Numerical Computation for MRE (Magnetic Resonance Elasticity) by Applying Numerical Differentiation Method

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Abstract. Palpation is a standard medical method to detect of abnormalities of the human body, because a pathological state of soft tissues is often correlated with changes in stiffness. However, a pathological lesion may be undetectable by palpation if it is located deep in the body or if it is too small. Recently, new technics are developing to identify stiffness (Young's modulus). The observed data contain noise, it prevent to substitute the data for the mathematical model equation of the human body. Because the model equation includes the second differential terms. "Numerical differentiation" is a numerical method to determine the derivatives of an unknown function from the given noisy values of the unknown function at the scattered points. In this talk, we will apply this method to the noisy data, and introduce numerical results.

1 Introduction

Palpation is a standard medical method to detect of abnormalities of the human body, because a pathological state of soft tissues is often correlated with changes in stiffness.

However, a pathological lesion may be undetectable by palpation if it is located deep in the body or if it is too small. Recently, new technics are developing to identify stiffness(Young's modulus). The principle of them is that:"By giving vibration from outside of the body, we observe the propagation of the wave. By processing the observed data, the stiffness can be identified."

To observe the propagation of the wave, there are two ways. One is by using ultarsonic device, the other is MRI device. For either technique, it is usually derived by substituting the observed data for a mathematical model expressing the human body. The human body can be approximated by the elastic model, we have to calculate the second derivative of the data.

Because the observed data contain the noise, it is impossible to derivate them, directly. So far, by modifying data, for example taking average of neighbourhood, we get the Young's modulus. As for measuring a value of elasticity or the identification of the border of the pathological part, we can not get correct information by this method. We have to develop the method to calculate the derivation of the noisy data without modifying of data.

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The higher order for one-dimensional and the first order for two dimensional numerical differntiaton along the line of this method were given in [2] and [3], respectively. For the two dimensional case, the new ingredient was that the variational problem for the regularised minimisation problem is solved by using Green's function for the Laplacian with Dirichlet boundary condition and a scheme for computing the first order derivative was given in [3]. The numerical example showed that this method was efficient. But in many applications, it is necessary to compute higher order derivatives. In this talk, we will treat a problem concerned with MRE(Magnetic Resonance Elasticity). It give an example to apply a numerical differntiation for the second derivatives from noisy scattered data.

Mathematical setting and a mathematical result are introduced in the following.

2 Problem

Suppose we know $\Omega \subset \mathbf{R^2}$ is a bounded domain with piecewise C^2 boundary and $\rho = \rho(x) \in H^4(\Omega)$ is a function defined in Ω. Let N be a natural number and $\{x^i\}_{i=1}^N$ be a group of points in Ω. We assume that Ω is divided into N parts $\{\Omega_i\}_{i=1}^N$, and there is only one points of $\{x^i\}_{i=1}^N$in each part. For simplicity we also assume that the areas $|\Omega_i|$ of all $\Omega_i(1 \le i \le N)$ are the same. We denote by d_i the diameter of Ω_i and let $d = \max\{d_i\}$.

We will discuss the following problem:
Suppose that we know the approximate value $\tilde{\rho_i}$ at point x^i i.e.

$$|\tilde{\rho_i} - \rho(x^i)| \le \delta, \quad i = 1, 2, \cdots, N,$$

where $\delta > 0$ is a given constant called the error level.

We want to find a function $f_*(x)$ which approximates function $\rho(x)$ such that

$$\lim_{d \to 0, \delta \to 0} \|f_* - \rho\|_{H^2(\Omega)} = 0.$$

We treat this problem as the following optimisation problem by using Tikhonov regularisation method.

Problem 2.1. Define a cost function $\Phi(f)$

$$\Phi(f) = \frac{1}{N} \sum_{j=1}^{N} (f(x^i) - \tilde{\rho_j})^2 + \alpha \|\Delta^2 f\|^2_{L^2(\Omega)}, \quad f \in H$$

where $H = \{f : f \in H^4(\Omega), f|_{\partial\Omega} = \Delta f|_{\partial\Omega} = 0\}$, and $\alpha > 0$ is a regularisation parameter. Then, the problem is to find $f_* \in H$ such that $\Phi(f_*) \le \Phi(f)$ for every $f \in H$.

The existence and uniqueness of the minimiser of Problem is shown in [1].

Theorem 2.1. Problem 2.1 is equivalent to finding a unique solution $f_* \in H$ for the following variational problem:

$$\int_\Omega \Delta^2 f_* \Delta^2 h dx = -\frac{1}{\alpha N} \sum_{j=1}^{N} (f(x^i) - \tilde{\rho_j}) h(x^j) \tag{1}$$

for all $h \in H$. This equation is the Euler equation of Problem 2.1. Moreover, the minimiser of Problem 2.1 is unique.

To solve numerical differentiation problem, it is necessary to provide a scheme for construction f_*. For that, by a formal argument using Green's function of bi-harmonic operator, we derive a method how to construct f_*. It will be shown as a theorem that the constructed f_* by this method is the solution of (1).

We get the equation (2) by (1) (cf. [1])

$$\begin{aligned} \alpha f_*(x) + \frac{1}{N}\sum_{j=1}^{N}(f(x^i) - \tilde{\rho_j}) \int_{\Omega} G(x^j, y)G(y, x)dy \\ = 0, \end{aligned} \tag{2}$$

where $G(x, y)$ is a bi-harmonic Green's function. By defining

$$a_j(x) = \int_{\Omega} G(x^j, y)G(y, x)dy$$

and

$$c_j = -\frac{1}{\alpha N}(f_*(x^j) - \tilde{\rho_j}),$$

(2) becomes

$$f_*(x) = \sum_{j=1}^{N} c_j a_j(x).$$

Now the problem of constructing f_* reduces to computing the coefficient c_j from $\tilde{\rho_j}$. From the definition of a_j and c_j with $x = x^j (j = 1, 2, \cdots N)$, we obtain

$$\begin{aligned} c_j &= -\frac{1}{\alpha N}(f(x^j) - \tilde{\rho_j}) \\ &= -\frac{1}{\alpha N}(\sum_{k=1}^{N} a_k(x^j)c_k - \tilde{\rho_j}). \end{aligned} \tag{3}$$

Let $\mathbf{A}$ be a (N, N) matrix which is defined by

$$\mathbf{A} = \left(\alpha N \delta_{ij} + a_i(x^j)\right),$$

where δ_{ij} is Kroneker's delta. And let $\mathbf{c}$ and $\mathbf{b}$ be vectors

$$\mathbf{c} = (c_j), \qquad \mathbf{b} = (b_j).$$

Then (1) becomes the linear equations

$$\mathbf{Ac} = \mathbf{b}.$$

Solving this equations, we will obtain coefficients c_j. Then we get f_*. To make sure f_* is the solution of our problem, we give the following theorem.

Theorem 2.2. Suppose function

$$f_*(x) = \sum_{k=1}^{N} a_k(x) c_k$$

where $\{c_j\}_{j=1}^{N}$ is the solution of linear system (3), then f_* is the solution of Problem 2.1.

3 Numerical Results

Here, we will see our method, the numerical differentiation, is effective. We insert a noise to the numerical data that are given by numerical computation of direct problem. Figure 1 is comparison with the elastic coefficient(stiffness) that we set in the direct problem and the numerical result of our method. Reconstruction is performed with good precision.

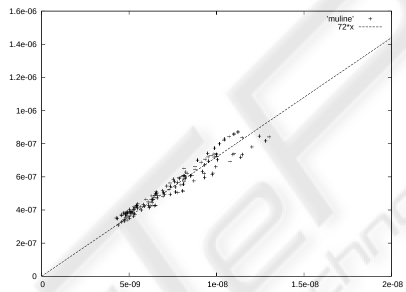

Fig. 1. Without noise.

Figure 2, we insert the noise(1%). For comparison, we show the result which is given by ordinary differential method Figure 3.

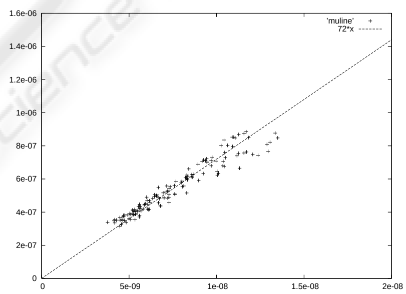

Fig. 2. Numerical differential method with 1% noise.

Figure4, 10 noise is inserted to the data. We can reconstruct the elastic modulus. We can see our method is robust to the noise.

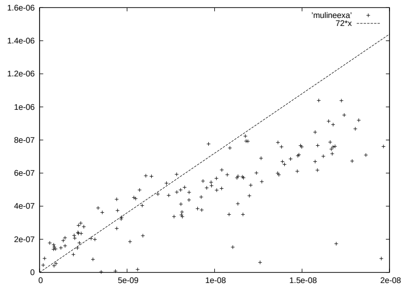

Fig. 3. Ordinary differential method with 1% noise.

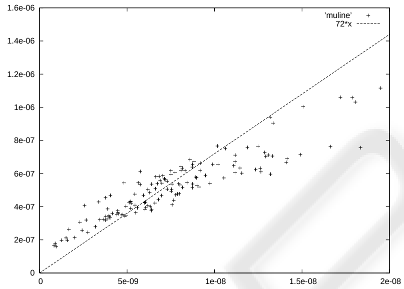

Fig. 4. Numerical differential method with 10% noise.

4 Conclusions

The real observed data contain the noise. It is impossible to derive the stiffness(Young's modulus), directly. Usually, by modifying the data, it is given. As for measuring a correct value of elasticity or the identification of the border of the pathological part, we have to develop the method to calculate the differential coefficient of the noisy data without modifying of data.

In this note, we proposed the numerical differential method and introduced numerical results. We can see our method is robust to the noise. However, we do not know the noise level of the real observed data. It is necessary to devise how to choose parameters with the real data.

References

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