

# OFF-LINE ROBUSTIFICATION OF PREDICTIVE CONTROL FOR UNCERTAIN SYSTEMS

## *A Sub-optimal Tractable Solution*

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**Abstract:** An off-line technique enabling to robustify an initial Model Predictive Control (MPC) for multivariable systems via the convex optimization of a Youla parameter is presented. Firstly, a multivariable predictive controller is designed for a nominal system and then robustified towards unstructured uncertainties, while guaranteeing stability properties over a specified polytopic domain of uncertainties. This condition leads to verify a Bilinear Matrix Inequality (BMI) for each vertex of the polytopic domain. This BMI can be mathematically relaxed to semi-definite programming (SDP) using a Sum of Squares (SOS) strategy, with a significant increase of the number of scalar decision variables. To overcome this inconvenient, an alternative tractable sub-optimal solution for the BMI is proposed, based on the elaboration of a stable solution obtained by minimization of the complementary sensitivity function.

## 1 INTRODUCTION

During the latest years, the robustness aspect of Model Predictive Control (MPC) has been considered both within online strategies (Kothare *et al.*, 1996; Goulart and Kerrigan, 2007; Camacho and Bordons, 2004) and off-line approaches (Wan and Kothare, 2003; Rossiter, 2003; Rodríguez and Dumur, 2005). Mixed methods computing off-line a set of controllers have been developed, leaving on-line only the selection of the current controller (Olaru and Dumur, 2004; Lee and Kouvaritakis, 2006).

This paper presents an off-line robustification procedure for model predictive control applied to multivariable (possibly non-square) uncertain systems. It considers both unstructured and polytopic uncertainties. Firstly, a predictive controller for a nominal system is designed. Secondly, the robustification problem under unstructured uncertainties is considered. This leads to a convex optimization of a multivariable Youla parameter solved with Linear Matrix Inequalities (LMIs) techniques, as described in (Stoica *et al.*, 2007). Thirdly, the robust stability of the controlled system towards system polytopic uncertainties is considered. Since the polytopic domain is chosen as a convex polytope, this implies checking the stability only for the vertices of the polytope (Kothare *et al.*, 1996). This condition leads

to satisfy a Bilinear Matrix Inequality (BMI) for all vertices of the polytopic domain. This problem can be transformed into semi-definite programming (SDP) using Sum of Squares (SOS) relaxations described in (Scherer and Hol, 2006), with a significant increase of the number of scalar decision variables. To avoid this increase of the computing time, this paper proposes a sub-optimal tractable solution based on the minimization of the complementary sensitivity function which permits to enlarge the stability domain. A feasible solution for each vertex can be found, the stability conditions for all the vertices of the polytopic domain being then explicitly integrated.

The most interesting result is that this robustification technique permits to guarantee the stability property on the entire polytopic uncertain domain, even if the initial MPC controller may be unstable for some regions of the polytopic domain.

This paper is organized as follows. The main steps leading to a MIMO MPC and the related class of stabilizing controllers are presented in Section 2. The robustification procedure under unstructured and polytopic uncertainties is detailed in Section 3. Finally, some concluding remarks are given in Section 4.

## 2 CLASS OF STABILIZING MPC

This section briefly presents the main steps leading to an initial stabilizing multivariable MPC in state-space formalism and the class of stabilizing controllers obtained via the Youla parameter. More details can be found in (Stoica *et al.*, 2007). Let us consider a discrete time MIMO LTI system with  $m$  inputs and  $p$  outputs, characterized by the 4-uplet  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$  of the state-space representation.

In order to cancel the steady-state errors, an integral action on the control vector is added, leading to an extended state-space description:

$$\begin{cases} \mathbf{x}_e(k+1) = \mathbf{A}_e \mathbf{x}_e(k) + \mathbf{B}_e \Delta \mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}_e \mathbf{x}_e(k) \end{cases} \quad (1)$$

Minimizing the quadratic objective function (2) gives the expression of the control signal. The following notations are used:  $\mathbf{y}_r$  - the setpoint;  $\tilde{\mathbf{Q}}_J$ ,  $\tilde{\mathbf{R}}_J$  - the weighting matrices. The future control increments  $\Delta \mathbf{u}(k+i)$  are supposed to be 0 for  $i \geq N_u$ . The same output prediction horizons ( $N_1$ ,  $N_2$ ) and the same control horizon  $N_u$  are applied for all input/output transfers.

$$J = \sum_{i=N_1}^{N_2} \|\hat{\mathbf{y}}(k+i) - \mathbf{y}_r(k+i)\|_{\tilde{\mathbf{Q}}_J(i)}^2 + \sum_{i=0}^{N_u-1} \|\Delta \mathbf{u}(k+i)\|_{\tilde{\mathbf{R}}_J(i)}^2 \quad (2)$$

The predicted output  $\hat{\mathbf{y}}(k)$  is derived from:

$$\hat{\mathbf{y}}(k+i) = \mathbf{C} \mathbf{A}^i \hat{\mathbf{x}}(k) + \sum_{j=0}^{i-1} \mathbf{C} \mathbf{A}^{i-j-1} \mathbf{B} \mathbf{u}(k+j) \quad (3)$$

with  $\hat{\mathbf{x}}(k)$  obtained from the following observer:

$$\hat{\mathbf{x}}_e(k+1) = \mathbf{A}_e \hat{\mathbf{x}}_e(k) + \mathbf{B}_e \Delta \mathbf{u}(k) + \mathbf{K}[\mathbf{y}(k) - \mathbf{C}_e \hat{\mathbf{x}}_e(k)] \quad (4)$$

An analytical minimization of (3) rewritten in a matrix form, as described in (Maciejowski, 2001), leads to the following control signal (Fig. 1):

$$\Delta \mathbf{u}(k) = \mathbf{F}_w \mathbf{w}(k) - \mathbf{L} \hat{\mathbf{x}}_e(k) \quad (5)$$

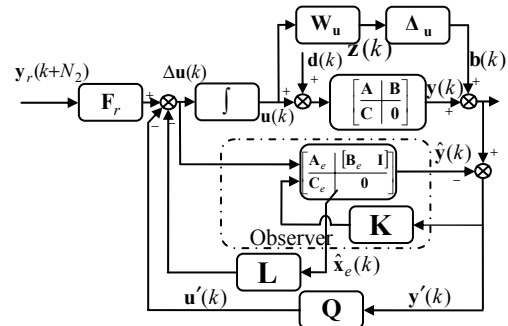


Figure 1: Robustified MIMO MPC via  $\mathbf{Q}$  parametrization.

The structure of the control gain matrix  $\mathbf{L} = [\mathbf{L}_1 \ \mathbf{L}_2]$  and the setpoint pre-filter  $\mathbf{F}_w$  are the same as in (Stoica *et al.*, 2007). The expression (5) provides an initial stabilizing controller. A possible way leading to the class of all stabilizing controllers is to use the Youla-Kučera parameter coupled with this control law. It is well known from the literature (Boyd and Barratt, 1991; Maciejowski, 1989) that any stabilizing controller can be represented by a state-space feedback controller coupled with an observer and a Youla (also called  $\mathbf{Q}$ ) parameter.

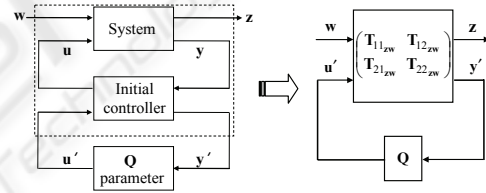


Figure 2: Class of stabilizing controllers with  $\mathbf{Q}$  parameter.

The first step is to add supplementary inputs  $\mathbf{u}'$  and outputs  $\mathbf{y}'$  with a zero transfer between them ( $\mathbf{T}_{22_{zw}} = 0$  in Fig. 2), which permits the connection of the  $\mathbf{Q}$  parameter between  $\mathbf{y}'$  and  $\mathbf{u}'$  without restricting the closed-loop stability. As a result, the closed-loop function between  $\mathbf{w}$  and  $\mathbf{z}$  is linearly parametrized by the  $\mathbf{Q}$  parameter, allowing convex specification (Boyd and Barratt, 1991):

$$\mathbf{T}_{zw} = \mathbf{T}_{11_{zw}} + \mathbf{T}_{12_{zw}} \mathbf{Q} \mathbf{T}_{21_{zw}} \quad (6)$$

where  $\mathbf{T}_{11_{zw}}, \mathbf{T}_{12_{zw}}, \mathbf{T}_{21_{zw}}$  depends on the considered input/output ( $\mathbf{w}/\mathbf{z}$ ) transfer.

### 3 ROBUSTNESS VIA THE YOULA PARAMETRIZATION

A procedure enhancing robustness of the previous multivariable MPC in terms of the Youla parameter is presented in the particular case of the maximization of the robust stability under additive unstructured uncertainties, while guaranteeing stability properties over a specified polytopic domain of uncertainties. It will be shown that the global robustification problem is a necessary trade-off between both robustification aspects.

#### 3.1 Robust Stability under Unstructured Uncertainties

Along with the small gain theorem (Maciejowski, 1989; Zhou *et al.*, 1996), a necessary and sufficient condition for the robust stability under unstructured uncertainties  $\Delta_u$  (Fig 3) is formulated as the following  $H_\infty$  norm minimization:

$$\min_{\mathbf{Q} \in \mathfrak{RH}_\infty} \|\mathbf{T}_{zw}\|_\infty \quad (7)$$

where  $\mathfrak{RH}_\infty$  is the space of stable transfers and  $\mathbf{T}_{zw}$  also contains the weighting factors.

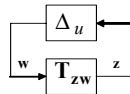


Figure 3: Unstructured uncertainty.

The minimization (7) may be more specifically formulated using the following theorem.

**Theorem** (Clement and Duc, 2000; Boyd *et al.*, 1994): A discrete time system given by the state-space representation  $(\mathbf{A}_{cl}, \mathbf{B}_{cl}, \mathbf{C}_{cl}, \mathbf{D}_{cl})$  is stable and admits a  $H_\infty$  norm lower than  $\gamma$  if and only if:

$$\exists \mathbf{X}_1 = \mathbf{X}_1^T > 0 / \begin{bmatrix} -\mathbf{X}_1^{-1} & \mathbf{A}_{cl} & \mathbf{B}_{cl} & \mathbf{0} \\ \mathbf{A}_{cl}^T & -\mathbf{X}_1 & \mathbf{0} & \mathbf{C}_{cl}^T \\ \mathbf{B}_{cl}^T & \mathbf{0} & -\gamma \mathbf{I} & \mathbf{D}_{cl}^T \\ \mathbf{0} & \mathbf{C}_{cl} & \mathbf{D}_{cl} & -\gamma \mathbf{I} \end{bmatrix} < 0 \quad (8)$$

where the notation “ $> 0$ ”/“ $< 0$ ” refers to a strictly positive/negative definite matrix. There exist appropriate techniques to transform the expression (8) into a LMI (Clement and Duc, 2000; Scherer, 2000). The decision variables should be  $\mathbf{X}_1$ ,  $\gamma$  and the  $\mathbf{Q}$  parameter included in the closed-loop matrices (Stoica *et al.*, 2007). As a result, the optimization

problem is formulated as the minimization of  $\gamma$  subject to this first LMI constraint:

$$\min_{LMI_0} \gamma \quad (9)$$

To restrict the search of the  $\mathbf{Q}$  parameter which initially varies in the infinite-dimensional space  $\mathfrak{RH}_\infty$ , a sub-optimal solution is to consider for each input/output pairs  $(i, j)$  a finite-dimensional subspace generated by an orthonormal base of discrete stable transfer functions (such as a polynomial or FIR filter). This MIMO Youla parameter can be obtained in the state-space formalism using a fixed pair  $(\mathbf{A}_Q, \mathbf{B}_Q)$  and searching only for the variable pair  $(\mathbf{C}_Q, \mathbf{D}_Q)$ .

#### 3.2 Robust Stability under Polytopic Uncertainties

The main result is the robustification procedure under polytopic uncertainties. Consider the following time-varying system, as a generalization of the polytopic system (Kothare *et al.*, 1996):

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) \end{cases} \quad (10)$$

where  $[\mathbf{A}(k) \ \mathbf{B}(k) \ \mathbf{C}(k)] \in \Omega$  and the polytope  $\Omega$  (Fig. 4) represents the convex hull  $Co$  defined by the  $l$  vertices  $[\mathbf{A}_i \ \mathbf{B}_i \ \mathbf{C}_i]$ .

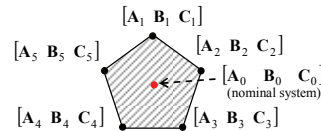


Figure 4: Polytopic uncertainty representation ( $l = 5$ ).

As  $\Omega$  is a polytope (convex set), guaranteeing the stability of (10) on the entire space  $\Omega$  means to guarantee the stability for all the vertices of the polytope (Kothare *et al.*, 1996). This is equivalent to satisfy the following condition (Boyd *et al.*, 1994) for each vertex  $i = \overline{1, l}$  of the domain  $\Omega$ :

$$\begin{bmatrix} -\mathbf{X}_2 & \mathbf{X}_2 \mathbf{A}_{cl,i} \\ \mathbf{A}_{cl,i}^T \mathbf{X}_2 & -\mathbf{X}_2 \end{bmatrix} < 0, \quad \mathbf{X}_2 = \mathbf{X}_2^T > 0 \quad (11)$$

This expression is bilinear in its decision variables  $\mathbf{X}_2$  and the  $\mathbf{Q}$  parameter included in  $\mathbf{A}_{cl,i}$ . The global robustification problem towards both unstructured and polytopic uncertainties is achieved

by minimizing  $\gamma$  subject to the constraints  $LMI_0$  and  $BMI_i$  (11):

$$\min_{LMI_0, BMI_i, i=1, \bar{l}} \gamma \quad (12)$$

But this is a difficult problem since it involves BMI expressions, in addition containing decision variables (the  $\mathbf{Q}$  parameter) jointly with a LMI. The challenge is to try to find a sub-optimal solution.

A first mathematical approach based on Sum of Squares (SOS) for relaxing the BMIs (12) is developed in the literature by (Scherer and Hol, 2006). But this relaxation technique leads to a huge number of scalar decision variables (that Matlab<sup>TM</sup> cannot deal with it for the moment) due to the size of SOS matrices. Hence it cannot be used within the presented robustification procedure.

For this reason, a second sub-optimal tractable solution (in three steps) of solving these BMIs is proposed. Firstly, in order to enlarge the polytopic domain around the nominal system, the minimization of the complementary sensitivity function is added to (9). This is equivalent to add the minimization of the transfer between  $\mathbf{b}$  and  $\mathbf{y}$  (Fig. 1) to (9). This minimization is then transformed into a LMI added to the first one (9):

$$\min_{LMI_0, LMI_{CS}} c_1 \gamma + c_2 \gamma_{CS} \quad (13)$$

choosing appropriate coefficients  $c_1, c_2$ . Solving the optimization problem (13) leads to a  $\mathbf{Q}$  parameter that will be used in the second step of the robustification procedure. In fact, the minimization (13) is recomputed until the resulting stability domain includes at least the polytopic domain of uncertainties, by selecting appropriate weightings  $c_1, c_2$ . The expression (13) offers the possibility to increase the stability domain, but does not offer any information about the limits of this domain. To explicitly include the considered polytopic domain, the second and third steps must be followed.

In order to find a sub-optimal solution of (11), the second step is to search  $\mathbf{X}_2$  using the  $\mathbf{Q}$  parameter obtained with (13). This can be achieved for instance by minimizing the trace of  $\mathbf{X}_2$  subject to the  $LMI_i$  ( $i = \bar{1}, \bar{l}$ ) derived from the BMIs (11), which permits to choose  $\mathbf{X}_2$  in order to enlarge the stability domain:

$$\min_{LMI_i, i=1, \bar{l}} tr(\mathbf{X}_2) \quad (14)$$

Thirdly, the value obtained for  $\mathbf{X}_{2,i}$  is used in the final step of the optimization problem which

decision variables are  $\mathbf{X}_1, \gamma$  and the  $\mathbf{Q}$  parameter included in the closed-loop matrices from  $LMI_0$  and  $LMI_i$ :

$$\min_{LMI_0, LMI_i, i=1, \bar{l}} \gamma \quad (15)$$

where  $LMI_i$  are the relaxations of the BMIs (11) for the vertices  $\mathbf{A}_i$ , while fixing the variable  $\mathbf{X}_2$ . The optimization (15) gives a Youla parameter that will guarantee the stability of the controlled system for all the vertices of the polytopic domain.

## 4 CONCLUSIONS

This paper has proposed an off-line methodology which improves the robustness of an initial stabilizing predictive controller via the convex optimization of the Youla parameter. This procedure deals with the stability robustness aspect of the nominal system towards unstructured uncertainties (solved with LMI tools), while guaranteeing the stability under a considered polytopic uncertain domain (leading to BMIs). In order to find a sub-optimal solution for these BMIs, a new method presenting a sub-optimal technique of solving this non-convex problem is proposed: one matrix variable is fixed using the minimization of the complementary sensitivity function, while looking for the other matrix variable. This provides computationally tractable solutions.

The main advantage of this robustification technique under polytopic uncertainties is that guaranteeing the BMI stability condition robustly stabilizes the controlled system for the entire polytopic domain, even if the system coupled with the initial predictive controller is unstable in some points of the polytopic domain. This offers a possible way of increasing the polytopic domain for which the stability is guaranteed.

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