in addition, each leaf stores all triangles it intersects. Figure 2 shows the PM3T-quadtree built on the same triangle mesh used for the PM2T-quadtree in Figure 1. The depth of a PM3T-quadtree is inversely proportional to the minimum distance between two vertices in the mesh since the subdivision criterion is only based on the position of vertices.

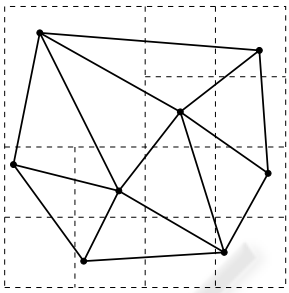


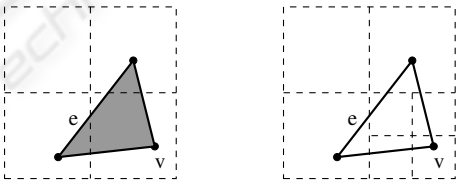
Figure 2: Example of a PM3-Triangle quadtree.

Note that, in a PM2T-quadtree, all triangles intersecting a leaf block are incident in a common vertex, while this is not true for a PM3T-quadtree. Thus, we are not able to find all such triangles just from the vertex, by using topological relations, as we did in the PM2T-quadtree. Therefore, we need an explicit list of the triangles intersecting each leaf. The data structure for the PM3T-quadtree stores, for each non-empty leaf block, its location code, the vertex it contains, and a list of the triangles that it intersects. Vertices are encoded with their coordinates and elevations, while for each triangle only the Triangle-Vertex (TV) relation is encoded, i.e., its three vertex indexes. The storage cost of this data structure is equal to $3N + 3T + 3L_N + 2T_L$, where T_L denotes the total length of all triangle lists. Note that the number of leaves L_N of a PM3T-quadtree is lower than that of a PM2T-quadtree (as shown in Section 4.2), while $T_L \geq T \simeq 2N$.

4.2 Structure of the Subdivision

We compare the domain subdivisions provided by the PM2T-quadtree, the PM2 quadtree and the PM3T-quadtree, on the same triangle mesh.

The domain subdivision induced by a PM2-quadtree may be finer than the one induced by a PM2T-quadtree. Consider a leaf block b with an associated vertex v. If b is intersected by the third edge e of a triangle t incident in v, this causes a subdivision of b in the PM2-quadtree. On the contrary, the PM2T-quadtree does not subdivide b because it considers edge e as part of triangle t (see Figure 3). Note that this situation can only happen near the boundary of the domain of the triangle mesh. In the interior of the mesh, such a leaf node b would be subdivided in the PM2T-quadtree as well, because of the other triangle adjacent to t along edge e. On the other hand, the subdivision induced by a PM2-quadtree cannot be coarser than the one induced by a PM2T-quadtree. The subdivision rule of the PM2T-quadtree is a triangle-based version of the edge-based rule used in the PM2-quadtree. Whenever the PM2T-quadtree splits a block because of a triangle t intersecting it, necessarily at least one edge of t intersects the same block, thus splitting it in the PM2-quadtree as well (see (Facinoli et al., 2007) for a complete proof).



(a) (b)

Figure 3: The PM2-quadtree (b) produces a finer subdivision than the PM2T-quadtree (a).

The subdivision induced by a PM2T-quadtree is finer than the subdivision induced by a PM3T-quadtree on the same triangle mesh (see Figures 1 and 2). This is due to the fact that, in a PM3T-quadtree, the triangles associated with a leaf block do not necessarily need to share a common vertex.

Figures 4 and 5 compare the depth and the number of leaf blocks in the three indexes, expressed as a function of the number of the mesh vertices, for synthetic data sets. Table 1 shows the results for real data. The PM2-quadtree is at least as deep as the PM2T-quadtree, and the number of its leaf blocks is about 25% more than in a PM2T-quadtree, on average. As expected, a PM3T-quadtree is less deep than the other two indexes and has fewer leaf blocks. In particular, the PM3T-quadtree depth is about 90%, and its leaves are about 30%, than in the PM2T-quadtree.

On the other hand, our experiments in (Facinoli et al., 2007) show that the number of triangles intersecting a leaf block are about 150% higher in a PM3T-quadtree than in a PM2T-quadtree. This leads to larger memory requirements for the PM3T-quadtree, in spite of the coarser subdivision (see Section 4.3).

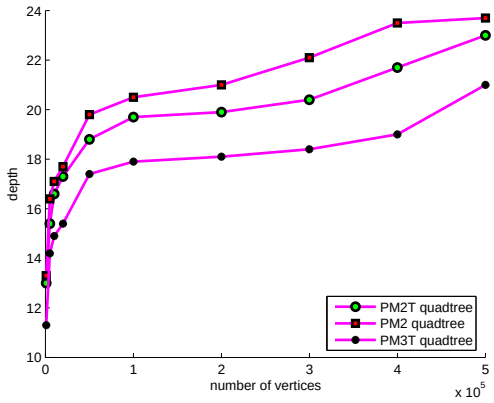


Figure 4: Depth of the tree on synthetic data.

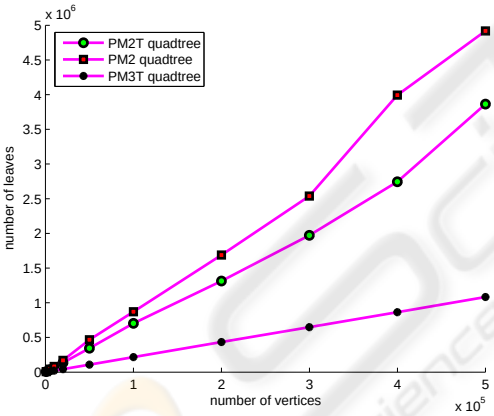


Figure 5: Number of leaf nodes on synthetic data.

Table 1: Depth of the tree, number of leaf nodes, and storage costs on real data.

index	depth	leaves	storage (Mb)
Rome			
PM2T	22	3,857,305	84.98
PM3T	12	1,397,315	95.56
PM2	12	3,895,022	166,72
Dolomites			
PM2T	23	4,659,781	87.86
PM3T	12	1,146,715	116.96
PM2	13	4,707,102	237.58

4.3 Costs of the Data Structures

We consider here the space requirements of the data structures for the PM2T-quadtree, the PM2-quadtree, and the PM3T-quadtree.

Figure 6 compares the storage costs on synthetic data. Results on real data are shown in Table 1.

The total cost of a PM2T-quadtree is about 3/4 than the total cost of a PM3T-quadtree. This happens even if the domain subdivision in the PM2T-quadtree is finer than in a PM3T-quadtree (see Subsection 4.2). Our experiments in (Facinoli et al., 2007) have shown that the cost of storing the leaf nodes in a PM2T-quadtree is almost 1/3 than in a PM3T-quadtree, because the leaves of a PM2T-quadtree do not store any triangle list.

The storage cost of the PM2T-quadtree is about 40% of that of the PM2-quadtree, because the latter has a finer domain subdivision, and furthermore stores a list of edges in each leaf node. Our experiments in (Facinoli et al., 2007) show that around 90% of space in a PM2-quadtree is used for storing the leaf nodes.

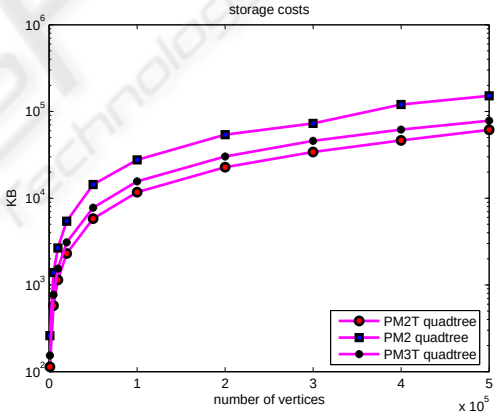


Figure 6: Storage costs (in Kbytes) for synthetic data.

4.4 Performance in Queries

We consider here the two triangle-based indexes, namely the PM2-Triangle quadtree and the PM3-Triangle quadtree.

The basic geometric query we consider here is *point location*, i.e., finding the triangle(s) of the mesh containing a given query point.

Other common queries are the *window query* and the *range query*, i.e., finding all the triangles intersected by a rectangular or a circular area. Thanks to the fact that a PM2T-quadtree represents the connectivity of the mesh, such queries can be answered

by performing a point location for the centroid of the query region, and then navigating the triangle mesh through adjacency relations, until all other involved triangles have been found. The cost is equal to the cost of point location plus a number of point-in-triangle tests linear in the number of reported triangles.

The algorithm for point location query descends the tree down to the leaf b containing the query point p, and then performs a point-in-triangle test for each triangle associated with b. To verify if a triangle contains p, we need to test which if p defines a left or right turn with the three edges of the triangle. If the triangles form a fan, some triangle edges can be skipped because already tested while processing the previous triangle of the fan. The most time consuming operation is computing a turn (since it involves floating-point arithmetics).

Point location on a PM2-quadtree is more complex since the triangles are not represented. It requires finding the edges of the triangle containing the point, which implies nearest neighbor computations.

Table 2: Results for the point location query on real data.

Index	Visited Triangles		Visited Nodes		Computed Turns	
	avg	var	avg	var	avg	var
Rome						
PM2T	1.60	0.30	11.3	4.9	4.80	0.70
PM3T	2.55	3.02	10.50	3.83	6.66	14.50
Dolomites						
PM2T	2.0	0.72	11.90	0.10	4.8	2.5
PM3T	3.10	2.32	10.90	0.10	7.70	11.34

We consider the number of nodes visited in the tree traversal, the number of triangles checked for inclusion and the number of turns computed (see Table 2). The upper bound to the number of visited nodes is the depth of the tree (see Figure 4). The number of visited triangles and the number of turns computed depend on the number of triangles incident in the leaf node containing the query point p.

The number of visited nodes is lower in the PM3T-quadtree, since it has a lower depth. The number of tested triangles and turn computations, as well as their variances, are lower in the PM2T-quadtree. This means that the PM2T-quadtree performs better, although the tree is deeper, since turn computations have a dominat cost. Specifically, a lower variance implies that the worst case performs better.

5 CONCLUDING REMARKS

We have presented a new spatial index for triangle meshes, called PM2-Triangle quadtree. We have defined and implemented a compact representation for a PM2T-quadtree based on the complete separation of the combinatorial information on the triangle mesh (needed to navigate in the mesh), and the structure of the spatial index (used to reduce searching space in queries). We have tested the efficiency of the PM2T-quadtree through experiments.

Our current and future work is in spatial indexes for tetrahedral meshes, for application to visualization of 3D scalar fields and to finite element meshes. The PM2-Triangle quadtree can be generalized into a PM2-Tetrahedron octree, and a data structure for it can be defined based on the same idea of separating the structure of the mesh and the structure of the spatial index.

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