AN EXTENSION TO THE BEZIER SUB-DIVISION METHOD TO COMPLETELY APPROXIMATE CURVES AND SURFACES

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Abstract: Sub-division splines generate a number of new control points calculated fron the old control points. Both control polygons/grids define the same curve/surface. At each iteration the resulting new points are much greater in number than the old points and lie nearer to the actual curves. After a number of iterations, the generated points lie on the actual curve, very close to each other, and by displaying them on a computer screen the result is a smooth curve/surface. This paper describes a method, which is an extension to the Bezier sub-division method, where the resulting curve is an approximation curve which interpolates only the first and the last control points. The method is also derived for surfaces.

1 INTRODUCTION

The most popular methods for curve-fitting are based on approximation – that is, the generated curve passes near the original points approximating their shape – like Bézier splines (Bezier, 1970, 1974, 1977), B-splines (Bartels et al, 1987), Beta-splines (Barsky, 1981, 1986), ν-Spline (Nielson, 1986), NURBS (Bartels et al, 1987; Piegl, 1990), Kochanek spline (Kochanek, 1984), Catmull-Rom spline (Catmull and Rom, 1974) and many others.

Subdivision methods have also attracted a large amount of research due to their ability to generated complex surfaces defined by an arbitrary topology of points, which are not based on a regular rectangular mesh and even still today the most important methods are those of Catmull and Clark (Catmull and Clark, 1978) and Doo and Sabin (Doo and Sabin, 1978).

This paper describes a subdivision method that subdivides a Bézier curve and then generalizes it to produce an approximated curve that interpolates only the first and the last original points. Section 2 describes the Bezier sub-division method, section 3 derives its extension, and section 4 concludes.

2 THE BÉZIER SUBDIVISION METHOD

The cubic Bézier curve defined by the control points V_0, V_1, V_2 and V_3 is given by Eq.1 for $0 \le u \le 1$.

$$Q(u) = V_0(1-u)^3 + V_1 3u(1-u)^2 + V_2 3u^2(1-u) + V_3 u^3 \qquad (1)$$

The same curve can also be generated by another two Bezier curves, $Q\left(\frac{u}{2}\right)$ and $Q\left(\frac{u}{2}+\frac{1}{2}\right)$, where the first is defined by the control points S_0 , S_1 , S_2 , S_3, and the second by T_0 , T_1 , T_2 , T_3 where $S_3 = T_0$, as illustrated in Fig.1. The control points S_0 , S_1 , S_2 , $S_3 = T_0$, T_1 , T_2 , T_3 are called new points and they can be calculated by the old points V_0, V_1, V_2 and V_3 by Eq.1 and Eq.2 (Bartels et al, 1987).

The new points are twice as much as the original points, (in-fact double minus one, since S_3 and T_0 is the same point) and they lie nearer to the actual curve than the original control polygon V_0, V_1, V_2, V_3. Then from the new control points another set of points (double minus one in size than the previous points) can be calculated that are even nearer to the actual curve. After a number of iterations the new points lie on the actual curve, very close to each other, and by displaying them on a computer screen the result is a smooth curve.

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$$\left.\begin{aligned}
S_0 &= V_0\\
S_1 &= \tfrac{1}{2}(V_0+V_1)\\
S_2 &= \tfrac{1}{4}\left(V_0+2V_1+V_2\right)=\frac{1}{2}\left(\frac{(V_0+V_1)}{2}+\frac{(V_1+V_2)}{2}\right)\\
S_3 &= \tfrac{1}{8}\left(V_0+3V_1+3V_2+V_3\right)=\frac{1}{4}\left(\frac{(V_0+V_1)}{2}+2\frac{(V_1+V_2)}{2}+\frac{(V_2+V_3)}{2}\right)
\end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned}
T_0 &= \tfrac{1}{8}\left(V_0+3V_1+3V_2+V_3\right)=\frac{1}{4}\left(\frac{(V_0+V_1)}{2}+2\frac{(V_1+V_2)}{2}+\frac{(V_2+V_3)}{2}\right)\\
T_1 &= \tfrac{1}{4}\left(V_1+2V_2+V_3\right)=\frac{1}{2}\left(\frac{(V_1+V_2)}{2}+\frac{(V_2+V_3)}{2}\right)\\
T_2 &= \tfrac{1}{2}(V_2+V_3)\\
T_3 &= V_3
\end{aligned}\right\} \quad (3)$$

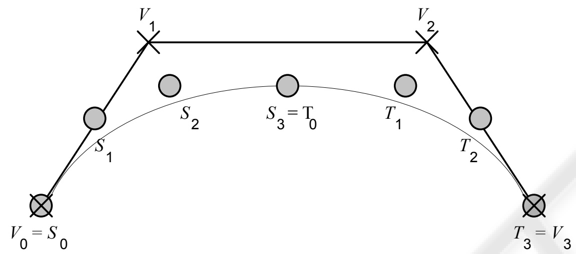

Figure1: The old and new points.

Consecutive segments in a composite Bézier curve are C^1 continuous if the penultimate control vertex of the first curve, the shared endpoint and the second vertex of the next curve are collinear and equally spaced. In Fig.2 the unprimed vertices define one curve segment and the primed vertices define another. Because V_2, $V_3 = V_0'$ and V_1' are collinear and $|V_3 - V_2| = |V_1' - V_0'|$ the composite curve will be C^1 continuous.

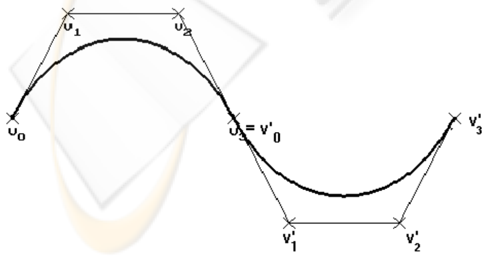

Figure2: Two Bézier segments.

From Eq.1 and Eq.2 it yields

$$\begin{aligned}
S_3 = T_0 &= \frac{1}{4}\left(\frac{(V_0+V_1)}{2}+2\frac{(V_1+V_2)}{2}+\frac{(V_2+V_3)}{2}\right)\\
&= \frac{\frac{1}{2}\left(\frac{(V_0+V_1)}{2}+\frac{(V_1+V_2)}{2}\right)+\frac{1}{2}\left(\frac{(V_0+V_1)}{2}+\frac{(V_1+V_2)}{2}\right)}{2}\\
&= \frac{S_2+T_1}{2} \qquad (4)
\end{aligned}$$

which means that S_3 (which is the same point as T_0) is the midpoint of S_2 and T_1 and therefore the required condition for continuity between successive Bézier segments at each step is satisfied. Fig.3 illustrates a Bezier curve after 1, 2 and 6 subdivisions. It can be noted that the actual curve interpolates the first and fourth control point of each segment.

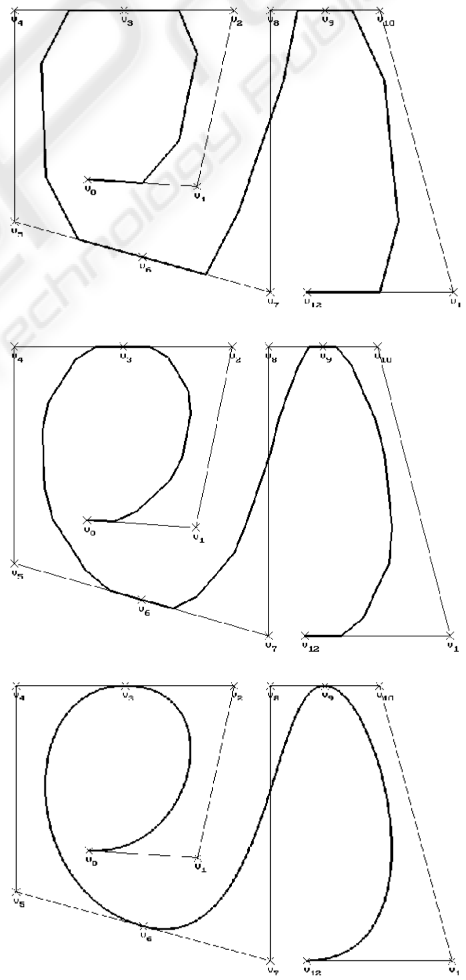

Figure 3: A recursive Bézier curve after 1, 2 and 6 subdivisions.

3 GENERATING AN APPROXIMATION CURVE

The new points can be divided into four different categories (Fig.4). The *V*-points which correspond to an old point at the two ends of the segment and are equal to the old point, the *V'*-points which also correspond to an old edge but not at the ends of the segment, the *E*-points that correspond to an old edge that one of the old points sharing the edge is at the ends of the segment, and the *E'*-points that corresponds to an old edge that none of the two old vertices sharing the edge is at the ends of the segment. Additionally, it can be emphasized that the *E'*-points lie on the actual curve.

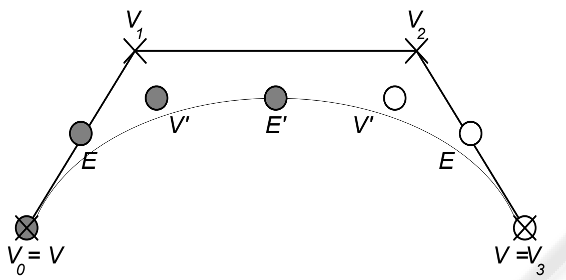

Figure 4: The new control polygon resulted from the original one after one iteration.

It has been derived in (Savva and Clapworthy, 1998) that if we use *V'*-points and *E'*-points everywhere on a curve except at the corner points then the resulting curve becomes an approximation curve that interpolates only the first and the last control points. Also the penultimate control vertex of the each previous segment, the shared endpoint and the second vertex of the next segment does not need to be collinear and equally spaced for continuity, and actually the resulting curve is has C^2 continuity everywhere. This is illustrated in Fig.5.

Eq.4 shows that $\frac{S_2 + T_1}{2} = S_3$. This condition is satisfied after the first iteration. But if we add the midpoints of all the edges in the initial control polygon then the resulting curve is similar as above (Fig.5) but it gives a better approximation to the control points as shown in Fig.6.

The same method is also derived for surfaces. The resulting surface is an approximation to the control grid but interpolates the corner points of the surface.

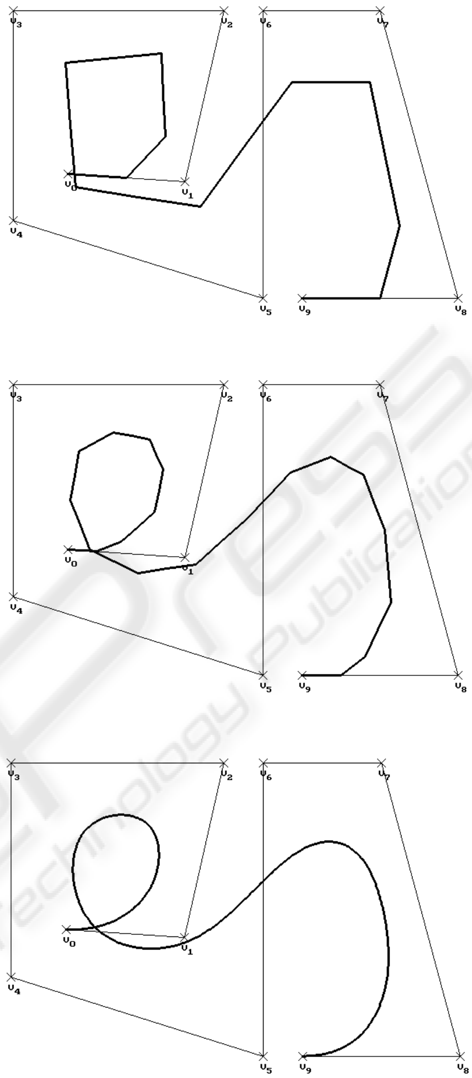

Figure 5: An approximated curve that interpolates the first and last control points after 1, 2 and 6 sub-divisions.

4 CONCLUSIONS

A Bezier curve interpolates the first and last control point for each segment and in order to achieve C^1 continuity between successive segments the penultimate control vertex of the first segment, the shared endpoint and the second vertex of the next curve must be collinear and equally spaced. Despite the fact that there is only C^1 continuity at these points, having to make the three control points collinear makes it difficult to be used in modelling applications.

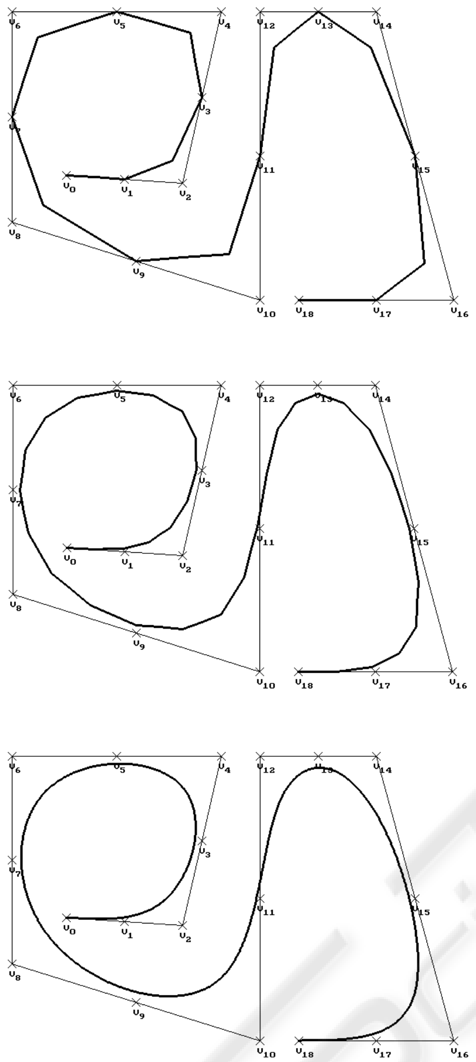

Figure 6: Adding the edge midpoints after 1, 2 and 6 subdivisions.

This paper describes an extension to the Bezier sub-division scheme. The resulting curve is an approximation curve that interpolates only the first and the last control points and the curve has C2 continuity everywhere.

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