FUZZY INDUCED AGGREGATION OPERATORS IN DECISION MAKING WITH DEMPSTER-SHAFER BELIEF STRUCTURE

José M. Merigó and Montserrat Casanovas

Department of Business Administration, University of Barcelona, Av. Diagonal 690, 08034 Barcelona, Spain

Abstract: We develop a new approach for decision making with Dempster-Shafer theory of evidence when the available information is uncertain and it can be assessed with fuzzy numbers. With this approach, we are able to represent the problem without losing relevant information, so the decision maker knows exactly which are the different alternatives and their consequences. For doing so, we suggest the use of different types of fuzzy induced aggregation operators in the problem. As a result, we get new types of fuzzy induced aggregation operators such as the belief structure – fuzzy induced ordered weighted averaging (BS-FIOWA) operator. We also develop an application of the new approach in a financial decision making problem.

1 INTRODUCTION

The Dempster-Shafer (D-S) theory of evidence (Dempster, 1967; Shafer, 1976) provides a unifying framework for representing uncertainty because it includes the situations of risk and ignorance as special cases. For further reading on the D-S theory, see (Yager and Liu, 2008).

Usually, when using the D-S theory it is assumed that the available information are exact numbers (Engemann et al., 1994; Merigó and Casanovas, 2007; Yager, 1992; 2004). However, this may not be the real situation found in the decision making problem because often, the available information is vague or imprecise and it is not possible to analyze it with exact numbers. Then, a better approach may be the use of fuzzy numbers (FN) because it considers the best and worst possible scenarios and a lot of others that could occur. When using FNs, we will follow the ideas of (Chang and Zadeh, 1972; Dubois and Prade, 1980; Kaufmann and Gupta, 1985).

Going a step further, the aim of this paper is to suggest the use of different types of fuzzy induced aggregation operators for aggregating the information in decision making with D-S theory. The reason for using various types of aggregation operators is that we want to show that the fuzzy decision making problem with D-S theory can be modelled in different ways depending on the interests of the decision maker. We will use the fuzzy induced ordered weighed averaging (FIOWA) operator because it provides a parameterized family of aggregation operators that include the fuzzy maximum, the fuzzy minimum, the fuzzy average (FA), the fuzzy weighted average (FWA) and the fuzzy OWA (FOWA), among others. Then, we will get a new aggregation operator that we will call the belief structure - FIOWA (BS-FIOWA) operator. We also develop an application of this new model in a business decision making problem.

In order to do so, the remainder of the paper is organized as follows. In Section 2, we briefly describe some basic concepts. In Section 3, we present the new approach about using fuzzy induced aggregation operators in decision making with D-S theory. Finally, in Section 4 we develop an application of the new approach.

2 PRELIMINARIES

2.1 Fuzzy Numbers

The FN was introduced by (Chang and Zadeh, 1972). Since then, it has been studied by a lot of authors such as (Kaufmann and Gupta, 1985).

A FN is a fuzzy subset (Zadeh, 1965) of a universe of discourse that is both convex and normal (Kaufmann and Gupta, 1985). Note that the FN may be considered as a generalization of the interval number (Moore, 1966) although it is not strictly the

same because the interval numbers may have different meanings.

In the literature, we find a wide range of FNs (Kaufmann and Gupta, 1985). For example, a trapezoidal FN (TpFN) A of a universe of discourse R can be characterized by a trapezoidal membership function $A = (\underline{a}, \overline{a})$ such that

$$
\begin{aligned}
\underline{a}(\alpha) &= a_1 + \alpha(a_2 - a_1), \\
\overline{a}(\alpha) &= a_4 - \alpha(a_4 - a_3).
\end{aligned}
\tag{1}
$$

where $\alpha \in [0, 1]$ and parameterized by (a_1, a_2, a_3, a_4) where $a_1 \leq a_2 \leq a_3 \leq a_4$, are real values. Note that if $a_1 = a_2 = a_3 = a_4$, then, the FN is a crisp value and if $a_2 = a_3$, the FN is represented by a triangular FN (TFN). Note that the TFN can be parameterized by (a_1, a_2, a_4).

2.2 Fuzzy Induced OWA Operator

The FIOWA (or FN-IOWA) operator was introduced by S.J. Chen and S.M. Chen (2003). It is an extension of the OWA operator (Yager, 1988; Yager and Kacprzyk, 1997) that uses uncertain information represented by FNs. It also uses a reordering process different from the values of the arguments. In this case, the reordering step is based on order inducing variables. It is defined as follows.

Definition 1. *Let Ψ be the set of FN. A FIOWA operator of dimension n is a mapping FIOWA: $\Psi^n \rightarrow \Psi$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then*:

$$
FIOWA(\langle u_1, \tilde{a}_1 \rangle, \ldots, \langle u_n, \tilde{a}_n \rangle) = \sum_{j=1}^{n} w_j b_j
\tag{2}
$$

where b_j is the $\tilde{a}_i$ value of the FIOWA pair $\langle u_i, \tilde{a}_i \rangle$ having the jth largest u_i, u_i is the order inducing variable and $\tilde{a}_i$ is the argument variable represented in the form FN.

2.3 Dempster-Shafer Theory of Evidence

The D-S theory provides a unifying framework for representing uncertainty as it can include the situations of risk and ignorance as special cases. It is defined as follows.

Definition 2. *A D-S belief structure defined on a space X consists of a collection of n nonnull subsets of X, B_j for j = 1,…,n, called focal elements and a mapping m, called the basic probability assignment, defined as, m: $2^X \rightarrow [0, 1]$ such that*:

(1) $\quad m(B_j) \in [0, 1]$.

(2) $\quad \sum_{j=1}^{n} m(B_j) = 1$. $\qquad$ (3)

(3) $\quad m(A) = 0, \forall\, A \neq B_j.$.

3 FIOWA OPERATORS IN DECISION MAKING WITH D-S THEORY OF EVIDENCE

In this Section, we describe the process to follow when using fuzzy induced aggregation operators in decision making with D-S theory.

3.1 Decision Making Approach

A new method for decision making with D-S theory is possible by using FN aggregation operators in the problem. Going a step further, we see that it is possible to use fuzzy induced aggregation operators such as the FIOWA operator. Note it is also possible to consider other cases such as the use of different types of fuzzy induced generalized means and fuzzy induced quasi-arithmetic means. The motivation for using FNs appears because sometimes, the available information is not clear and it is necessary to assess it with another approach such as the use of FNs. Although the information is uncertain and it is difficult to take decisions with it, at least we can represent the best and worst possible scenarios and the possibility that the internal values of the fuzzy interval will occur. The decision process can be summarized as follows.

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, \ldots, A_q\}$ with states of nature $\{S_1, \ldots, S_n\}$. $\tilde{a}_{ih}$ is the uncertain payoff, given in the form of FNs, to the decision maker if he selects alternative A_i and the state of nature is S_h. The knowledge of the state of nature is captured in terms of a belief structure m with focal elements $B_1, \ldots, B_r$ and associated with each of these focal elements is a weight $m(B_k)$. The objective of the problem is to select the alternative which gives the best result to the decision maker. In order to do so, we should follow the following steps:

Step 1: Calculate the uncertain payoff matrix.
Step 2: Calculate the belief function m about the states of nature.

Step 3: Calculate the attitudinal character of the decision maker $\alpha(W)$ (Yager, 1988).

Step 4: Calculate the collection of weights, *w*, to be used in the FIOWA aggregation for each different cardinality of focal elements. (Merigó, 2007; Yager, 1988; 1993).

Step 5: Determine the uncertain payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of *i* and *k*. Hence $M_{ik} = \{a_{ih} \mid S_h \in B_k\}$.

Step 6: Calculate the fuzzy induced aggregated payoff, $V_{ik} = \text{FIOWA}(M_{ik})$, using Eq. (2), for all the values of *i* and *k*.

Step 7: For each alternative, calculate the generalized expected value, C_i, where:

$$C_i = \sum_{r=1}^{r} V_{ik} m(B_k) \qquad (4)$$

Step 8: Select the alternative with the largest C_i as the optimal.

3.2 Using FIOWA Operators in Belief Structures

Analyzing the aggregation in *Steps* 6 and 7 of the previous subsection, it is possible to formulate in one equation the whole aggregation process. We will call this process the belief structure – FIOWA (BS-FIOWA) aggregation. It can be defined as follows.

Definition 3. *A BS-FIOWA operator is defined by*

$$C_i = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \qquad (5)$$

where w_{j_k} is the weighting vector of the kth focal element such that $\sum_{j=1}^{n} w_{j_k} = 1$ and $w_{j_k} \in [0,1]$, b_{j_k} is the j_kth largest of the $\tilde{a}_{i_k}$ and the $\tilde{a}_{i_k}$ are FNs, and $m(B_k)$ is the basic probability assignment.

Note that q_k refers to the cardinality of each focal element and *r* is the total number of focal elements.

The BS-FIOWA operator is monotonic, commutative, bounded and idempotent.

Note that it is possible to distinguish between descending (BS-DFIOWA) and ascending (BS-AFIOWA) orders.

3.3 Families of BS-FIOWA Operators

By using a different manifestation in the weighting vector of the FIOWA operator, we are able to develop different families of FIOWA and BS-FIOWA operators. As we can see in Definition 3, each focal element uses a different weighting vector in the aggregation with the FIOWA operator. Therefore, the analysis needs to be done individually.

For example, the maximum is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Max}\{\tilde{a}_i\}$. The fuzzy minimum is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Min}\{\tilde{a}_i\}$. The FA is found when $w_j = 1/n$, for all $\tilde{a}_i$. The FWA is obtained if $u_i > u_{i+1}$, for all *i*, and the FOWA operator is obtained if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the *j*th largest of $\tilde{a}_i$.

Other families of FIOWA operators could be used in the BS-FIOWA operator such as the step-FIOWA, and the olympic-FIOWA, among others. For more information, see (Merigó, 2007).

For example, the step-FIOWA operator is found when $w_k = 1$ and $w_j = 0$, for all $j \neq k$. The olympic-FIOWA operator is found if $w_1 = w_n = 0$, and for all others $w_j = 1/(n-2)$.

Finally, if we assume that all the focal elements use the same weighting vector, then, we can refer to these families as the BS-fuzzy maximum, the BS-fuzzy minimum, the BS-FA, the BS-FWA, the BS-S-FIOWA, the BS-olympic-FIOWA, etc.

4 APPLICATION IN FINANCIAL DECISION MAKING

In the following, we are going to develop an application of the new approach in a decision making problem. We will analyze the selection of financial strategies where an enterprise is looking for its optimal financial strategy for the next year. Note that other applications could be developed such as the selection of human resources, etc.

Assume an enterprise is planning its financial strategy for the next year and considers 4 possible financial strategies to follow: $\{A_1, A_2, A_3, A_4\}$.

In order to evaluate these financial strategies, the group of experts considers that the key factor is the economic situation of the company for the next year. After careful analysis, the experts have considered five possible situations that could happen in the future: S_1 = Very bad, S_2 = Bad, S_3 = Regular, S_4 = Good, S_5 = Very good.

Table 1: Fuzzy payoff matrix.

	S_1	S_2	S_3	S_4	S_5
A_1	(50,60,70)	(30,40,50)	(30,40,50)	(60,70,80)	(40,50,60)
A_2	(10,20,30)	(20,30,40)	(50,60,70)	(50,60,70)	(80,90,100)
A_3	(30,40,50)	(50,60,70)	(40,50,60)	(40,50,60)	(40,50,60)
A_4	(60,70,80)	(40,50,60)	(30,40,50)	(30,40,50)	(30,40,50)

Depending on the uncertain situations that could happen in the future, the experts establish the payoff matrix. As the future states of nature are very imprecise, the experts cannot determine exact numbers in the payoff matrix. Instead, they use FNs to calculate the future benefits of the enterprise depending on the state of nature that happens in the future and the financial strategy selected. Note that in this example the experts use TFN. Then, they can calculate the best and worst possible scenarios and represent all the internal results with a membership level. The results are shown in Table 1.

After careful analysis of the information, the experts have obtained some probabilistic information about which state of nature will happen in the future. This probabilistic information is represented by the following belief structure about the states of nature.

Focal element
$B_1 = \{S_1, S_2, S_3\} = 0.3$
$B_2 = \{S_3, S_4, S_5\} = 0.3$
$B_3 = \{S_2, S_3, S_4, S_5\} = 0.4$

The attitudinal character of the enterprise is very complex because it involves the opinion of different members of the board of directors. Therefore, the experts use order inducing variables for analysing the attitudinal character of the enterprise. The results are shown in Table 2.

Table 2: Inducing variables.

	S_1	S_2	S_3	S_4	S_5
A_1	7	6	4	9	2
A_2	1	5	7	9	3
A_3	4	3	8	6	5
A_4	2	5	6	7	8

Table 3: Fuzzy aggregated results.

	FA	FWA	FOWA	FIOWA	AFIOWA
V_{11}	(36.6,46.6,56.6)	(36,46,56)	(36,46,56)	(36,46,56)	(38,48,58)
V_{12}	(43.3,53.3,63.3)	(43,53,63)	(42,52,62)	(43,53,63)	(45,55,65)
V_{13}	(40,50,60)	(42,52,62)	(38,48,58)	(39,49,59)	(41,51,61)
V_{21}	(26.6,36.6,46.6)	(29,39,49)	(25,35,45)	(25,35,45)	(29,39,49)
V_{22}	(60,70,80)	(62,72,82)	(59,69,79)	(62,72,82)	(59,69,79)
V_{23}	(50,60,70)	(53,63,73)	(47,57,67)	(50,60,70)	(50,60,70)
V_{31}	(40,50,60)	(40,50,60)	(39,49,59)	(41,51,61)	(40,50,60)
V_{32}	(40,50,60)	(40,50,60)	(40,50,60)	(40,50,60)	(40,50,60)
V_{33}	(42.5,52.5,62.5)	(42,52,62)	(42,52,62)	(43,53,63)	(43,53,63)
V_{41}	(43.3,53.3,63.3)	(42,52,62)	(42,52,62)	(45,55,65)	(42,52,62)
V_{42}	(30,40,50)	(30,40,50)	(30,40,50)	(30,40,50)	(30,40,50)
V_{43}	(32.5,42.5,52.5)	(32,42,52)	(32,42,52)	(33,43,53)	(32,42,52)

Table 4: Fuzzy generalized expected value.

	FA	FWA	FOWA	FIOWA	AFIOWA
A_1	(40,50,60)	(40.5,50.5,60.5)	(38.6,48.6,58.6)	(39.3,49.3,59.3)	(41.3,51.3,61.3)
A_2	(46,56,66)	(48.5,58.5,68.5)	(44,54,64)	(46.1,56.1,66.1)	(46.4, 56.4, 66.4)
A_3	(41,51,61)	(40.8,50.8,60.8)	(40.5,50.5,60.5)	(41.5,51.5,61.5)	(41.2,51.2,61.2)
A_4	(35,45,55)	(34.4,44.4,54.4)	(34.4,44.4,54.4)	(35.7,45.7,55.7)	(34.4,44.4,54.4)

The experts establish the following weighting vectors for the FIOWA:

Weighting vector
$W_3 = (0.3, 0.3, 0.4)$
$W_4 = (0.2, 0.2, 0.3, 0.3)$
$W_5 = (0.1, 0.2, 0.2, 0.2, 0.3)$

With this information, we can obtain the aggregated payoffs. The results are shown in Table 3.

Once we have the aggregated results, we have to calculate the fuzzy generalized expected value. The results are shown in Table 4.

As we can see, depending on the fuzzy aggregation operator used, the results and decisions may be different. Note that in this case, our optimal choice is the same for all the aggregation operators but in other situations we may find different decisions between each aggregation operator. A further interesting issue is to establish an ordering of the financial strategies. Note that this is very useful when the decision maker wants to consider more than one alternative. As we can see, depending on the aggregation operator used, the ordering of the financial strategies may be different. Note that in this example the results are clear being A_2 the optimal choice and the ordering: $A_2 \rangle A_3 \rangle A_1 \rangle A_4$ excepting for the AFIOWA operator, where the ordering is: $A_2 \rangle A_1 \rangle A_3 \rangle A_4$.

5 CONCLUSIONS

We have studied the D-S theory of evidence in decision making with uncertain information represented in the form of FNs. With this approach, we have been able to assess the information in a more complete way because in this model we consider the different scenarios that could happen in the problem. For doing so, we have used different types of fuzzy induced aggregation operators in the decision process such as the FIOWA operator. Then, we have obtained the BS-FIOWA operator.

We have also developed an application of the new approach in a business decision making problem about selection of financial strategies. We have seen the usefulness of this approach about using probabilities and FIOWAs in the same problem. We have also seen that depending on the fuzzy induced aggregation operator used the results may lead to different decisions.

In future research, we expect to develop further extensions to this approach by adding new

characteristics in the problem and applying it to other decision making problems.

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