RICCATI SOLUTION FOR DISCRETE STOCHASTIC SYSTEMS WITH STATE AND CONTROL DEPENDENT NOISE

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Abstract: In this paper we present the Riccati solution of linear quadratic control problems with input and state dependent noise which is encountered during our previous study to the adaptive critic solution for systems characterized by functional uncertainty. Uncertainty of the system equations is quantified using a state and control dependent noise model. The derived optimal control law is shown to be of cautious type controllers. The derivation of the Riccati solution is via the principle of optimality. The Riccati solution is implemented to linear multi dimensional control problem and compared to the certainty equivalent Riccati solution.

1 INTRODUCTION

The objective of this paper is to introduce the Riccati solution of stochastic linear quadratic systems with input and state dependent noise which is encountered during our previous study (Herzallah, 2007) of the adaptive critic solution to stochastic systems characterized by functional uncertainty. The problem of stochastic linear quadratic control is discussed in (Rami et al., 2001) and is shown to have different form than that of traditional linear quadratic control. However, the work in (Rami et al., 2001) discusses models with multiplicative white noise on both the state and control and it is for continuous time systems. In the current paper the optimal control law for stochastic discrete linear quadratic systems characterized by functional uncertainty will be derived. This yields a cautious type controller which takes into consideration model uncertainty when calculating the optimal control law.

The optimization problem of the linear stochastic control with state and control dependent noise is to find a feedback control which minimizes the following quadratic cost function (Herzallah, 2007):

$$L = \sum_{k=0}^{N-1} U(\mathbf{x}(k), \mathbf{u}(k)) + \psi[\mathbf{x}(N)], \qquad (1)$$

where

$$U(\mathbf{x}(k), \mathbf{u}(k)) = \mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + \mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k) \qquad (2)$$

$$\psi[\mathbf{x}(N)] = \mathbf{x}^T(N)\mathbf{C}\mathbf{x}(N) + \mathbf{Z}\mathbf{x}(N) + U_0, \qquad (3)$$

subject to the system equation given by

$$\mathbf{x}(k+1) = \tilde{\mathbf{G}}\mathbf{x}(k) + \tilde{\mathbf{H}}\mathbf{u}(k) + \tilde{\eta}(k+1), \qquad (4)$$

where N is the time horizon, $\mathbf{x} \in R^n$ represents the state vector of the system, $\mathbf{u} \in R^m$ denotes the control action, $U(\mathbf{x}(k), \mathbf{u}(k))$ is a utility function, $\psi[(\mathbf{x}(N)$ is the weight of the performance measure due to the final state, and $\tilde{\eta}(k+1)$ is an additive noise signal assumed to have zero mean Gaussian distribution of covariance matrix $\tilde{\mathbf{P}}$. If the matrices $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{H}}$ were known and the system was noiseless, the solution of this problem is well known (Anderson and Moore, 1971; Ogata, 1987). The optimal control is a linear function of $\mathbf{x}$ which is independent of the additive noise $\tilde{\eta}(k+1)$. This solution is also applicable in the presence of independent noise $\tilde{\eta}(k+1)$, because the covariance matrix $\tilde{\mathbf{P}}$, of the noise term is independent of $\mathbf{u}$(k).

In the current paper the optimal control for systems with unknown models will be derived. It has been shown that systems with unknown functions should be formulated in an adaptive control framework which is known to have functional uncertainties (Fel'dbaum, 1960; Fel'dbaum, 1961). This means that state and control dependent noise always accompany systems with unknown models. In the literature, three different methods were used to handle the control problem of systems characterized by functional uncertainty. The first method is the heuristic equivalent control method (Åström and Wittenmark, 1989; Guo and Chen, 1991; Xie and Guo, 1998; Yaz, 1986) in which the control is found by solving for the equivalent deterministic system and then simply replace the

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unknown variables by their estimates. The second method is the cautious control method (Goodwin and Sin, 1984; Apley, 2004; Campi and Prandini, 2003; Herzallah and Lowe, 2007) which takes the uncertainty of the estimates into consideration when calculating the control but do not plan for any probing signals to reduce the future estimation of uncertainty. The last but most efficient method is the dual control method (Fel'dbaum, 1960; Fel'dbaum, 1961; Fabri and Kadirkamanathan, 1998; Filatov and Unbehauen, 2000; Maitelli and Yoneyama, 1994) which takes uncertainty of the estimates into consideration when estimating the control and at the same time plan to reduce future estimation of uncertainty.

The Riccati solution in this paper is for the more general systems of equation (4), where the parameters of the system equation are unknown and where the noise term is state and control dependent. The parameters of the model are to be estimated on–line based on some observations. Not only the model parameters are to be estimated on–line, but also the state dependent noise which characterizes uncertainty of the parameters estimate and allows estimating the conditional distribution of the system output or state. The conditional distribution of the system output will be estimated by the method used in (Herzallah, 2007). The optimal control is again linear in $\mathbf{x}$, but is now rather critically dependent on the parameters of the estimated uncertainty of the error $\tilde{\eta}(k+1)$. This in turn, yields a cautious type controller which takes into consideration model uncertainty when calculating the optimal control law. A numerical example is provided and the result is compared to the certainty equivalent controller.

The Riccati solution will be introduced soon, but first we give a brief discussion about estimating model uncertainty which we need for the derivation of the Riccati solution of the cautious controller.

2 BASIC ELEMENTS

As a first step to the optimization problem, the conditional distribution of the system output or state needs to be estimated. According to theorem 4.2.1 in (Gersho and Gray, 1992), the minimum mean square error (MMSE) estimate of a random vector $\mathcal{Z}$ given another random vector $\mathcal{X}$ is simply the conditional expectation of $\mathcal{Z}$ given $\mathcal{X}$, $\hat{\mathcal{Z}} = E(\mathcal{Z} \mid \mathcal{X})$. For the linear systems discussed in this paper, a generalized linear model is used to model the expected value of the system output,

$$\hat{\mathbf{x}}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \tag{5}$$

The parameters of the generalized linear model are then adjusted using an appropriate gradient based method to optimize a performance function based on the error between the plant and the linear model output. The stochastic model of the system of equation (4) is then shown (Herzallah, 2007) to have the following form:

$$\mathbf{x}(k+1) = \hat{\mathbf{x}}(k+1) + \eta(k+1), \tag{6}$$

where $\eta(k+1)$ represents an input dependent random noise.

Another generalized linear model which has the same structure and same inputs as that of the model output is then used to predict the covariance matrix, $\mathbf{P}$ of the error function $\eta(k+1)$,

$$\mathbf{P} = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k). \tag{7}$$

where $\mathbf{A}$ and $\mathbf{B}$ are partitioned matrices and are updated such that the error between the actual covariance matrix and the estimated one is minimized.

Detailed discussion about estimating the conditional distribution of the system output can be found in (Herzallah, 2007; Herzallah and Lowe, 2007).

3 RICCATI SOLUTION AND MAIN RESULT

In this section we derive the Riccati solution of the infinite horizon linear quadratic control problem characterized by functional uncertainty. We show here that the optimal control law is a state feedback law which depends on the parameters of the estimated uncertainty, and that the optimal performance index is quadratic in the state $\mathbf{x}(k)$ which also dependent on the estimated uncertainty. The derivation is based on the principal of optimality and is for finite horizon control problem which is known to be the steady state solution for an infinite horizon control problem. Hence, by the principal of optimality the objective is to find the optimal control sequence which minimizes Bellman's equation (Bellman, 1961; Bellman, 1962)

$$J[(\mathbf{x}(k)] = U(\mathbf{x}(k), \mathbf{u}(k)) + \gamma < J[\mathbf{x}(k+1)] >, \tag{8}$$

where $< . >$ is the expected value, $J[\mathbf{x}(k)]$ is the cost to go from time k to the final time, $U(\mathbf{x}(k), \mathbf{u}[x(k)])$ is the utility which is the cost from going from time k to time $k+1$, and $< J[\mathbf{x}(k+1)] >$ is assumed to be the average minimum cost from going from time $k+1$ to the final time. The term γ is a discount factor $(0 \leq \gamma \leq 1)$ which allows the designer to weight the relative importance of present versus future utilities.

Using the general expressions of Equations (2), (6) and (5) in Bellman's equation and taking $\gamma = 1$, yields

$$J[(\mathbf{x}(k)] = \mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k) + \mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + \\ < J[\mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) + \eta(k+1)] > . \quad (9)$$

The true value of the cost function J is shown in (Herzallah, 2007) to be quadratic with the following form

$$J(\mathbf{x}) = \mathbf{x}^T\mathbf{M}\mathbf{x} + \mathbf{S}\mathbf{x} + U_0. \quad (10)$$

Making use of this result in equation (9) yields

$$\begin{aligned} J[(\mathbf{x}(k)] &= \mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k) + \mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) \\ &+ < [\mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) + \eta(k+1)]^T \\ &\mathbf{M}(k+1)[\mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) + \eta(k+1)] \\ &+ \mathbf{S}(k+1)[\mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) + \eta(k+1)] > . \end{aligned} \quad (11)$$

Evaluating the expected value of the third term of equation (11) and using equation (7) yields

$$\begin{aligned} J[(\mathbf{x}(k)] &= \mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k) + \mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) \\ &+ \mathbf{x}^T(k)\mathbf{G}^T\mathbf{M}(k+1)\mathbf{G}\mathbf{x}(k) \\ &+ 2\mathbf{x}^T\mathbf{G}^T\mathbf{M}(k+1)\mathbf{H}\mathbf{u}(k) \\ &+ \mathbf{u}^T(k)\mathbf{H}^T\mathbf{M}(k+1)\mathbf{H}\mathbf{u}(k) \\ &+ \mathrm{tr}[\mathbf{A}\mathbf{M}(k+1)]\mathbf{x}(k) + \mathrm{tr}[\mathbf{B}\mathbf{M}(k+1)]\mathbf{u}(k) \\ &+ \mathbf{S}(k+1)[\mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k)]. \end{aligned} \quad (12)$$

Minimization of the explicit performance index defined in Equation (12) leads to the control law specified in the following theorem.

Theorem 1: The control law minimizing the performance index $J[(\mathbf{x}(k)]$ of Equation (12), is given by

$$\mathbf{u}^* = -\mathbf{K}(k)\mathbf{x}(k) - \mathbf{Z}(k), \quad (13)$$

where

$$\mathbf{K}(k) = \left[\mathbf{R} + \mathbf{H}^T\mathbf{M}(k+1)\mathbf{H}\right]^{-1}\left[\mathbf{H}^T\mathbf{M}(k+1)\mathbf{G}\right] \quad (14)$$

$$\mathbf{Z}(k) = \left[\mathbf{R} + \mathbf{H}^T\mathbf{M}(k+1)\mathbf{H}\right]^{-1}\frac{1}{2}\left[\mathrm{tr}[\mathbf{B}\mathbf{M}(k+1)] + \mathbf{S}(k+1)\mathbf{H}\right]. \quad (15)$$

Proof. This theorem can be proved directly by deriving Equation (12) with respect to the control and setting the derivative equal to zero. Note that the optimal control law consists of two terms: the linear term in $\mathbf{x}$ which is equivalent to the linear term obtained in deterministic control problems and an extra constant term which gives cautiousness to the optimal control law. Note also that the evaluation of the optimal control law requires evaluating the matrix, $\mathbf{M}(k+1)$ and the vector, $\mathbf{S}(k+1)$. This evaluation can be obtained by evaluating the optimal cost function $J[(\mathbf{x}(k)]$.

The optimal cost function $J[(\mathbf{x}(k)]$ can be obtained by substituting Equation (13) in (12). This yields the quadratic cost function defined in the following theorem.

Theorem 2: The optimal control law defined in Equation (13) yields a quadratic cost function of the following form

$$J[(\mathbf{x}(k)] = \mathbf{x}^T(k)\mathbf{M}(k)\mathbf{x}(k) + \mathbf{S}(k)\mathbf{x}(k) + U_0, \quad (16)$$

where

$$\begin{aligned} \mathbf{M}(k) &= \mathbf{Q} + \mathbf{G}^T\mathbf{M}(k+1)\mathbf{G} \\ &- \mathbf{G}^T\mathbf{M}(k+1)\mathbf{H}\mathbf{F}^{-1}\mathbf{H}^T\mathbf{M}(k+1)\mathbf{G} \end{aligned} \quad (17)$$

$$\begin{aligned} \mathbf{S}(k) &= \mathrm{tr}[\mathbf{A}\mathbf{M}(k+1)] - \Big\{\mathrm{tr}[\mathbf{B}\mathbf{M}(k+1)] \\ &+ \mathbf{S}(k+1)\mathbf{H}\Big\}\mathbf{F}^{-1}\mathbf{H}^T\mathbf{M}(k+1)\mathbf{G} \\ &+ \mathbf{S}(k+1)\mathbf{G}. \end{aligned} \quad (18)$$

and where

$$\mathbf{F}^{-1} = [\mathbf{R} + \mathbf{H}^T\mathbf{M}(k+1)\mathbf{H}]^{-1}. \quad (19)$$

Equation (17) is called the Riccati equation. It is similar to that obtained for the certainty equivalent controller. Equation (18) is dependent on the solution of the Riccati equation. It provides cautiousness to the optimal quadratic controller, therefore, will be referred to as the equation of cautiousness. According to equation (3), the optimal cost at $k = N$ equal to $\psi[(\mathbf{x}(N)]$. This means that $\mathbf{M}(N) = \mathbf{C}$ and $\mathbf{S}(N) = \mathbf{Z}$. Hence equation (17) and (18) can be solved uniquely backward from $k = N$ to $k = 0$. That is $\mathbf{M}(N), \mathbf{M}(N-1), \ldots, \mathbf{M}(0)$ and $\mathbf{S}(N), \mathbf{S}(N-1), \ldots, \mathbf{S}(0)$ can be obtained starting from $\mathbf{M}(N)$ and $\mathbf{S}(N)$ which are known.

To reemphasize, the matrix $\mathbf{M}(k)$ has an equivalent form similar to that obtained for deterministic control problems. However the optimal control law and the cost function are dependent on the values of the vector $\mathbf{S}(k)$ as well as on $\mathbf{M}(k)$.

For infinite horizon control problems, the optimal control solution becomes a steady state solution of the finite horizon control (Ogata, 1987; Anderson and Moore, 1971). Hence $\mathbf{K}(k)$, $\mathbf{Z}(k)$, $\mathbf{M}(k)$, and $\mathbf{S}(k)$ become constant and defined as follows

$$\mathbf{K} = \left[\mathbf{R} + \mathbf{H}^T\mathbf{M}\mathbf{H}\right]^{-1}\left[\mathbf{H}^T\mathbf{M}\mathbf{G}\right] \quad (20)$$

$$\mathbf{Z} = \left[\mathbf{R} + \mathbf{H}^T\mathbf{M}\mathbf{H}\right]^{-1}\frac{1}{2}\left[[\mathrm{tr}\mathbf{B}\mathbf{M}] + \mathbf{H}\mathbf{S}\right] \quad (21)$$

$$\mathbf{M} = \mathbf{Q} + \mathbf{G}^T\mathbf{M}\mathbf{G} - \mathbf{G}^T\mathbf{M}\mathbf{H}\mathbf{F}^{-1}\mathbf{H}^T\mathbf{M}\mathbf{G} \quad (22)$$

$$\begin{aligned} \mathbf{S} &= \mathrm{tr}[\mathbf{A}\mathbf{M}] - \left[\mathrm{tr}[\mathbf{B}\mathbf{M}] + \mathbf{S}\mathbf{H}\right]\mathbf{F}^{-1}\mathbf{H}^T\mathbf{M}\mathbf{G} \\ &+ \mathbf{S}\mathbf{G}. \end{aligned} \quad (23)$$

In implementing the steady state optimal controller, the steady state solution of the Riccati equation as well as the equation of cautiousness should be obtained. Since the Riccati equation of the cautious controller derived in this paper has similar form to that of the certainty equivalent controller, standard methods proposed in (Ogata, 1987) can be implemented to obtain the solution of the Riccati equation. To obtain the solution of the steady state equation of cautiousness given by Equation (23),

$$\mathbf{S} = \mathrm{tr}[\mathbf{AM}] - \Big\{ \mathrm{tr}[\mathbf{BM}] + \mathbf{SH} \Big\} \mathbf{F}^{-1}\mathbf{H}^T\mathbf{MG} + \mathbf{SG},$$

we simply start with the non steady state equation of cautiousness which was given by Equation (18),

$$\begin{aligned} \mathbf{S}(k) &= \mathrm{tr}[\mathbf{AM}(k+1)] - \Big\{ \mathrm{tr}[\mathbf{BM}(k+1)] \\ &+ \mathbf{S}(k+1)\mathbf{H} \Big\} \mathbf{F}^{-1}\mathbf{H}^T\mathbf{M}(k+1)\mathbf{G} \\ &+ \mathbf{S}(k+1)\mathbf{G}, \end{aligned} \quad (24)$$

by substituting the steady state matrix $\mathbf{M}$ and reversing the direction of time, we modify Equation (24) to read

$$\begin{aligned} \mathbf{S}(k+1) &= \mathrm{tr}[\mathbf{AM}] - \Big\{ \mathrm{tr}[\mathbf{BM}] \\ &+ \mathbf{S}(k)\mathbf{H} \Big\} \mathbf{F}^{-1}\mathbf{H}^T\mathbf{MG} + \mathbf{S}(k)\mathbf{G}. \end{aligned} \quad (25)$$

Then beginning the solution with $\mathbf{S}(0) = \mathbf{0}$, iterate Equation (25) until a stationary solution is obtained.

4 SIMULATION EXAMPLE

To numerically test and demonstrate the Riccati solution of the cautious controller, the theory developed in the previous section is applied here to 2–inputs 3–outputs control problem described by the following stochastic equation

$$\mathbf{x}(k+1) = \mathbf{Gx}(k) + \mathbf{Hu}(k) + \mathbf{w}(k+1), \quad (26)$$

where

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.16 & 0.84 & 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$$

$$E[\mathbf{w}(k+1)\mathbf{w}^T(k+1)] = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}.$$

Note that although the added noise to the system of Equation (26) is not state and control dependent, the estimated noise is state and control dependent reflecting the fact that the estimated model is not exact. The performance index to be minimized is specified so as to keep the state values near the origin. That is

$$J = \sum_{k=0}^{\infty} [\mathbf{x}^T(k)\mathbf{Qx}(k) + \mathbf{u}^T(k)\mathbf{Ru}(k)], \quad (27)$$

where $\mathbf{Q} = \mathbf{I}$ and $\mathbf{R} = \mathbf{I}$. Three generalized linear models were used to provide a prediction for the states, x_1, x_2 and x_3. The covariance matrix of the state vector is assumed to be diagonal, therefore, one generalized linear model with three outputs was used to provide a prediction for the variance of the error of estimating the first state x_1, the second state x_2 and the third state x_3.

For comparison purposes, the optimal control law is calculated by assuming the certainty equivalence method where conventional Riccati solution is used to estimate the optimal control, and by taking uncertainty measure into consideration where the proposed Riccati solution is used to estimate the optimal control. The same noise sequence and initial conditions were used in each case. The generalized linear models were never subjected to an initial off–line training phase. Closed loop control was activated immediately, with the initial parameter estimates selected at random from a zero mean, isotropic Gaussian, with variance scaled by the fan-in of the output units. The output of the two methods is shown in Figure 1. As expected, the figure shows that the certainty equivalence controller initially responds crudely, exhibiting large transient overshoot because it is not taking into consideration the inaccuracy of the parameter estimates. Only after the initial period, when the parameters of the system model converge, does the control assume good tracking. On the other hand, the cautious controller does not react hastily during the initial period, knowing that the parameters estimate are still inaccurate.

5 CONCLUSIONS

In this paper, we have derived the Riccati solution for more general systems with unknown functionals and state dependent noise. The Riccati solution of this paper is suitable for deterministic and stochastic control problems characterized by functional uncertainty. The optimal control is of cautious type and takes into consideration model uncertainty.

The derived Riccati equation in this paper has similar form to that of the certainty equivalent controller. However, as a result of considering the uncertainty on the models, the derived optimal control law has been

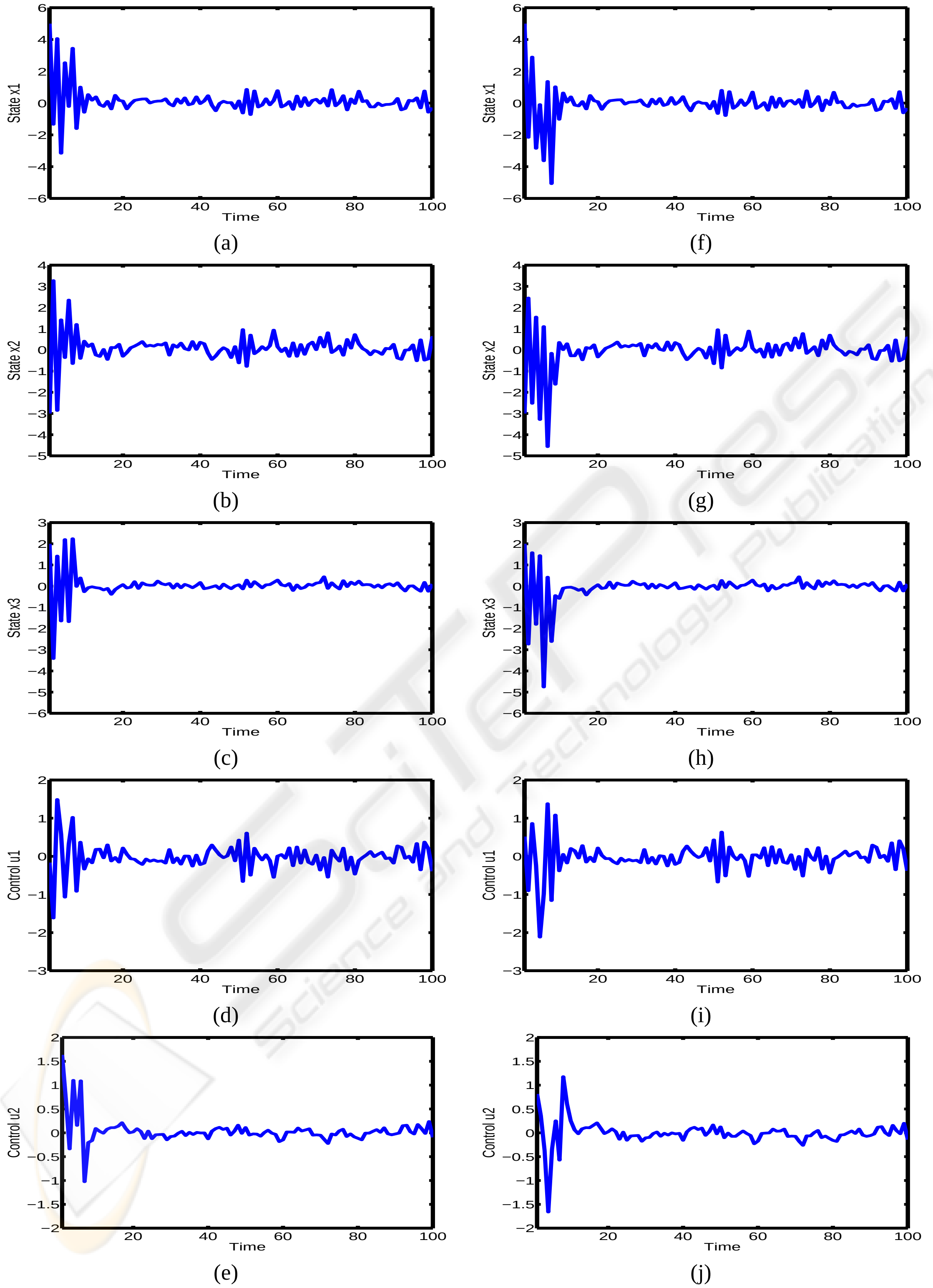

Figure 1: Controlled multi dimensional stochastic system: (a) State 1 using the proposed Riccati solution. (b) State 2 using the proposed Riccati solution. (c) State 3 using the proposed Riccati solution. (d) Control 1 using the proposed Riccati solution. (e) Control 2 using the proposed Riccati solution. (f) State 1 using the conventional Riccati solution. (g) State 2 using the conventional Riccati solution. (h) State 3 using the conventional Riccati solution. (i) Control 1 using the conventional Riccati solution. (j) Control 2 using the conventional Riccati solution.

shown to have an extra term which depends on the estimated uncertainty. A simulation example has illustrated the efficacy of the cautious controller.

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