ONE MODIFICATION OF THE CUSUM TEST FOR DETECTION EARLY STRUCTURAL CHANGES

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Abstract: Structural shifts in time series can occur as consequences of complex processes, arising in system. An ignorance of such structural changes can cause an associated regression model misspecification. In practice, early detection and response to outbreaks, causing the changes in a process, is highly important. The famous CUSUM test of Brown, Durbin & Evans, has a poor power in detecting the structural breaks in parameters occurring early (and also late) in the sample. In this paper, we propose CUSUM-similar test which, due to the transformation of recursive residuals forces the detection of temporal dependence structure in linear regression model and has a larger power for the early structural breaks. Here our interest centres on the detection of single breaks occurring in parameters of the linear model. Distribution and other probabilistic characteristics of the transformed residuals are provided, the boundaries for the new test are derived. The new test can be considered then as a complement to the standard CUSUM test.

1 INTRODUCTION

Structural shifts in time series can occur as consequences of complex processes, arising in system (Basseville and Nikiforov, 1993). An ignorance of such structural changes can cause an associated regression model misspecification. The evidence of the parameters instability in linear models can be detected by the number of corresponding diagnostic tests. Particularly, the famous and important of them include fluctuation tests, with the CUSUM and CUSUMQ tests of Brown, Durbin & Evans (BDE tests) (R. L. Brown and Evans, 1975), standing first on the list. These tests are easy for implementing and based on the calculating of the cumulative sums of recursive residuals (CUSUM) and the cumulative sums of the squares of recursive residuals (CUSUMQ) for regression. The CUSUM test is generally used to detect the systematic movements of parameters, whereas CUSUMQ test tends to capture the sudden, haphazard movements.

A number of extensions of the standard BDE CUSUM tests have been developed, like the CUSUM and CUSUM of squares tests using OLS (ordinary least squares) residuals instead of recursive residuals (Proberger and Kraemer, 1992), the CUSUM tests with lagged dependent variables in regression (W. Proberger and Alt, 1989), the CUSUM tests with non-stationary regressors (Inder and Hao, 1996), the MOSUM and MOSUM of squares tests putting emphasis on moving averages rather than cumulative sums (C.-S. J. Chu and Kuan, 1995a), the moving-estimates test (C.-S. J. Chu and Kuan, 1995b), etc.

It is a well-known fact, that, CUSUM tests, both with recursive and OLS residuals, have a poor power in detecting the structural breaks in parameters occurring early and late in the sample, as well as the ones orthogonal to the mean regressor (see (W. Kraemer and Alt, 1988); (Kraemer and Sonnberg, 1986), pp. 50-51; (Proberger and Kraemer, 1990); (Proberger and Kraemer, 1992)). A modification of recursive residuals CUSUM test, which is robust to the later problem, was proposed in (Luger, 2001). The reason for earlier problem is that CUSUM has no chance to cumulate for the such kind of breaks. In particular, some improving, due to alternative test boundaries developing, was suggested, in (Zeileis, 2000) for OLS CUSUM test.

In practice, early detection and response to outbreaks, causing the changes in a process, is highly important. In this paper, we propose a CUSUM-similar test which, due to the transformation of recursive residuals forces the detection of temporal dependence structure in linear regression model and has

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a larger power for the early structural breaks. Here our interest centres on the detection of single breaks occurring in parameters of the linear model. Distribution and other probabilistic characteristics of the transformed residuals are also provided, the boundaries for the new test are derived. The new test can be considered then as a complement to the standard CUSUM test. The paper is organized as follows. Section 2 presents the BDE's original formulation of the CUSUM test, discusses its main features. Section 3 offers our modification of the CUSUM testing procedure. In the Section 4 we conduct comparison of two tests via Monte Carlo simulation study. Section 5 summarizes and concludes.

2 STANDARD CUSUM TEST

Consider the linear regression model

$$y_t = x_t'\beta_t + \varepsilon_t \ , t = 1, ..., T, \tag{1}$$

where y_t is the tth observation of the dependent variable, x_t is the $k \times 1$ vector of covariates at time t, β_t are unknown parameters, and ε_t are independent and normally distributed with zero mean and variance σ^2. We are interested in testing for a discrete jump in at least one of the β_t at the unknown time point. The null hypothesis can be formulated as

$$H_0: \ \beta_t = \beta, t = 1, ..., T, \tag{2}$$

with alternative H_1 that at time T^* at least one of the β_t changes its value:

$$H_1: \ \beta_t = \beta, t = 1, ..., T^* < T; \beta_t \neq \beta, \ t = T^* + 1, ..., T. \tag{3}$$

The CUSUM test suggested by Brown, Durbin and Evans (BDE) is a standard and commonly used diagnostic test for this kind of situations. BDE's CUSUM test uses the standardized residuals defined as

$$w_t = \frac{y_t - x_t'\widehat{\beta}_{t-1}}{\sqrt{1 + x_t \left(X_{t-1}'X_{t-1}\right)^{-1} x_t'}} \text{ for } t = k+1, k+2, ..., T, \tag{4}$$

where $\widehat{\beta}_{t-1}$ is the OLS-estimator of β based on the first $t-1$ observations, $\widehat{\beta}_{t-1} = \left(X_{t-1}'X_{t-1}\right)^{-1} X_{t-1}Y_{t-1}$, and X_{t-1} and Y_{t-1} are the $(t-1) \times k$ and $(t-1) \times 1$ matrices that obtain by stacking x_s and y_s, respectively, for $s = 1, 2, ..., t-1$.

The advantage of working with w_ts defined by (4): it can be shown, that under H_0 (2), they are independent normally distributed with zero mean and variance σ^2.

BDE CUSUM test is based on the cumulated sums of standardized residuals:

$$C_t = \frac{1}{\widehat{\sigma}} \sum_{s=k+1}^{t} w_s, \tag{5}$$

where $\widehat{\sigma}$ is OLS-estimate of σ, $\widehat{\sigma^2} = \frac{1}{T-k} \sum_{s=k+1}^{t} w_s^2$. We cannot obtain the explicit distribution of C_t. But, under null hypothesis of parameter stability, the expected value and variance of the statistic C_t should be equal to zero and the number of normalized residuals being summed. Hence, the continuous Brownian motion process Z_t can be considered as a good approximation of the discrete path of C_t. Under H_0 the sequence C_t is thus a sequence of the approximately normal variables, where $E(C_t) = 0$, $Var(C_t) = t - k$, $Cov(C_t, C_r) = \min(t, r) - k$. Confidence bounds for the cumulated sums C_t are then obtained by plotting the two straight lines connecting the points $k \pm a\sqrt{T-k}$ and $T \pm 3a\sqrt{T-k}$, where a is a parameter depending on the α - significance level chosen, and is calculated on the basis of results for the Gaussian process. Namely, $a = 1.143$ for $\alpha = 0.01$; $a = 0.948$ for $\alpha = 0.05$; $a = 0.850$ for $\alpha = 0.1$ (see (R. L. Brown and Evans, 1975)).

Unfortunately, the CUSUM test suffers from low power, which decreases dramatically if the change point is close to the early beginning or to the end of the sample. In the next section we consider one modification of the CUSUM test based on the sconced cumulative sums.

3 MODIFICATION OF CUSUM TEST

Let's take into account the temporal structure of the residuals w_t, and construct a new sequence of random variables u_t, $t = k+1, ..., T$:

$$u_t = w_t + c \ w_t \ I\{w_t \ w_{t-1} > 0\}, \quad w_k = 0. \tag{6}$$

where, as before, $w_t \sim N\left(0, \sigma^2\right)$ and independent, $I\{\cdot\}$ is indicator function, and c is some constant, $c > 0$, which can be considered as a penalty magnitude for upward or downward trends in CUSUM values. We will obtain now a distribution of the variables u_t.

Theorem. Let w_t are independent identically distributed random variables, having a common symmetric about zero distribution, where $F_w(x)$ is a probability function, and $c > 0$. Then

$$F_{u_t}(x) = \frac{1}{2}\left[F_{w_t}(x) + F_{w_t}\left(\frac{x}{1+c}\right)\right]. \qquad (7)$$

Proof. We can perform $F_{u_t}(x)$ as a sum of the following probabilities:

$$\begin{aligned} F_{u_t}(x) &= P[u_t < x] = P[w_t + c\, w_t\, I\{w_t\, w_{t-1} > 0\} < x] = \\ & P[w_t + c\, w_t \ < x\, ;\ w_t\, w_{t-1} > 0] + \\ & P[w_t < x\, ;\ w_t\, w_{t-1} \le 0]. \end{aligned} \qquad (8)$$

For simplicity let's consider two cases:
1) $x \le 0$. Then (8) can be written as:
$P[w_t + c\, w_t \ < x\, ;\ w_{t-1} < 0]$ + $P[w_t \le x\, ;\ w_{t-1} \ge 0] = P\left[w_t < \frac{x}{1+c}\, ;\ w_{t-1} < 0\right]$ + $P[w_t \le x\, ;\ w_{t-1} \ge 0] = F_{w_t}\left(\frac{x}{1+c}\right) F_{w_t}(0)$ + $F_{w_t}(x)(1 - F_{w_t}(0)) = F_{w_t}\left(\frac{x}{1+c}\right) F_{w_t}(0) + F_{w_t}(x) -$ $F_{w_t}(x) F_{w_t}(0) = \frac{1}{2}\left[F_{w_t}\left(\frac{x}{1+c}\right) + F_{w_t}(x)\right]$.
2) $x > 0$. (8) has a form:
$P[w_t + c\, w_t \ < 0\, ; w_t\, w_{t-1} > 0]$ + $P[0 \le w_t + c\, w_t < x\, ;\ w_t\, w_{t-1} > 0] +$ $P[w_t \ < x\, ; w_t\, w_{t-1} \le 0] = P[w_t < 0\, ;\ w_{t-1} < 0]$ + $P\left[0 \le w_t < \frac{x}{1+c}\, ;\ w_{t-1} > 0\right]$ + $P[w_t \ < x\, ; w_t\, w_{t-1} \le 0] = F_{w_t}(0) F_{w_t}(0)$ + $\left(F_{w_t}\left(\frac{x}{1+c}\right) - F_{w_t}(0)\right) F_{w_t}(0) +$ $P[w_t \ < 0\, ; w_{t-1} \ge 0] + P[0 \le w_t < x\, ;\ w_{t-1} < 0] =$ $F_w(0) F_w(0) +$ $\left(F_{w_t}\left(\frac{x}{1+c}\right) - F_{w_t}(0)\right) F_{w_t}(0) + F_{w_t}(0)(1 - F_{w_t}(0)) +$ $(F_{w_t}(x) - F_{w_t}(0)) F_{w_t}(0) = \frac{1}{4} + \frac{1}{2}\left(F_{w_t}\left(\frac{x}{1+c}\right) - \frac{1}{2}\right) +$ $\frac{1}{4} + \frac{1}{2}\left(F_{w_t}(x) - \frac{1}{2}\right) = \frac{1}{2}\left[F_{w_t}\left(\frac{x}{1+c}\right) + F_{w_t}(x)\right]$.

Thus, the CDF (PDF) of the random variables u_t is a mixture of two normal distributions. It follows from the Theorem that u_ts have zero expectation, $E(u_t) = 0$, and variance $Var(u_t) = \frac{\sigma^2}{2}\left(1 + (1+c)^2\right)$. The joint CDF function obtained can be obtained by analogy with (7).

Covariance, $Cov(u_t, u_{t+1})$= $E[(u_t - E(u_t))(u_{t+1} - E(u_{t+1}))]$=$E(u_t\, u_{t+1})$, equals to zero. In addition, variables u_ts have zero skewness $s(u_t) = 0$, and kurtosis $\kappa(u_t) = \frac{6\left[1+(1+c)^4\right]}{\left(1+(1+c)^2\right)^2}$, revealing fatter tails.

The new statistic, which we will call "penalized" CUSUM (PCUSUM), can be written as following:

$$C_t^p = \frac{1}{\widehat{\sigma_p}} \sum_{s=k+1}^{t} u_s = \frac{1}{\widehat{\sigma^p}}\left(\sum_{s=k+1}^{t} w_s + c \sum_{s=k+1}^{t} w_s I\{w_s\, w_{s-1} > 0\}\right), \qquad (9)$$

where $\widehat{\sigma_p}^2 = \frac{1}{T-k} \sum_{s=k+1}^{t} u_s^2$. It is follows from the properties of u_t, that $E\left(C_t^p\right) = 0$ but $\widehat{\sigma_p} \ge \widehat{\sigma}$, thus, we expect that the PCUSUM test has a chance to cumulate well and to detect structural shifts only in very beginning of the sample.

Like the original CUSUM test (5), the modified test (9) is only an asymptotic test and has no any certain distribution. Obviously, under $c \to 0$, the sequence $C_{k+1}^p, ..., C_t^p$ may be approximated by the Brownian motion process mentioned in Section 1, and the bounds for cumulated sums C_t^p are calculated then as the ones for C_t. But for larger values c this can not be applied. Hence, we have here a subproblem of simulation of the presented test boundaries.

We will partially implement the techniques used in (Tanizaki, 1995) for confidence intervals calculation. The simulation algorithm may be carried out as follows. Let us generate L replicates. In each replicate i, $i = 1, 2, ..., L$, we simulate $T - K$ random variables $u_{i,t}$, $t = k+1, k+2, ..., T$, pairwise dependent and uncorrelated, drawn from the mixture of two bivariate normal distributions (7). Then the cumulated sums, $s_{i,t}^p = \frac{1}{\widehat{\sigma^p}} \sum_{s=k+1}^{t} u_{i,s}$, are calculated. At the significance level α we have for statistic (9) that $P[L_{k+1}^p < C_{k+1}^p < U_{k+1}^p, ..., L_T^p < C_T^p < U_T^p] = 1 - \alpha$, where L_t^p and U_t^p are correspondingly the lower and upper bounds for the value C_t^p, and

$$P\left[L_{k+1}^p < C_{k+1}^p < U_{k+1}^p, ..., L_T^p < C_T^p < U_T^p\right] \neq \left(P\left[L_t^p < C_t^p < U_t^p\right]\right)^{T-k}$$

for any $t = k+1, ..., T$, since C_t^p are not independent. Let assume that $P\left[L_t^p < C_t^p < U_t^p\right] = 1 - \alpha_p$, and denote $\alpha = f(\alpha_p)$, where f is some unknown function (in the case of independence one has $f(x) = 1 - x^{T-k}$), which we will obtain by simulation. Applying the Newton-Raphson algorithm, we calculate our α_p following the scheme:

$$\alpha_p^{(j)} = \alpha_p^{(j-1)} + d^{(j)}\left(\alpha - f\left(\alpha_p^{(j-1)}\right)\right),$$

where j is iteration number, $d^{(j)} = \delta d^{(j-1)}$ with $\delta = 0.5$ and $d^{(0)} = 1$, $\alpha_p^{(0)} = \alpha$. The convergence criterion is $\left|\alpha_p^{(j)} - \alpha_p^{(j-1)}\right| < 0.0001$. Repeat, that the function $f\left(\alpha_p^{(j)}\right)$ is derived at jth iteration as following: $f\left(\alpha_p^{(j)}\right)$ equals to the number of sequences $\left\{s_{i,k+1}^p, ..., s_{i,T}^p\right\}_{i=1}^{L}$ within intervals $\left(L_{i,k+1}^p, U_{i,k+1}^p\right)^{(j)}$,..., $\left(L_{i,T}^p, U_{i,T}^p\right)^{(j)}$ divided by L, where intervals $\left(L_{i,t}^p, U_{i,t}^p\right)^{(j)}$ are obtained for the value $\alpha_p^{(j)}$. Note, that unlike the standard CUSUM statistic, which has symmetric lower and upper boundaries, we don't claim here $L_t^p = -U_t^p$ for PCUSUM statistic. To be fair to the standard

CUSUM, we use a linear regression to obtain the linear boundaries from the curved ones: $\widetilde{L}_t^p = a_{1L} + a_{2L}t$ and $\widetilde{U}_t^p = a_{1U} + a_{2U}t$, where a_{1L}, a_{1U} are intercepts, a_{2L}, a_{2U} are slopes, $\widetilde{L}_t^p$, $\widetilde{U}_t^p$ are the best fitting lines.

4 MONTE CARLO STUDY

In this section we present a Monte Carlo study of the new CUSUM test in comparison with the classical one. In order to see the performance of C_t^p, we consider the model (1), where the matrix of independent variables x_t has the following design:

$$X^{MC} = \left\{ \left[1, (-1)^t\right]' \right\}_{t=1}^{T}, \qquad (10)$$

which was also used in simulation study in (Inder and Hao, 1996) and (Kraemer and Sonnberg, 1986). Values ε_t, as before, are independent normal, generated with parameters 0 and 1. We simulate our responses y_t for three sample sizes, $T = 20$ (small sample), $T = 50$ (medium sample) and $T = 500$ (large sample), with $\beta = [10, 2]'$ under the null hypothesis of parameters constancy. The significance level $\alpha = 0.05$, the computed empirical levels α_p for samples under $c = \{0.2;\ 0.5;\ 1.5\}$ are correspondingly $\alpha_p = \{0.0075; 0.0049; 0.0054\}$ ($T = 20$), $\alpha_p = \{0.0055; 0.0036; 0.0029\}$ ($T = 50$), $\alpha_p = \{0.0011; 0.0013; 0.0016\}$ ($T = 500$).

Empirical (actual) test sizes, based on the simulated data, at the nominal size of 5%, and $c = \{0.2;\ 0.5;\ 1.5\}$ in PCUSUM test, are presented in the Table 1. Generally estimated sizes, calculated as rejection rates under null hypothesis, with Monte-Carlo replications number $N = 5000$, are either below the nominal size for CUSUM test, resulting in more "liberal" test, or almost equal to the nominal size. However, the empirical sizes for PCUSUM test under $c = 1.5$ are larger than 0.05, declaring the more "sensitive" test.

Table 1: Empirical Sizes of the Tests, $\alpha = 0.05$.

Test	$T = 20$	$T = 50$	$T = 500$
CUSUM	0.0154	0.0276	0.0458
PCUSUM, $c = 0.2$	0.0338	0.0453	0.0476
PCUSUM, $c = 0.5$	0.0512	0.0510	0.0444
PCUSUM, $c = 1.5$	0.0686	0.0586	0.0646

Now, we introduce at time $T^* = [\lambda T]$, where λ can take any values between 0 and 1, some structural shift. Let's consider a single structural shift in parameters β is given by $\Delta\beta = \frac{b_0}{\sqrt{T}}[\cos\phi,\ \sin\phi]'$, where ϕ is the angle between $\Delta\beta$ and mean regressor $r = \left[\frac{1}{T}\sum_{t=1}^{T} x_{1t}, \frac{1}{T}\sum_{t=1}^{T} x_{2t}\right]' = [1, 0]'$, b_0 is a constant determining the intensity of the shift , $\|\Delta\beta\| = \frac{|b_0|}{\sqrt{T}}$. We will take a number of different values of b_0 and ϕ, namely, $b_0 = \{-12;\ -8;\ -6;\ -3; 3; 6; 8; 12\}$ (positive and negative values, as we don't claim the boundaries of PCUSUM to be symmetric) and $\phi = \{0\}$. In other words, we are testing two hypotheses: H_0 : parameters are constant for $1 \le t \le T$ against H_1 : parameters have two different constant values, for $1 \le t < T^*$ and $T^* \le t \le T$.

It is possible to show that, as we have expected, the linear boundaries for PCUSUM test become narrower than CUSUM test boundaries only at the beginning of the sample. Hence, it makes sense to choose λ corresponding to the structural changes at the beginning of the sample, $\lambda = 0.3$, for example. Empirical power was calculated as a probability that the test statistic under the alternative hypothesis exceeded the significance threshold calculated from the distribution under the null hypothesis (a frequency of the null hypothesis rejection under the alternative hypothesis). The obtained power plots, under $N = 1000$, $\lambda = 0.3$, $\alpha = 5\%$, $c \in \{0.2; 0.5; 1.5\}$ for $T = 20$, $T = 50$ and $T = 500$ are presented on the Figures 1, 2 and 3, correspondingly.

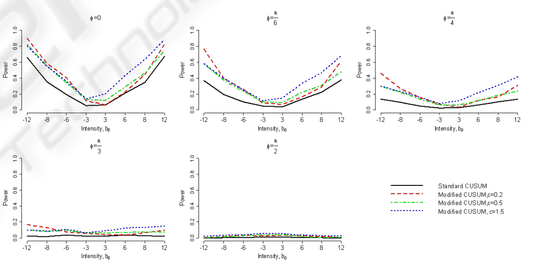

Figure 1: Power plots under $T = 20$, $\lambda = 0.3$.

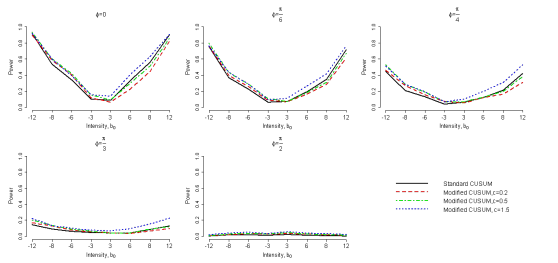

Figure 2: Power plots under $T = 50$, $\lambda = 0.3$.

The obtained simulation results for the small sample with $T = 20$ reveal that the PCUSUM outperforms the CUSUM. For small values of shift intensity b_0 and for $\phi = 90$ this superiority is quite insufficient, for all samples. For the medium sample $T = 50$, PCUSUM with $c = 1.5$ has higher power ev-

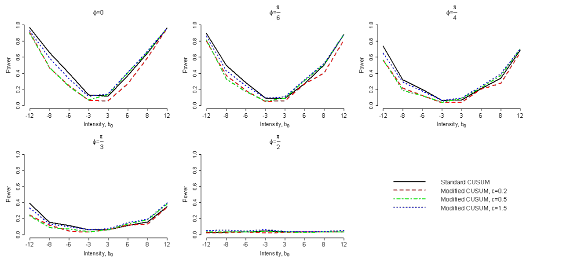

Figure 3: Power plots under $T = 500$, $\lambda = 0.3$.

erywhere, but PCUSUM with $c = 0.5$ and $c = 0.2$ - only for negative values of b_0. In big sample with $T = 500$ an advantage is exhibited only by PCUSUM with $c = 1.5$, for positive b_0s. By virtue of the boundaries non-symmetry, the results are different for positive and negative values of b_0 also among PCUSUM tests: for example, for $T = 20$ PCUSUM with larger parameter c outperforms PCUSUM with smaller one for positive b_0s, but for negative b_0s PCUSUM with smaller value of c remains more effective.

5 CONCLUSIONS

In this paper we proposed a modified, based on the penalized residuals, version of the standard BDE CUSUM test for single structural breaks in parameters of linear regression. The new test, PCUSUM, is recommended as a complement to the standard CUSUM test for better detecting the structural shifts occurring early in the samples. Simulation results have shown, that the modified CUSUM test has the better chance to cumulate parameter breaks, occurred at the beginning of the sample.

The subjects of eventual future research are:

- closer examination of properties of the PCUSUM test boundaries, their comparison with derived curved boundaries of the standard CUSUM test;
- adoption of the modified residuals (6) by CUSUM of squares test (CUSUMQ) for testing structural changes in variance (serial correlation and heteroscedasticity);
- appropriate transformation of the proposed new test for both early and late structural breaks (inclusion of the more lagged residuals in (6), etc).

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