A MIN-PLUS APPROACH FOR TRAFFIC FLOW MODELING

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Abstract: In this paper we propose a modeling method for traffic flow phenomena based on the min-plus algebra. We adopt a modular approach by dividing roadways as elementary stretches which can be combined in order to get a model for a complex infrastructure. The approach is flexible in the sense that different scales can be considered for each elementary model. In fact, whatever its size, each roadway stretch is here studied as a min-plus linear system and is modeled by its impulse response in min-plus algebra. In this first step in studying traffic flow, we focus on modeling detailing the adopted methodology. We also present simulations to validate the approach.

1 INTRODUCTION

Traffic is a non-linear phenomenon complex to predict or simulate. In spite of that, many models for traffic flow were studied since the late fifties, some at a macroscopic level (for example models based on gas kinetic, see (Helbing, 1996)), others at a microscopic level (for example models based on cellular automata, see (Nagel and Schreckenberg, 1992)).

In this paper we propose a modeling method for traffic flow phenomena based on the min-plus algebra. Previous works have used min-plus algebra to study road traffic (Lolito et al., 2005), (Farhi et al., 2007). In these papers, authors present a microscopic model based on Petri nets whose dynamics are written using min-plus algebra. This approach is not far from cellular automata approach (Nagel and Schreckenberg, 1992) since roadways are divided in stretches containing at most one vehicle, and individual movement of a vehicle in a stretch is conditioned by its availability (no vehicle on this section).

The proposed approach is more flexible in the sense that different scales can be considered for the model. We also adopt a modular approach by dividing roadways as elementary stretches which can be combined in order to get a model for a complex infrastructure. But the scale of a stretch can be larger. In fact, whatever its size, each roadway stretch is here studied as a min-plus linear system and is modeled by its impulse response in min-plus algebra. In the extreme, a stretch can be sized to contain a single vehicle. It can also correspond to a more macroscopic element (*e.g.* a roadway which is several kilometers long). Doing so, we expect that descriptions of various phenomena (inherent to various roadway configurations such as intersections, traffic lights,...) can be modulated according to their complexity and that large problems can be tackled. Whereas analytical results were derived in (Lolito et al., 2005), (Farhi et al., 2007), we here focus on modeling of traffic flow. In fact, this first paper aims only at stating the adopted modeling methodology. We present simulations to validate the approach.

The paper is organized as follows.

In section 2, we show that vehicular traffic flow can be considered as a min-plus linear system, that is a linear system over min-plus algebra. Representation of such a system thanks to its impulse response is recalled.

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In section 3, we propose a methodology to model traffic flow as a min-plus linear system. A particular 2 inputs-2 outputs system is presented as a generic representation for a wide-variety of elementary roadway stretches. Then we show how such elementary models can be composed to obtain a model for a succession of roadway sections.
In section 4, two examples are presented. The first one concerns a basic road section (without any intersection). The proposed model is simple but rough in the sense that it leads to considered that an infinity of vehicles can simultaneously run on the section. The second example is a refinement (taking into account the limited capacity), and the composition of two elementary models is experimented. In both cases, we derive from simulation the *fundamental diagram* that links the flow to the density of vehicles on the road.
In section 5, we discuss characteristics of the proposed model comparatively with traffic flow models in the literature.

2 PRELIMINARIES

In this section, we first explain why vehicular traffic flow on a road section can be studied as a linear system over min-plus algebra. Then we recall that min-plus linear systems, and in particular traffic flow, can be represented by their impulse responses.

2.1 Traffic on a Roadway Stretch is a Min-plus Linear System

As usual when studying complex systems, we consider a roadway network as an assembly of road sections. Each roadway stretch will be seen as a min-plus linear system, and so, a complex infrastructure will be studied as the system resulting from the assembly of corresponding elementary subsystems. In other words, each road section is considered to be a simple min-plus linear system whose input and output correspond to the flows of vehicles respectively entering and leaving the section.

Let us recall that min-plus linear systems are systems for which the property of linearity, also called "principle of superposition", can be applied to the two binary operations min and $+$ of the min-plus algebra (see for example (Gaubert, 1992)).

Definition 2.1 (Signal, Min-plus Linear System) *A signal u is defined as a map from $\mathbb{Z}$ to $\mathbb{R} \cup \{-\infty\}$.*
A system $\mathcal{S}$ is called min-plus linear if for all signal u, v, and $\forall t \in \mathbb{Z}$,

$$[\mathcal{S}(\min(u,v))](t) \quad = \quad \min([\mathcal{S}(u)](t), [\mathcal{S}(v)](t)), \quad (1)$$

and $\forall u, \forall a \in \mathbb{R}, \forall t \in \mathbb{Z}$,

$$[\mathcal{S}(a+u)](t) \quad = \quad a + [\mathcal{S}(u)](t). \quad (2)$$

in which $\mathcal{S}(u)$ is the system output signal in response to input u.

To study a road section as a min-plus linear system, let us give the following meanings for its input and output :

- the input u is a counter of vehicles entering the road section: $u(t)$ denotes the cumulated number of vehicles having entered the road section up to time t,
- the output y is a counter of vehicles leaving the road section: $y(t)$ denotes the cumulated number of vehicles having left the road section up to time t.

It is assumed that vehicles are conserved along a road section[1], *i.e.*, $u(t)$ is always equal to the sum of $y(t)$ and the number of vehicles on the road section at time t.

Under this assumption, the amount of vehicles out of the road section with $\min(u, v)$ as input flow, is claimed to be equal to the minimum of quantity of vehicles out of the road section obtained with u and v considered separately.
On the one hand, since $min(u, v)$ corresponds to a less dense flow than the ones given separately by u and v, vehicles of the flow given by $min(u, v)$ cross the road section at least as fast as the ones in flows u and v considered separately. So we deduce the following inequality:

$$\forall t\,,\; [\mathcal{S}(min(u,v))](t) \geq min([\mathcal{S}(u)](t), [\mathcal{S}(v)](t)), \quad (3)$$

On the other hand, causality of the system induces the converse inequality. More precisely, since

$$\forall t,\; [min(u,v)](t) \leq u(t),$$

we have

$$\forall t,\; [\mathcal{S}(min(u,v))](t) \leq [\mathcal{S}(u)](t),$$

that is the amount of vehicles out the road section is at any time t greater with u than with $min(u, v)$ as input flow. With similar arguments we have $\forall t,\; [\mathcal{S}(min(u,v))](t) \leq [\mathcal{S}(v)](t)$, and we deduce that

$$[\mathcal{S}(min(u,v))](t) \leq min([\mathcal{S}(u)](t), [\mathcal{S}(v)](t)). \quad (4)$$

Inequalities (3) and (4) satisfied by vehicular traffic flow on a roadway stretch correspond to the first condition (1) defining a min-plus linear system.

[1] Additions or withdrawals of vehicles *via* entry/exit lanes will be taken into account as additional inputs/outputs in the model.

Still considering that no vehicle can disappear along a road section, the amount of vehicles out of the road section with $a+u$ as input flow, is claimed to be equal to the sum of a with the amount of vehicles out of the road section with u as input flow. Furthermore, in the input flow $a+u$ at time t, the a vehicles can be considered to have been added to flow u for a sufficiently long time so that they have already crossed the road section. From these observations, we deduce that a road traffic system satisfies the second condition (2) defining a min-plus linear system.

2.2 Min-plus Representations for Roadway Stretches

We have shown that road sections considered in §2.1 could be studied as min-plus linear systems. Then, they can be represented by their impulse responses (see for example (Gaubert, 1992), (Lahaye, 2000)).

Definition 2.2 (Impulse Response) *Let $\mathcal{S}$ be a min-plus linear system, there exists a unique mapping h, called impulse response, such that $y = \mathcal{S}(u)$ is expressed as:*

$$\forall u, \forall t\,;\; y(t) \triangleq \min_{s\leq t}\{h(s)+u(t-s)\} = (h\otimes u)(t).$$

The system output is nothing but the *inf-convolution* - which plays the role of convolution in min-plus linear systems theory - between its impulse response and the system input u.

In the following, we may rather consider a lower approximation denoted β of an impulse response h. Such an approximation is analogous to the service curve usually used in Network Calculus theory (see (Cruz, 1991a), (Cruz, 1991b), (Boudec and Thiran, 2001)). Considering that:

$$\forall t\,,\quad h(t) \geq \beta(t), \tag{5}$$

we have by isotony of the convolution product ($\otimes$):

$$\forall t\,, y(t) = (h\otimes u)(t) \geq (\beta\otimes u)(t),$$

which means that $(\beta\otimes u)(t)$ is a lower approximation of the system output, that is $(\beta\otimes u)(t)$ gives a minimal flow of vehicles leaving the road section. Such a lower approximation β is used in particular when exact identification of h is not possible. We show in the following lemma that a mapping β' such that $\forall t\,, (\beta'\otimes u)(t) \leq y(t)$ is a lower approximation of the impulse response h of the system.

Lemma 2.1 *Let β' be a mapping such that $\forall u, y \geq \beta'\otimes u$ with $y = h\otimes u$, then we have: $\forall t\,, h(t) \geq \beta'(t)$.*

Proof Let us define the particular signal $\delta(t)$ as:

$$\delta(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{6}$$

Then we easily check that $y = h\otimes u \geq \beta'\otimes u$ implies $h \geq \beta'$ by taking $u = \delta$.

Finally, let us give an interpretation to mappings h and β. An input equal to the signal δ defined by (6), comes down to considering that

- no vehicle is in the system before $t = 0$,
- an infinity of vehicles are available to enter the system as soon as $t > 0$.

Then we have with $u = \delta$

$$\begin{aligned} y(t) = (h\otimes u)(t) &= \min_{s\leq t}\{h(s)+u(t-s)\} \\ &= \min(\underbrace{h(s)+(+\infty)}_{s<t}, \underbrace{h(t)+0}_{s=t}) \\ &= h(t) \end{aligned}$$

So for all t, $h(t)$ can be interpreted as the number of vehicles having crossed the corresponding road section up to time t while an infinity of vehicles could enter the section from time 0. In other words, $h(t)$ can be interpreted as the maximum number of vehicles that can be "served" during the time interval $[0,t]$. Since $h(t) \geq \beta(t)$, $\forall t$, $\beta(t)$ can be interpreted as a lower approximation of the maximum number of vehicles that can cross the section during $[0,t]$.

3 PROPOSED MODELING METHODOLOGY

In this section, a modeling methodology is proposed for roadways as min-plus linear systems. In a first place, we select a particular min-plus linear system to represent any road section. This elementary representation is intended to be sufficiently generic to model various roadway stretches (with specific parameters). These constitute the elementary bricks we shall combine to build models for larger infrastructures. The way these bricks are assembled is explained in a second place.

3.1 A Generic Model for Elementary Road Sections

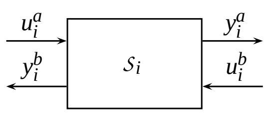

Figure 1: Generic system $\mathcal{S}_i$ with 2 inputs and 2 outputs proposed to represent any road section.

We propose to model any road section as a min-plus linear system with two inputs and two outputs interpreted as the following counters:

- $u_i^a(t)$ denotes the cumulated number of vehicles having entered the section indexed i up to time t,
- $u_i^b(t)$ denotes the cumulated number of vehicles authorized to leave the section i up to time t,
- $y_i^a(t)$ denotes the cumulated number of vehicles having left the section indexed i up to time t,
- $y_i^b(t)$ denotes the cumulated number of vehicles authorized to enter the section i up to time t.

Signals $u_i^b(t)$ and $y_i^b(t)$ will be used to take into account specific phenomena and/or the mutual influences between successive sections. They are intended to enable to model a wide variety of roadway stretches. For examples:

- For a section ending with a traffic light, the input signal $u_i^b(t)$ will be used to traduce the successive light phases.
- Output $y_i^b(t)$ will be suitable to model the effect of a congestion on the upstream sections.

The min-plus linear system $\mathcal{S}_i$ admits the following representation

$$\left\{ \begin{array}{lcl} y_i^a & = & \Theta_i u_i^a \oplus \Sigma_i u_i^b \\ y_i^b & = & \Phi_i u_i^a \oplus \Gamma_i u_i^b \end{array} \right. \tag{7}$$

in which notation $\oplus$ stands for the point-wise minimum of signals, that is $(u \oplus v)(t) = min(u(t), v(t))$, $\forall t$. The impulse responses Θ_i, Σ_i, Φ_i and Γ_i traduce respective influences of the inputs on the two outputs.

3.2 Model for a Succession of Elementary Road Sections

An important feature of linear systems is that they can be cascaded in series, in parallel or put in feedback and then we always get a linear system. In this section, we explain how proposed generic models should be assembled in order to get a model for successive road sections. In particular, we give the representation obtained for two road sections cascaded in series.

Let us consider two successive road sections indexed i and $i+1$, and modeled by min-plus linear systems $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$. They are respectively represented by

$$\left\{ \begin{array}{lcl} y_i^a & = & \Theta_i u_i^a \oplus \Sigma_i u_i^b \\ y_i^b & = & \Phi_i u_i^a \oplus \Gamma_i u_i^b \end{array} \right. , \tag{8}$$

and

$$\left\{ \begin{array}{lcl} y_{i+1}^a & = & \Theta_{i+1} u_{i+1}^a \oplus \Sigma_{i+1} u_{i+1}^b \\ y_{i+1}^b & = & \Phi_{i+1} u_{i+1}^a \oplus \Gamma_{i+1} u_{i+1}^b \end{array} \right. . \tag{9}$$

Cascading $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ comes down to merging inputs and outputs of $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ in the following way (see figure (2)):

$$u_{i+1}^a = y_i^a, \tag{10}$$

$$u_i^b = y_{i+1}^b. \tag{11}$$

In other words, we merely consider that:

- the cumulated number of vehicles leaving section i (given by y_i^a) is equal to the cumulated number of vehicles entering section $i+1$ (given by u_{i+1}^a),
- the cumulated number of vehicles authorized to enter section $i+1$ (that is y_{i+1}^b) is equal to the cumulated number of vehicles authorized to leave section i (*i.e.* u_i^b).

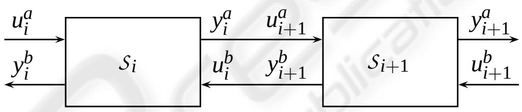

Figure 2: Proposed assembly for $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ modelling two successive road sections.

Min-plus linear system theory has shown that the resulting system when cascading is also a min-plus linear system (see (Gaubert, 1992) or (Lahaye, 2000)). In the following, we explicit the representation for this system, denoted $\mathcal{S}$, that is the corresponding impulse responses Θ, Σ, Φ and Γ.

Equations (8) and (9) lead to:

$$\left\{ \begin{array}{lcl} y_{i+1}^a & = & \Theta_{i+1} u_{i+1}^a \oplus \Sigma_{i+1} u_{i+1}^b \\ y_i^b & = & \Phi_i u_i^a \oplus \Gamma_i u_i^b \end{array} \right. . \tag{12}$$

From equations (10) and (11) we have:

$$\begin{array}{lcl} u_{i+1}^a & = & y_i^a \\ & = & \Theta_i u_i^a \oplus \Sigma_i u_i^b \\ & = & \Theta_i u_i^a \oplus \Sigma_i y_{i+1}^b \\ & = & \Theta_i u_i^a \oplus \Sigma_i \Phi_{i+1} u_{i+1}^a \oplus \Sigma_i \Gamma_{i+1} u_{i+1}^b \\ & = & (\Sigma_i \Phi_{i+1})^* \Theta_i u_i^a \\ & & \oplus (\Sigma_i \Phi_{i+1})^* \Sigma_i \Gamma_{i+1} u_{i+1}^b \end{array} \tag{13}$$

The star notation a^* (often referred to as "Kleene star operation") stands for $\bigoplus_{n \in \mathbb{N}} \underbrace{a \otimes a \ldots \otimes a}_{\text{n times}}$. The last equality gives the least solution to previous implicit equation (see (Baccelli et al., 1992, §4.5)). Selecting the least solution means that we are interested in the earliest functioning of the system $\mathcal{S}$. On the other hand, we have:

$$\begin{aligned} u_i^b &= y_{i+1}^b \\ &= \Phi_{i+1} u_{i+1}^a \oplus \Gamma_{i+1} u_{i+1}^b \\ &= \Phi_{i+1} y_i^a \oplus \Gamma_{i+1} u_{i+1}^b \\ &= \Phi_{i+1} \Theta_i u_i^a \oplus \Phi_{i+1} \Sigma_i u_i^b \oplus \Gamma_{i+1} u_{i+1}^b \\ &= (\Phi_{i+1}\Sigma_i)^* \Phi_{i+1} \Theta_i u_i^a \\ &\quad \oplus (\Phi_{i+1}\Sigma_i)^* \Gamma_{i+1} u_{i+1}^b \end{aligned} \tag{14}$$

Substituting signals u_{i+1}^a and u_i^b by (13) and (14) in equations (12) leads to obtain outputs of $\mathcal{S}$ (that is y_{i+1}^a, y_i^b) in response to its inputs (that is u_i^a, u_{i+1}^b):

$$\left\{ \begin{array}{rl} y_{i+1}^a = & \Theta_{i+1}\left[(\Sigma_i\Phi_{i+1})^*\Theta_i u_i^a \oplus (\Sigma_i\Phi_{i+1})^*\Sigma_i\Gamma_{i+1}u_{i+1}^b\right] \\ & \oplus \Sigma_{i+1} u_{i+1}^b \\ y_i^b = & \Phi_i u_i^a \\ & \oplus \Gamma_i \left[(\Phi_{i+1}\Sigma_i)^*\Phi_{i+1}\Theta_i u_i^a \oplus (\Phi_{i+1}\Sigma_i)^*\Gamma_{i+1}u_{i+1}^b\right] \end{array} \right.$$

$$\Leftrightarrow$$

$$\left\{ \begin{array}{rl} y_{i+1}^a = & (\Sigma_i\Phi_{i+1})^*\Theta_{i+1}\Theta_i u_i^a \\ & \oplus \left[\Sigma_{i+1} \oplus (\Sigma_i\Phi_{i+1})^*\Theta_{i+1}\Sigma_i\Gamma_{i+1}\right] u_{i+1}^b \\ y_i^b = & \left[\Phi_i \oplus (\Phi_{i+1}\Sigma_i)^*\Phi_{i+1}\Theta_i\Gamma_i\right] u_i^a \\ & \oplus (\Phi_{i+1}\Sigma_i)^*\Gamma_{i+1}\Gamma_i u_{i+1}^b \end{array} \right.$$

4 EXAMPLES

We have shown in section 2 how road traffic can be studied as a min-plus linear system. In this section, we apply the modeling methodology proposed at section 3 to model and simulate two kinds of elementary roadway stretches.

4.1 Elementary Stretch of a Roadway

As a first example we consider a simple stretch of roadway without any facilities: no traffic light, no intersection, no entry and no exit lanes… Furthermore we assume that the stream of vehicles on the considered road section is not affected by upstream and downstream traffic. According to the methodology proposed in section 3, the road section is described as a min-plus linear system $\mathcal{S}_1$ represented by:

$$\left\{ \begin{array}{rcl} y_1^a &=& \Theta_1 u_1^a \oplus \Sigma_1 u_1^b \\ y_1^b &=& \Phi_1 u_1^a \oplus \Gamma_1 u_1^b \end{array} \right.$$

with

$$\begin{array}{ll} \Theta_1 = h_1 & \Sigma_1 = \varepsilon \\ \Phi_1 = \varepsilon & \Gamma_1 = \varepsilon \end{array}$$

in which ε denotes the "null signal" (in this case the "null impulse response"), with respect to the additive law $\oplus$ (corresponding to the pointwise min). That is:

$$\forall t, \varepsilon(t) = +\infty$$

and we have for all signal u

$$\forall t, \varepsilon(t) \oplus u(t) = u(t) \oplus \varepsilon(t) = min(u(t), +\infty) = u(t).$$

And so we simply have

$$y_1^a = h_1 \otimes u_1^a.$$

As a reminder, $h_1(t)$ (respectively $\beta_1(t)$) denotes the maximal number (resp. a lower approximation of the maximal number) of vehicles that can cross the corresponding road section up to time t while an infinity of vehicles could enter the section from time 0. In future works such models h_1 and β_1 are intended to be obtained through an identification procedure using road traffic counting experiments. Previous works (see for example (Menguy et al., 2000)) have investigated such identification problems for linear systems on idempotent semi-rings (such as the min-plus algebra). By anticipation, we consider here a mapping β_1 as defined on figure 3. The latency (equal to T seconds) corresponds to time spent to cross the road section for an isolated vehicle (not slowed down by other vehicles). The slopes at $t > T$ are then related to the flow of vehicles on the road section knowing that $\beta_1(t)$ vehicles have passed the section up to time t. The chosen shape for β_1 take care of the fact that vehicles run slower in relation to the density of traffic.

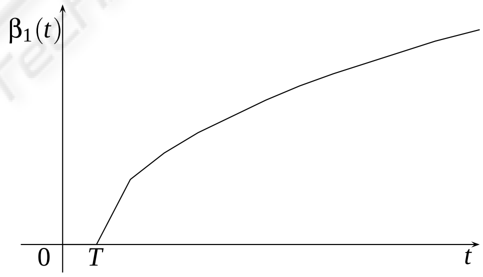

Figure 3: Mapping β_1 used to model a simple roadway stretch.

Thanks to the considered min-plus linear representation, traffic can be simulated, notably by using a C++ library developed by the COINC research group (COINC, 2009). This software enables to define signals as functions composed of segments (such as β_1 in figure 3) and implement the expected operations on signals: min-plus convolution ($\otimes$), pointwise min ($\oplus$), Kleene star operation ($*$), …. We here have used the library to compute the signal $\beta_1 \otimes u_1^a \geq y_1^a$ with u_1^a defined as ramp signal with slope rate r. This means to considering that vehicles enter indefinitely the section from time 0 with a constant input flow r (*i.e.* r vehicles enter per second). Doing so with r varying from a value lower than the asymptotic rate of β_1 to a value greater to it (see figure 4), we expect to study the behaviour for all traffic conditions.

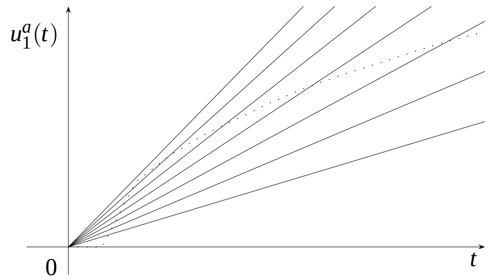

Figure 4: Input signals u_1^a considered for the simulations.

For each simulation (with a given value of r), we have computed for all $t \in [0,200]$

- the so-called "backlog", that is the amount of vehicles on the section at time t, given by

$$u_1(t) - y_1(t)$$

- the "virtual delay", that is the travel time for vehicle(s) entered in the section at time t if they don't overtake vehicles entered before. The virtual delay is given by

$$\inf\{\tau \geq 0 : u_1(t) \leq y_1(t+\tau)\}.$$

These values are respectively proportional to the vehicles density K and the flow Q. We then obtain the *fundamental diagram* of figure 5.This diagram gives the relation between the flow and the vehicles density for the car traffic on a road. It has been observed empirically and derived theoretically in the case of a unique road or a regular system of roads (see for example (Helbing, 2001)).

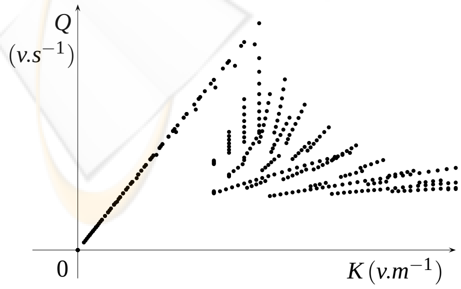

Figure 5: Fundamental diagram obtained for a simple road section.

Let us mention that with this model if signal u_1^a is such that $\lim_{t\to\infty} u_1^a(t) - \beta_1(t) = +\infty$, then since $y_1^a(t) = \min_{s\leq t}(\beta_1(t-s) + u_1^a(s)) \leq \beta_1(t)$ we have $\lim_{t\to\infty} u_1^a(t) - y_1^a(t) \geq \lim_{t\to\infty} u_1^a(t) - \beta_1(t) = +\infty$. This means that if the arrival of vehicles on the road exceeds its service capacity then the section will asymptotically contain an infinity of vehicles. The next example notably shows how to take into account the intrinsic limited capacity of a section.

4.2 Succesion of Elementary Stretches with Limited Capacities

We now consider two successive road sections as represented on figure 6. These roadway stretches modeled by β_1 and β_2 are also supposed to be without any facilities (no traffic light, ...), but W_1 and W_2 have been added to limit their capacities. More precisely, the maximum amount of vehicles that section 1 can contain is supposed to correspond to the integer W_1. We then have

$$\forall t, u_1^a(t) = min(u(t), y_1^a(t) + W_1),$$

in which $u_1^a(t)$ denotes the amount of vehicles likely to enter section 1 up to time t. This leads to

$$\forall t, u_1^a(t) \leq y_1^a(t) + W_1 \;\Leftrightarrow\; u_1^a(t) - y_1^a(t) \leq W_1$$

which shows that the number of vehicles on the section given by $u_1^a(t) - y_1^a(t)$ is then well bounded by W_1.
Referring to equations (7) defining the generic model, we deduce for $i = 1,2$ that

$$\begin{array}{ll} \Theta_i = h_i & \Sigma_i = e \\ \Phi_i = W_i h_i & \Gamma_i = W_i \end{array}.$$

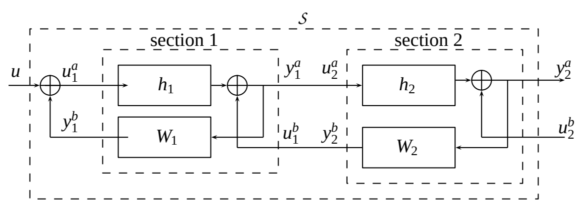

Figure 6: Two successive road sections.

From results detailed in section 3.2, we can compute the representation for the system resulting from section 1 and 2 in cascade. Defining two mappings β_1 and β_2 comparable (but different slopes) as that of figure 3, we have simulated the system as in section 4.1. We then obtain two fundamental diagrams:

- diagram of figure 7 obtained for flows between u and y_2^a, with $u(t)$ denoting the amount of vehicles which have been candidates for entering section 1 up to time t;

- diagram of figure 8 obtained for flows between u_1^a and y_2^a, with $u_1^a(t)$ denoting the amount of vehicles having entered section 1 up to time t.

Figure 7: Fundamental diagram for two successive sections with limited capacities (flows between u and y_2^a).

Figure 8: Fundamental diagram for two successive sections with limited capacities (flows between u_1^a and y_2^a).

5 DISCUSSION ON THE PROPOSED (MIN,+) MODEL

For about fifty years, mathematical description of traffic flow has been a lively subject of research and debate for traffic engineers. This has resulted in a broad scope of models describing different aspects of traffic flow operations which can be classified according to various criteria: level of detail, application area and scale of application, deductive or inductive description of phenomena,...(see for example the survey (Hoogendoorn and Bovy, 2001)). In this section, we discuss the location of the proposed model in these classifications.

Traffic flow can be described either by considering the time-space behavior of individual drivers under the influence of vehicles in their proximity (microscopic models), the behaviour of drivers without explicitly distinguishing their time-space behavior (mesoscopic models), or from the viewpoint of the collective vehicular flow (macroscopic models). According to the level-of-detail, variables have different natures. In a microscopic model variables describe individually behaviors and interactions of the systems entities (i.e. vehicles and drivers). In macroscopic flow models, the traffic stream is represented in an aggregate manner using characteristics as flow-rate, density, and velocity. In our model, we manipulate functions which count every vehicle (from which we have derived aggregated characteristics such as flow-rate and density), but their behavior is not necessarily described individually through the approximation β of the impulse response (unless the road section is sized such that it contains only one vehicle). In that sense, our model could be considered as an intermediate approach which, according to the size of the modeled sections, fluctuates between microscopic and macroscopic approaches. Our description of observed phenomena is also somewhat intermediate. In fact, the choice of β and the limitation of capacity on sections come under a deductive approach of phenomena which are known or which can be guessed. But the mapping β should be the fitted thanks to an inductive approach, that is using input/output data from real systems.

Although it has only be used to simulate traffic flow in the present paper, we expect that analytical results can be derived from the proposed model. In fact, for more than two decades a new system theory has been developed for systems linear over idempotent semi-rings (such as min-plus algebra): numerous results have been proposed for performance evaluation, control,... (see (Cohen, 2006) for a recent survey). In future works, some of these results should be applied/adapted to study vehicular traffic flow.

In this paper, we have considered elementary stretches of roadways. On the one hand, we expect that the chosen framework is sufficiently generic to model a wide variety of more complex road sections (with or without entry/exit lane, intersections, ...). On the other hand, the way to aggregate elementary models is simple enough to consider that models for large infrastructures can be obtained.

6 CONCLUSIONS

We have proposed a modeling method for traffic flow. Each road section is modeled by its impulse response in min-plus algebra. This model has been used to simulate and derive the fundamental diagram for elementary road stretches.

Future works will concern an identification method to build an approximation β of the impulse response from real road traffic data. We also plan to study more complex roadway stretches (for example phenomena associated to a traffic light), and to adapt existing results from linear system theory over min-plus algebraic to derive analytical results on traffic flow.

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