OFF-LINE ROBUSTIFICATION OF EXPLICIT MPC LAWS
The Case of Polynomial Model Representation

Pedro Rodríguez-Ayerbe and Sorin Olaru
Department of Automatic Control, Supélec, 3 rue Joliot Curie, F91192 Gif-sur-Yvette, France

Keywords: Piecewise affine controller, Robustification, Youla-Kucera parameter, Model Predictive Control.

Abstract: The paper deals with the predictive control for linear systems subject to constraints, technique which leads to nonlinear (piecewise affine) control laws. The main goal is to reduce the sensitivity of these schemes with respect to the model uncertainties and avoid in the same time a fastidious on-line optimisation which may reduce the range of application. In this idea a two stage predictive strategy is proposed, which synthesizes in a first instant an analytical (continuous and piecewise linear) control law based on the nominal model and secondly robustify the central controller (the controller obtained when no constraint is active). This robustification is then expanded to all the space of the piecewise structure by means of its corresponding noise model.

1 INTRODUCTION

The model predictive control (MPC) laws are optimization based techniques which allow constraints handling from the design stage. The analytical formulation of the optimum and its on-line evaluation avoids a challenging optimization from the point of view of the real-time control environment. Solutions in this direction exist at least for two important classes of problems (linear and quadratic) subject to linear constraints due to the Abadie constraint qualification (Goodwin *et al*., 2004). It must be said that these are in fact a part of a larger class of multiparametric convex programs (Bemporad *et al*., 2002b) for which exact or approximate algorithms exist (TØndel *et al*., 2003, Seron *et al*, 2003, Olaru and Dumur, 2004; Bemporad and Filippi, 2006).

In the case of robust predictive control laws, the model uncertainties and the disturbances can be taken into account at the design stage. A popular technique in this sense is the use of a min-max criterium (in the case when the extreme combination of disturbances or uncertainties are known) (Kerrigan and Maciejowski, 2004; Bemporad *et al*., 2002a) which comes finally to the resolution of a single multiparametric linear program. The structure of this ultimate optimization is however quite complex and large prediction horizons cannot be handled due to the exponential growth of disturbances realization to be taken into account. In a slightly different manner, by constructing an estimation mechanism (Goodwin *et al*., 2004) for the constrained variables, one can obtain alternatively a robust control structure, but the multiparametric optimization remains intricate.

A first study on the robustness improvements for the explicit affine feedback policy constructed upon constrained predictive control strategies was presented in (Olaru and Rodriguez-Ayerbe, 2006). The simplest way to proceed is to consider an observer of the state variables (Goodwin *et al*., 2004), the dimension of the state space being preserved and the piece-wise structure of controller unchanged. The same observer can be used for all feasible regions and can be viewed as noise characterisation of the model. Nevertheless, the observer does not describe the entire class of stabilizing controllers. The present paper presents an improved result based on the Youla-Kučera parametrization which spans the space of stabilizing controllers. For a two-degree of freedom controller, one has access to all the stabilizing controllers that preserve the same input/output behavior, so the Youla-Kučera parameter offers more degrees of freedom than the use of an observer.

The robustification is made such that the state space dimension of the controller is augmented. The main contribution here is the reconstruction of the noise model induced by the central Youla-Kučera

Rodríguez-Ayerbe P. and Olaru S. (2008).
OFF-LINE ROBUSTIFICATION OF EXPLICIT MPC LAWS - The Case of Polynomial Model Representation.
In *Proceedings of the Fifth International Conference on Informatics in Control, Automation and Robotics - SPSMC*, pages 96-101
DOI: 10.5220/0001481300960101

parameter, in order to use it to generate the corresponding robust piece-wise controller.

In the following, section 2 briefly recalls the predictive control and the Youla-Kučera parametrization. Section 3 details the explicit formulation of the control laws obtained in the constrained case. Section 4 contains the main contribution: the noise model of the Youla-Kučera parameter and the numerical examples are presented in section 5 and the final conclusions in section 6.

2 PREDICTIVE CONTROL

The Generalized Predictive Control (GPC) strategy, introduced in (Clarke *et al.*, 1987), uses for the prediction a CARIMA plant model:

$$A(q^{-1})y_t = B(q^{-1})u_{t-1} + \frac{C(q^{-1})\xi_t}{\Delta(q^{-1})} \quad (1)$$

with u, y the input and output, ξ a white noise, A and B polynomials in the backward shift operator of degrees n_a and n_b respectively, and $\Delta(q^{-1}) = 1 - q^{-1}$ the difference operator. The C polynomial is the model argument taking into account the noise influence on the system. In the GPC case the cost function to be minimized over a receding horizon is quadratic:

$$J = \sum_{j=N_1}^{N_2}\left[w_{t+j} - \hat{y}_{t+j}\right]^2 + \sum_{j=1}^{N_u}\lambda_j\left[\Delta u_{t+j-1}\right]^2 \quad (2)$$

where N_1, N_2 are the costing horizons, N_u the control horizon, λ_j the control weighting factor and w the set-point.

Using the model (1) and the solution of some Diophantine equations (Clarke *et al.*, 1987), this control strategy leads to two-degrees of freedom *RST* controller, implemented through a difference equation (Figure 1):

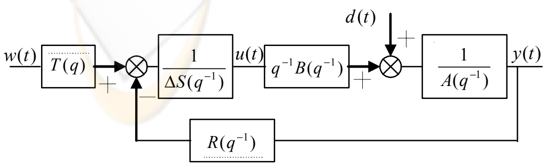

Figure 1: Two-degrees of freedom GPC controller.

In (Yoon and Clarke, 1995) the relation between the RST controller obtained with C=1 and $C \neq 1$ is studied. Considering R', S', T' the controller obtained with C=1 and $\overline{R}, \overline{S}, \overline{T}$ whose obtained with $C \neq 1$, the following relations are obtained:

$$\overline{R} = R'C + A\Delta M \quad \overline{S} = S'C - q^{-1}BM \quad \overline{T} = T'C \quad (3)$$

with:

$$M(q^{-1}) = \sum_{i=N_1}^{N_2}\alpha_i q^i (CE_i' - \overline{E}_i) \quad (4)$$

$$\mathbf{m} = \left[\alpha_{N1} \quad \ldots \quad \alpha_{N2}\right] \quad (5)$$

$\mathbf{m}$ being the first row of $(\mathbf{G}^{\mathrm{T}}\mathbf{G} + \mathbf{\Lambda})^{-1}\mathbf{G}^{\mathrm{T}}$.

The set of all stabilizing controllers for the system shown in Figure 1 is given by the Youla-Kučera parametrization as follows (Maciejowski, 1989):

$$Q = \frac{Q_{num}}{Q_{den}} \begin{cases} R = R'Q_{den} + \Delta A Q_{num} \\ S = S'Q_{den} - q^{-1}BQ_{num} \\ T = T'Q_{den} \end{cases} \quad (6)$$

where $Q(q^{-1})$ is a stable transfer function.

The choice of the Q parameter is a complex problem on its own but it is not the subject of the current paper. The methods presented in (Rodriguez and Dumur, 2005; Rossiter 2003; Ansay *et al.*, 1998; Yoon and Clarke, 1995; Kouvaritakis *et al.*, 1992) can be used for the choice of this parameter.

Comparing (3) and (6) it turns out that the controller for $C \neq 1$ is obtained for Q=M/C. As M depends of C as shown by (4), the robustification by the C polynomial has less degrees of freedom than the robustification by Youla-Kučera parameter (Yoon and Clarke, 1995).

3 EXPLICIT CONSTRAINED GPC LAWS

In the case when the GPC law is subject to constraints, the optimization has to be solved with respect to a feasible domain. If the considered constraints are stated on the control action, on the control increment, on the plant outputs or any other signal related by a CARIMA model to the control signal, then one can restate them in a form depending only on the control increment, leading to a set of linear constraints (Ehrlinger *et al.*, 1996):

$$\begin{cases} \mathbf{A}_{in}\ \mathbf{k}_u \leq \mathbf{B}_{in}\ \mathbf{p}_t + \mathbf{b}_{in} \\ \mathbf{A}_{eq}\ \mathbf{k}_u = \mathbf{B}_{eq}\ \mathbf{p}_t + \mathbf{b}_{eq} \end{cases} \quad (7)$$

involving the optimization argument $\mathbf{k}_u(t)$ and the vector of context parameters:

$$\mathbf{p}_t = \begin{bmatrix} \mathbf{y}_{past} & \{\gamma_{past}\}^{\mathrm{T}} & \Delta\mathbf{u}_{past}^{\mathrm{T}} & \mathbf{w}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \quad (8)$$

which regroups a finite sequence of past inputs $\Delta\mathbf{u}$, future setpoints $\mathbf{w}$, past outputs $\mathbf{y}$ and present and future values of the signals under constraints (noted for short γ_{past}).

It is interesting to observe that this set of constraints characterizes in fact a parameterized polyhedron (Olaru and Dumur, 2004) in the optimization argument space. The optimum will lie on a combination of its parameterized vertices and thus one can construct the explicit solution for the multiparametric optimization:

The use of a dual representation of the feasible domain and projection mechanisms (see Olaru and Dumur, 2004 and 2005) can be advantageous in order to express the optimality conditions if there exist unbounded feasible directions.

$$\min_{\mathbf{k}_u(t)} J$$
$$\text{subject to}: \begin{cases} \mathbf{A}_{in}\mathbf{k}_u(t) \leq \mathbf{B}_{in}\mathbf{p}_t + \mathbf{b}_{in} \\ \mathbf{A}_{eq}\mathbf{k}_u(t) = \mathbf{B}_{eq}\mathbf{p}_t + \mathbf{b}_{eq} \end{cases} \quad (9)$$

Subsequently, the predictive control law can be described explicitly upon the solution of (9) as a piecewise affine function of the vector of parameters (Seron *et al*, 2003; Bemporad *et al*., 2002; Olaru and Dumur, 2004).

$$\mathbf{k}_u^*(\mathbf{p}) = K_{Lin_i}\mathbf{p} + K_{tli} \quad \text{for} \quad \mathbf{p} \in CR_i \quad (10)$$

with CR_i, critical regions in the space of context parameters, The GPC subjet to constraints has a piecewise RST polynomial form:

$$S_i(q^{-1})\Delta u_t = -R_i(q^{-1})y_t + T_i(q)w_t + V_i\gamma_{past} + tl_i \quad (11)$$

for $\begin{bmatrix} \mathbf{y}_{past} & \{\gamma_{past}\}^{\mathrm{T}} & \Delta\mathbf{u}_{past}^{\mathrm{T}} & \mathbf{w}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in CR_i$.

The structure of such a piecewise controller is shown in Figure 2. Once the look-up table of local polynomial RST laws is available, an efficient positioning mechanism (based on a search tree) can be constructed such that the on-line evaluation routine can find the optimal control action according to the GPC philosophy (TØndel *et al*., 2003).

4 ROBUSTIFICATION

At this stage, it is assumed that the design of initial controller has been performed with N_1, N_2, N_u, λ adjusted to satisfy the required input/output behavior. The resulting picewise two-degrees of freedom *RST* controller will be denoted R_i', S_i', T_i', in the following sections.

The observer based robustification corresponds to the consideration of an observer of the state. In the case of GPC, this corresponds to a choice of a *C* polynomial. The roots of this polynomial correspond to the poles of the observer. The obtained piecewise controller can be implemented as in Figure 2, see also (Olaru and Rodriguez-Ayerbe, 2006, Camacho and Bordons, 2004 ; Bitmead *et al.*,1990).

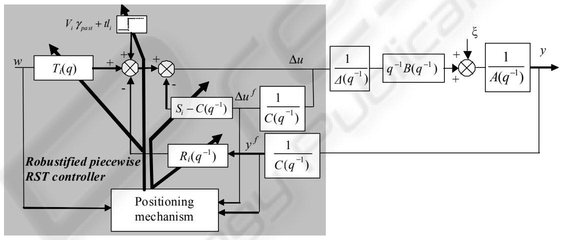

Figure 2: Piecewise RST formulation for the GPC law under constraints and robustification using the C polynomial.

The Youla-Kučera parameter based robustification has more degrees of freedom than the use of an observer. It allows to access the entire space of stabilizing controllers. The idea is thus to robustify the central RST law of the piecewise controller, that is, the law where the constraints are not activated, and to expand this robustification to the others RST laws of the initial piecewise controller. The choice of this parameter can be done using stability robustness and nominal performance specifications, see (Rodriguez and Dumur, 2005; Rossiter 2003; Ansay *et al*., 1998; Yoon and Clarke, 1995; Kouvaritakis *et al*., 1992). In the following the selection will be done according to these principles.

As the use of an observer corresponds to the consideration of a *C* polynomial for the noise influence on the CARIMA model (1), the idea is to find the corresponding noise model of the Youla-Kučera parameter. To do this, an extra polynomial is added in the model of the system. A $D(q^{-1})$ polynomial will appear as following:

$$A(q^{-1})y_t = B(q^{-1})u_{t-1} + \frac{C(q^{-1})\xi_t}{D(q^{-1})\Delta(q^{-1})} \quad (12)$$

With this new model, Diophantine equations are:

$$\begin{aligned}&\Delta(q^{-1})A(q^{-1})D(q^{-1})E_j(q^{-1})+q^{-j}F_j(q^{-1})=C(q^{-1})\\&G_j(q^{-1})C(q^{-1})+q^{-j}H_j(q^{-1})=B(q^{-1})E_j(q^{-1})D(q^{-1})\end{aligned} \tag{13}$$

Finding the relation between the controller obtained for C=D=1 and the one obtained for $C \neq 1$ and $D \neq 1$, we obtain something similar to (3). Considering R', S', T' the controller obtained with C=D=1 and $\widetilde{R}, \widetilde{S}, \widetilde{T}$ whose obtained with $C \neq 1$ and $D \neq 1$, the following relations are obtained:

$$\widetilde{R} = R'C + A\Delta\widetilde{M} \quad \widetilde{S} = S'C - q^{-1}B\widetilde{M} \quad \widetilde{T} = T'C \tag{14}$$

With, (see Appendix for structural details) :

$$\widetilde{M}(q^{-1}) = \sum_{i=N_1}^{N_2} \alpha_i q^i (CE_i' - \widetilde{E}_i D) \tag{15}$$

So, the D polynomial corresponding to the considered Youla-Kučera parameter must verify:

$$Q_{num}(q^{-1}) = \widetilde{M}(q^{-1}) = \sum_{i=N_1}^{N_2} \alpha_i q^i (CE_i' - \widetilde{E}_i D) \tag{16}$$

Once the corresponding noise model has been obtained, it can be used to regenerate the piecewise affine controller. The same input/output behaviour as for the initial one is assured, in the ideal case of no model errors. A modified close loop behaviour will be observed with respect to disturbance rejection, robustness, etc.

The resolution of (16) is a non linear problem that can be undertaken with standard optimization methods. Nevertheless, is not always possible to guarantee a *real* solution. The resolution of (16) and its limitations are raising interesting questions, research being currently conducted on this subject. From a practical point of view, any such limit case can be avoided by retuning the initial predictive control parameters or the robustification specification.

5 EXAMPLE

Consider the position control of an induction motor, with 1.0724 ms as sampling period

$$H(q^{-1}) = \frac{\theta(q^{-1})}{\tau_{\text{ref}}(q^{-1})} = \frac{10^{-4}(0.821q^{-1} + 0.8206q^{-2})}{(1-q^{-1})(1-0.998q^{-1})} \tag{17}$$

Constraints in control amplitude are considered: $\tau_{\text{ref}} \in [\tau_{\max}, -\tau_{\max}]$ and $\tau_{\max} = 1.8$. An initial GPC controller is designed with $C = D = 1$ with the following tuning parameters: $N_1 = 1$, $N_2 = 16$, $\lambda = 0.0001$ and $N_u = 2$. The position of the motor is obtained through an encoder of 14400 points per rotation, and the highs dynamics of the system (current loop, inverter dynamic, mechanic dynamics in high frequency) have been not identified.

This initial controller is obtained with (9). A piecewise linear controller with 9 regions is obtained. The central region corresponds to the case where no constraint is active. This controller will be noted R_0, S_0, T_0.

To robustify off-line this piecewise controller, the idea is to robustify the central one (R_0, S_0, T_0) and expand this robustification to other regions. In this way a Youla-Kučera parameter has been obtained by method described in (Rodriguez and Dumur, 2005). The following parameter is considered.

$$Q = \frac{-4196.2 + 10499.99q^{-1} + 8902.17q^{-2} + 2541.93q^{-3}}{1 - 3.565q^{-1} + 4.838q^{-2} - 2.973q^{-3} + 0.7q^{-4}} \tag{18}$$

With (6), we obtain the controller R_{0Q}, S_{0Q}, T_{0Q}.

Solving (16) with $C = Q_{den}$, the following D polynomial is obtained:

$$D = 1 - 0.873q^{-1} + 0.472q^{-2} - 0.018q^{-3} + 0.426q^{-4} \tag{19}$$

This value has been obtained by available optimization methods (classical Matlab routines in occurrence) as long as (16) represents a set of non linear equations difficult to solve analytically.

With this D polynomial, the optimization problem (9) can be solved but this time with matrices obtained from (13) for $C = Q_{den}$ and D as in (19). The solution of this new optimization problem leads to a new piecewise controller with 9 regions, as the initial one. The central controller of this piecewise controller correspond to R_{0Q}, S_{0Q}, T_{0Q}.

Figures 3 and 4 show the obtained simulations results for a filtered step reference considering a second order neglected dynamic in high frequency of the following characteristics: $\omega_0 = 1000 rad/s$ $\xi = 0.3$.

In these figures we can observe that the obtained behaviour is stable in the case of robustified controller and instable in the case of initial controller. So, the robustified controller has better behaviour towards uncertainties in high frequency and the continuity between regions is guaranteed.

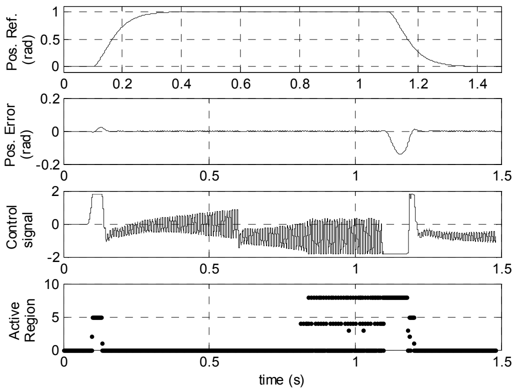

Figure 3: Position reference, position error, control signal and active region for the initial controller and uncertain model.

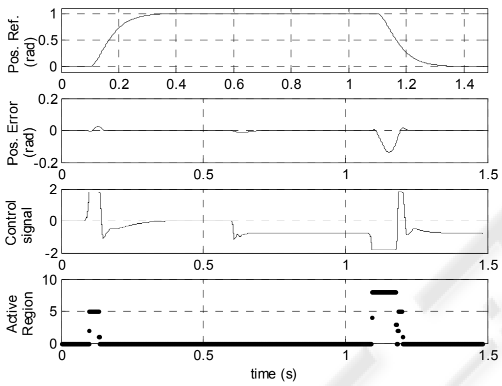

Figure 4: Position reference, position error, control signal and active region for the robustified controller and uncertain model.

6 CONCLUSIONS

The paper investigated the robustification methods for the control laws obtained in a constrained predictive control framework. The idea is to design in a first instance a piecewise polynomial controller which satisfy the basic demands in terms of tracking performances. In a second stage, the same predictive control structure (prediction horizon, weightings, etc.) is robustified using the model arguments accounting for the noise influence. The idea is similar to that of using a fixed observer, but exploring all the class of stabilizing controllers of the unconstrained system. This increases the number of degrees of freedom.

The robustification of initial unconstrained controller is made using the Youla-Kučera parametrization, and then this robustification is expanded to all the piecewise structure of the controller. For this, the noise model corresponding to the Youla-Kučera parameter is found, and use to regenerate the robust piecewise controller by preserving the same input/output behavior but being more robust.

The limitations of the method are in the existence of the corresponding noise model of the Youla-Kučera parameter. This is transparent in the resolution of a non linear equation system. The robustification being done off-line, any infeasibility can be handled by retuning the GPC parameters.

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APPENDIX

By solving the first Diophantine equation of (13) for C=D=1 and $C \neq 1$, $D \neq 1$, a relationship between the R' polynomial obtained for C=D=1 and the $\widetilde{R}$ obtained for $C \neq 1$, $D \neq 1$ is obtained:

$$\begin{cases} \Delta(q^{-1})A(q^{-1})E'_j(q^{-1}) + q^{-j}F'_j(q^{-1}) = 1 \\ \Delta(q^{-1})A(q^{-1})D(q^{-1})\widetilde{E}_j(q^{-1}) + q^{-j}\widetilde{F}_j(q^{-1}) = C(q^{-1}) \end{cases}$$

$$\Rightarrow \widetilde{F}_j = CF'_j + \Delta A q^j (CE'_j - D\widetilde{E}_j)$$

R is obtained as: $R = \sum_{i=N_1}^{N_2} \alpha_i F_i$ (see Yoon and Clarke 1995).

For $\widetilde{R}$:

$$\widetilde{R} = \sum_{i=N_1}^{N_2} \alpha_i \widetilde{F}_i = \sum_{i=N_1}^{N_2} \alpha_i (CF'_i + \Delta A q^i (CE'_i - D\widetilde{E}_i))$$

$$= CR' + A\Delta \sum_{i=N_1}^{N_2} \alpha_i q^i (CE'_i - D\widetilde{E}_i)$$

$$= CR' + A\Delta M$$

With the same development, the corresponding expression for $\widetilde{S}$ of (14) is obtained. Solving (16) comes to the identification of a D polynomial corresponding to $Q_{num}(q^{-1})$. It must be noted that the first i coefficients of E_{i+1} are the same than those of E_i polynomials. (E_i is a polynomial of degree i-1.) With this, the solution of D for the numerical example has been obtained solving the following non linear equations:

$$D = 1 + d_1 q^{-1} + d_2 q^{-2} + d_3 q^{-3} + d_4 q^{-4}$$
$$C = 1 + c_1 q^{-1} + c_2 q^{-2} + c_3 q^{-3} + c_4 q^{-4}$$
$$E' = 1 + e'_1 q^{-1} + \ldots + e'_{15} q^{-15}$$
$$\widetilde{E} = 1 + \widetilde{e}_1 q^{-1} + \ldots + \widetilde{e}_{15} q^{-15}$$
$$M = m_0 + m_1 q^{-1} + m_2 q^{-2} + m_3 q^{-3}$$

$$\left\{ \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_3 & c_4 & 0 \\ c_3 & c_4 & 0 & 0 \\ c_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & e'_1 & \cdots & & & e'_{15} \\ 0 & 1 & \cdots & & & e'_{14} \\ 0 & 0 & \ddots & & & e'_{13} \\ 0 & 0 & 0 & 1 & \cdots & e'_{12} \end{pmatrix} - \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ d_2 & d_3 & d_4 & 0 \\ d_3 & d_4 & 0 & 0 \\ d_4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \widetilde{e}_1 & \cdots & & & \widetilde{e}_{15} \\ 0 & 1 & \cdots & & & \widetilde{e}_{14} \\ 0 & 0 & \ddots & & & \widetilde{e}_{13} \\ 0 & 0 & 0 & 1 & \cdots & \widetilde{e}_{12} \end{pmatrix} \right\} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_{16} \end{pmatrix} - \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} \quad (20)$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & & & & 0 \\ c_1 & 1 & 0 & & & & & \vdots \\ c_2 & c_1 & 1 & \ddots & & & & \\ c_3 & c_2 & c_1 & \ddots & \ddots & & & \\ c_4 & c_3 & c_2 & c_1 & 1 & \ddots & & \\ 0 & c_4 & c_3 & \cdots & & \ddots & 0 & \vdots \\ \vdots & \vdots & & & & & 1 & 0 \\ 0 & \cdots & 0 & c_4 & c_3 & c_2 & c_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ e'_1 \\ \vdots \\ \\ \\ \\ \vdots \\ e'_{15} \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & \cdots & & & & 0 \\ d_1 & 1 & 0 & & & & & \vdots \\ d_2 & d_1 & 1 & \ddots & & & & \\ d_3 & d_2 & d_1 & \ddots & \ddots & & & \\ d_4 & d_3 & d_2 & d_1 & 1 & \ddots & & \\ 0 & d_4 & d_3 & \cdots & & \ddots & 0 & \vdots \\ \vdots & \vdots & & & & & 1 & 0 \\ 0 & \cdots & 0 & d_4 & d_3 & d_2 & d_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \widetilde{e}_1 \\ \vdots \\ \\ \\ \\ \vdots \\ \widetilde{e}_{15} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad (21)$$

With $\mathbf{x} = (d_1 \ \ d_2 \ \ d_3 \ \ d_4 \ \ \widetilde{e}_1 \ \ \cdots \ \ \widetilde{e}_{15})$, the following problem has been solved:

$$F(\mathbf{x}) = 0$$

With F defined by the matrix relations (20) and (21), and using a standard optimization routine.