THE (PROBABILISTIC) LOGICAL CONTENT OF CADIAG2
Rule-based Probabilistic Approach

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Abstract: *Cadiag*2 is a well-known rule-based expert system that aims at providing support for medical diagnose in internal medicine. *Cadiag*2 consists of a knowledge base in the form of a set of *if-then* rules that relate medical entities, in this paper interpreted as *conditional probabilistic statements*, and an inference engine constructed upon methods of *fuzzy set theory*. The aim underlying this paper is the understanding of the inference in *Cadiag*2. To that purpose we provide a (probabilistic) logical formalization of the inference of the system and check its adequacy with probability theory.

1 INTRODUCTION

*Cadiag*2 (Computer Assisted DIAGnosis) is a well-known rule-based expert system aimed at providing support in diagnostic decision making in the field of internal medicine. Its design and construction was initiated in the early 80's at the Medical University of Vienna by K.P. Adlassnig – see (Adlassnig et al., 1986), (Adlassnig et al., 1985), (Adlassnig, 1986) or (Leitich et al., 2002) for more on the origins and design of *Cadiag*2.

*Cadiag*2 consists of two fundamental pieces: the inference engine and the knowledge base. The inference engine is based on methods of approximate reasoning in *fuzzy set theory*, in the sense of (Zadeh, 1965) and (Zadeh, 1975). In fact *Cadiag*2 is presented in some monographs as an example of a *fuzzy expert system*, (Klir and Folger, 1988), (Zimmermann, 1991). The knowledge base, Φ_{Cad}, consists of a set of *if-then* rules intended to represent relationships between distinct medical entities: symptoms, findings, signs and test results on the one hand and diseases and therapies on the other. The number of rules in Φ_{Cad} is approximately 50.000. The vast majority of them are binary (i.e., they relate single medical entities) and only such rules are considered in this paper. The rules in Φ_{Cad} are defined along with a certain *degree of confirmation* which intuitively expresses the *degree* to which the antecedent *confirms* the consequent. For example,

> *IF suspicion of liver metastases by liver palpation THEN may be pancreatic cancer with degree of confirmation 0.30.*[1]

As mentioned in (Adlassnig, 1986) we can identify such degrees of confirmation with probabilities and the rules themselves with conditional probabilistic statements. In (Adlassnig, 1986) it is stated that such degrees of confirmation can be interpreted as *frequencies*. An interpretation in terms of *degrees of belief* of the doctor (or doctors) on the truth of the consequent given that the antecedent of the rule holds is also possible though. This fact motivates a probabilistic interpretation of *Cadiag*2's inference. Such an interpretation leads to the primary aim of this paper: formalise the inference in *Cadiag*2 on probabilistic grounds and check its adequacy with probability logic (Halpern, 2003) or, more generally, with probability theory. We shall not expect big surprises in this respect. The inference mechanism in *Cadiag*2 proceeds in a *compositional* way and thus it is bound to be probabilistically unsound (as will be clarified later). This was soon observed in earlier studies concerning the celebrated expert system MYCIN – see (Buchanan and Shortliffe, 1984) or (Shortliffe, 1976) for a description of MYCIN and (Hajek, 1988), (Hajek, 1989), (Hajek and Valdés, 1994), (Heckerman, 1986), (Valdés, 1992) for probabilistic approaches to it. How far is *Cadiag*2's inference from probabilistic soundness remains to be seen though.

It is worth mentioning here that, although the interest among theoretical AI researchers in rule-based

[1] This rule is mentioned as an example in (Adlassnig et al., 1986).

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expert systems seems to be lesser today than some years ago, rule-based expert systems are still very popular among AI engineers. Many *Cadiag*2-like systems are in use and more are being built for future implementation. Is is mainly for this reason that we believe that further analysis and understanding of *Cadiag*2-like systems is of relevance.

This paper is in some way a continuation of (Ciabattoni and Vetterlein, 2009). In (Ciabattoni and Vetterlein, 2009) the inference mechanism of *Cadiag*2 is formalised by means of a logical calculus, *CadL*, and compared to *t-norm-based* formalisms (Hajek, 1998). It is shown that *CadL* does not respond to any t-norm-based (or to any fragment of a t-norm-based) logic. As far as we know, (Ciabattoni and Vetterlein, 2009) constitutes the first attempt at formalising and understanding *Cadiag*2 in a logical way. The present paper is the second.

This paper is structured as follows. In Section 2 we give some basic definitions and introduce most of the notation used later in the other sections. In Section 3 the inference process in *Cadiag*2 is briefly described. In Section 4 the formal system *CadPL* is defined and analysed in the light of probability logic. *CadPL* is a formalization of the inference mechanism of *Cadiag*2 based on a probabilistic interpretation of it.

2 PRELIMINARY DEFINITIONS AND NOTATION

Throughout we will be working with a finite propositional language, $L = \{p_1, \ldots, p_n\}$. We will denote by *SL* its closure under classical connectives. Within the context of *Cadiag*2 the language L represents the set of medical entities in the system.

Let $L_{Lit} = \{p\ ,\neg p \mid p \in L\} \subset SL$, the set of literals of the language L.

Let $\Delta = \{\phi_1, \ldots, \phi_k\} \subseteq SL$. We will denote by $\bigwedge \Delta$ the sentence $\phi_1 \wedge \ldots \wedge \phi_k$.

Definition 1. *Let $w : SL \longrightarrow [0,1]$. We say that w is a probability function on L if the following two conditions hold, for all $\theta, \phi \in SL$:*

- *If $\models \theta$ then $w(\theta) = 1$.*
- *If $\models \neg(\theta \wedge \phi)$ then $w(\theta \vee \phi) = w(\theta) + w(\phi)$.*[2]

We define conditional probability from the notion of unconditional probability in the conventional way. For w a probability function on L and $\phi, \theta \in SL$,

$$w(\phi|\theta) = \frac{w(\phi \wedge \theta)}{w(\theta)}.$$

[2]Here (and throughout) $\models$ is classical entailment.

The statements we will be dealing with are primarily of the form '*the probability of* θ *given* ϕ *is equal to* η'. Let $\mathcal{FL}^{=}$ be the set of all statements of the form $P(\theta|\phi) = \eta$, for $\theta, \phi \in SL$ and $\eta \in [0,1]$. Occasionally we will refer to the set $\mathcal{FL}^{\geq}$, defined like $\mathcal{FL}^{=}$ but with '$\geq$' in place of '$=$'.

We will refer to ϕ in a statement of the form $P(\theta|\phi) = \eta$ as the *evidence* and to θ as the *uncertain entity* or *event*.

We will denote by $\mathcal{FL}_s^{=}$ the subset of conditional statements of $\mathcal{FL}^{=}$ where both the evidence and the uncertain entity are literals, i.e. sentences in L_{Lit}. By $\mathcal{FL}_c^{=}$ we will denote the subset of conditional statements of $\mathcal{FL}^{=}$ where the uncertain entity is a literal and the evidence consists of a conjunction of literals (we define $\mathcal{FL}_s^{\geq}$ and $\mathcal{FL}_c^{\geq}$ analogously).

The binary fragment of *Cadiag*2's knowledge base, Φ_{CadBin}, will be in principle regarded as a subset of $\mathcal{FL}_s^{=}$. That is arguably the most natural interpretation of Φ_{CadBin} when interpreting the rules probabilistically.

Let $\Theta \in \mathcal{FL}^{=}$ and w a probability function on L. We define satisfiability of Θ by w (denoted $\models_w \Theta$) in the obvious way. More specifically, for $\eta \in [0,1]$ and $\theta, \phi \in SL$,

$$\models_w P(\theta|\phi) = \eta \iff w(\theta|\phi) = \eta.$$

Satisfiability for statements in $\mathcal{FL}^{\geq}$ is defined analogously. Such notion of *satisfiability* is extended to subsets in $\mathcal{FL}^{=}$ and $\mathcal{FL}^{\geq}$ in its trivial way. We will sometimes identify the notion of *consistency* of a set of probabilistic statements with that of satisfiability.

Definition 2. *Let $\preceq$ be the partial ordering relation on $[0,1]$ defined as follows: For $a, b \in [0,1]$, $a \preceq b$ if and only if $0 < a \leq b$ or $0 < a < 1$ and $b = 0$.*

We define $\prec$ from $\preceq$ in the conventional way.

As we will see later, the definition of $\prec$ responds to the use of both 0 and 1 as maximal values in *Cadiag*2. The value 0 denotes certainty in the non-occurrence of an event or falsity of a statement and the value 1 denotes certainty in its occurrence or its truth.

For the next definition let

$$\mathbb{D} = [0,1] \times [0,1] - \{(0,1), (1,0)\}.$$

Definition 3. *The function $max^* : \mathbb{D} \longrightarrow \mathbb{R}$ is defined as follows, for $(a,b) \in \mathbb{D}$:*

$$max^*(a,b) = \begin{cases} a & if \ \ b \prec a \\ b & otherwise \end{cases}$$

The definition of max^* is extended to more than two arguments in its trivial way.

3 THE INFERENCE IN CADIAG2

In this section we describe very briefly a generalization of the inference mechanism in *Cadiag*2. A more detailed description and analysis of it can be found in (Ciabattoni and Vetterlein, 2009).

*Cadiag*2 formally distinguishes between three different types of rules: type *confirming to the degree d* (for $d \in [0,1]$), type *mutually exclusive* and type *always occurring* – see (Adlassnig et al., 1986), (Adlassnig et al., 1985), (Adlassnig, 1986) or (Daniel et al., 1997) for more on *Cadiag*2's rules. The last two types mentioned are *classical* in the sense that the degree of confirmation for the rules of these types is 1 and that the antecedent of such rules (or evidence in our settings) needs to be *fully* true (*degree of presence or of truth* 1, see below) in order for these rules to be triggered by the system. Such a distinction is not taken into consideration in this paper and it is in this sense that we say that our description of the inference mechanism of *Cadiag*2 is actually a generalisation of the *real* inference process. The inference engine in *Cadiag*2 gets started with a set of symptoms, findings, signs and diseases occurring in Φ_{CadBin} present in the patient. Let Γ be the set of such medical entities.

Cadiag2 starts with an assignment w_0 on Γ that gives a value in the interval $[0,1]$ to each entity in Γ. Such value is intended to represent the degree to which the entity is present in the patient. The intended interpretation of such values is based, in principle, on fuzzy set theory. However, other interpretations can also be suitable, at least to some extent. In fact, when defining the system *CadPL*, the interpretation to which we will commit will be probabilistic. The assignment w_0 is then extended to negative statements and logical equivalents according to the following rule:

> If $w_0(\phi) = \eta$ then $w_0(\neg\phi) = 1 - \eta$, for $\phi \in SL$ and $\eta \in [0,1]$.

After the initial assignment the inference rules in Φ_{CadBin} come into play. All the rules triggered by the sentences in Γ are used during the inference process. At each step in the inference process a rule is applied (that is done, in principle, in no particular order). At the first step in the inference a rule of the form $P(\theta|\phi) = \eta$ in Φ_{CadBin} is triggered, with $\eta \in [0,1]$ and $\theta, \phi \in L_{Lit}$. In order for that to happen ϕ or its negation needs to be in Γ and the value $w_0(\phi)$ has to be strictly positive. The application of the rule $P(\theta|\phi) = \eta$ generates a new assignment, w_1, on $\{\theta\}$. The value assigned to θ by it is calculated as the minimum between η and $w_0(\phi)$ and the value assigned to $\neg\theta$ and logical equivalents (if necessary for the inference) is calculated from $w_1(\theta)$ as mentioned above for w_0.

At the n^{th} step in the inference process a new rule of the form $P(\theta|\psi) = \eta$ in Φ_{CadBin} will be triggered, for $\eta \in [0,1]$ and $\theta, \psi \in L_{Lit}$. In order for $P(\theta|\psi) = \eta$ to be triggered ψ must have been assigned at least one value in $(0,1]$ either by the initial assignment, w_0, or by any other assignment on $\{\psi\}$ defined during the inference process at some previous step. At the n^{th} step the application of this new rule will generate a new assignment on $\{\theta\}$ that will give θ the minimum between η and the value of ψ considered for triggering the rule at this step in the inference (as above, this value needs to be strictly positive). If the strictly positive values generated for ψ before the n^{th} step are multiple the inference mechanism in *Cadiag*2 will call the rule $P(\theta|\psi) = \eta$ again in further steps, if it has not done so previously, until all the values for ψ have been used and all the possible values for θ generated. The assignment w_n is defined to $\neg\theta$ as mentioned above.

The inference process goes on until all the rules triggered by all the sentences in Γ and its negations have been used and all the possible assignments for the sentences involved in the inference have been generated. *Cadiag*2 yields as an outcome of the inference the set of medical entities in L occurring in the rules triggered by the evidence in Γ along with the *maximal value* (with respect to the ordering $\preceq$ defined in Section 2) assigned to them during the inference. If a sentence is assigned both value 0 and 1 along the inference process the system generates an error message.

It is worth mentioning that the original inference process in *Cadiag*2 works in a slightly different way. The update in the value of the distinct sentences involved in the inference is done as soon as two different values for the same sentence are produced by the system. The value chosen in the update for atomic sentences in L is the *maximal* one (with respect to the ordering $\preceq$). Notice though that this feature has a highly undesirable result (unless further restrictions on the rules or on the order in which the rules are applied are imposed), which is that the outcome of a run of the inference mechanism can depend on the order in which the rules are applied.

Such a drawback is easily avoided by assuming that the update is only done at the end of the process. There are other several undesirable features in *Cadiag*2's inference engine, most of them related to the maximal value 0 and negated propositions. Maybe the most evident concerning the maximal value 0 is that a medical entity that at some step along the inference process is assigned value 0 (that is to say, it is considered false with certainty or impossible) *triggers* any rules in which it occurs as evidence if any other value other than 0 is assigned to it along the inference process. For a deeper analysis of such aspects of the inference process in *Cadiag*2 see (Ciabattoni and Vetterlein, 2009).

We represent sentences together with the assignments generated for them at each step in the inference by pairs in $SL \times [0,1]$ along with a subscript indicating the step in the process at which such pairs have been generated. As mentioned above, a step in the inference process is given by the application of a rule in Φ_{CadBin} and the new assignments that it generates for the sentences involved in the rule.

Let $p \in L$ and $\eta \in [0,1]$ be the highest assignment to p in a run of the inference mechanism in *Cadiag*2. We will use the subscript max^* on the pair (p,η) – that is to say, $(p,\eta)_{max^*}$ – to denote that η is the *maximal* value assigned during the inference process for p (with respect to the ordering $\preceq$).

4 THE FORMAL SYSTEM CADPL

Some medical entities that occur in the rules of *Cadiag*2 represent statements that are *vague*. For example, in *Cadiag*2 we have a medical entity given by the following statement: '*reduced glucose in serum*'.[3] In such a statement the adjective '*reduced*' is vague.

*Cadiag*2 tackles vagueness by assigning values to medical entities in the interval $[0,1]$. Such values stand in principle for fuzzy membership within the context of fuzzy set theory – see (Adlassnig et al., 1986), (Adlassnig et al., 1985) or (Adlassnig, 1986). In this paper we consider the possibility of interpreting such values as probabilities, which can be done in quite intuitive ways given the nature of the statements we are dealing with.

Let us consider again the statement '*reduced glucose in serum*'. Let us assume that the value assigned by the evaluation system in *Cadiag*2 to the statement '*Patient A has reduced glucose in serum*' out of the evidence given by the corresponding measurement of the amount of glucose in Patient A is η, for some $\eta \in [0,1]$. As an example, we could interpret such value as the *degree of belief* that a medical doctor has in the truth of the statement given the evidence. As such η could be interpreted as a probability. The probabilistic interpretation is certainly favoured by the *discretization* applied to medical concepts in *Cadiag*2 (for example, the concept '*glucose in serum*' generates five distinct medical entities in *Cadiag*2: '*highly reduced glucose in serum*', '*reduced glucose in serum*', '*normal glucose in serum*', '*elevated glucose in serum*' and '*highly elevated glucose in serum*'). Notice that such an interpretation places us within the subjective probabilistic frame and thus, for the sake of coherence, the knowledge base Φ_{CadBin} should also be interpreted subjectively. Other interpretations are also possible though. For example, one could regard such values as the ratio given by the number of doctors that agree on the truth of the statement out of all the doctors involved in the assessment. In order to accommodate such values into a coherent probabilistic frame along with the statements in Φ_{CadBin} one could justify them as being *subjective* probabilities assessed by a *group* of experts – see (Genest and Zidek, 1986) or (Osherson and Vardi, 2006) for an analysis and justification of such concept.

Let $\phi \in L_{Lit}$ represent a medical entity present in the patient and assume that $\eta \in [0,1]$ is the initial value assigned to it at the start of a run of *Cadiag*2's inference process. We can formalise this by means of a probabilistic conditional statement of the form $P(\phi|\kappa) = \eta$ in $\mathcal{FL}^{=}$, where $\kappa \in SL$ is the evidence that supports the presence of ϕ in the patient. For simplicity the sentence κ will be assumed to be a literal in L_{Lit}.

Next we are going to define the formal system *CadPL*. Recall that the ultimate goal when doing so is to define a system which represents the inference process in *Cadiag*2 when interpreted from a probabilistic point of view. Although the inference in *Cadiag*2 can be closely related to probability theory (given the nature of the rules of inference in Φ_{CadBin}) it is not based on probabilistic methods and so the degree of freedom when choosing the rules of the system *CadPL* is high. We have chosen the rules by interpreting *in the most natural way* the steps along the inference process within a probabilistic frame. The main idea behind such interpretation consists of the identification of the inference process with the *propagation of evidence* facilitated by the rules in Φ_{CadBin}. For example, from $P(\phi|\kappa) = \eta$, where $k \in L_{Lit}$ is evidence supporting the presence of ϕ in the patient, and $P(\theta|\phi) = \zeta$ in Φ_{CadBin} we would infer $P(\theta|\kappa) = min(\eta,\zeta)$, where $min(\eta,\zeta)$ is the value (probability) assigned to θ given the evidence κ. We would have a *propagation* process of this nature for each single piece of evidence. The evidence would then be brought together in *Cadiag*2 by what we call the *Right conjunction* rule: given two outcomes of *Cadiag*2's inference process, say $P(p|\kappa_1) = \eta$ and $P(p|\kappa_2) = \zeta$, for $p \in L$ and $\kappa_1, \kappa_2 \in L_{Lit}$, *Cadiag*2 combines the evidence given by κ_1 and κ_2 by computing $P(p|\kappa_1 \wedge \kappa_2) = max^*(\eta,\zeta)$. The inference rules of *CadPL* that we next present formalise this interpretation.A theory $\mathcal{T}$ in *CadPL* is a finite subset of sentences in $\mathcal{FL}_s^{=}$.

[3] This example is extracted from (Adlassnig et al., 1986).

For what follows let $\mathcal{T} = \Omega \cup \Phi$, with

$$\Omega = \{P(\phi_1|\kappa_1) = \eta_1, \ldots, P(\phi_m|\kappa_m) = \eta_m\},$$

for some $m \in \mathbb{N}$.

Let $\mathcal{K}_\Omega = \{\kappa_1, \ldots, \kappa_m\}$ and $\Gamma = \{\phi_1, \ldots, \phi_m\}$.

The set Ω is intended to represent the initial assignment in the inference process, Φ the set of rules of the system, Γ the initial set of medical entities present in the patient and $\mathcal{K}_\Omega$ the evidence in support of the presence of the corresponding medical entities in Γ.

The formal system *CadPL* is defined by the following inference rules:

Inference rules

- *Reflexivity rule*
  For $\phi \in L_{Lit}$, $\kappa \in \mathcal{K}_\Omega$ and $\eta \in [0,1]$,
  $$\frac{P(\phi|\kappa) = \eta \ \in \Omega}{\mathcal{T} \vdash P(\phi|\kappa) = \eta}$$
- *Negation rule*
  For $\phi \in L_{Lit}$, $\psi \in SL$ and $\eta \in [0,1]$,
  $$\frac{\mathcal{T} \vdash P(\phi|\psi) = \eta}{\mathcal{T} \vdash P(\neg\phi|\psi) = 1 - \eta}$$
- *Equivalence rule*
  For $\psi, \phi, \theta \in SL$ and $\eta \in [0,1]$,
  $$\frac{\psi \equiv \phi \qquad \mathcal{T} \vdash P(\phi|\theta) = \eta}{\mathcal{T} \vdash P(\psi|\theta) = \eta}$$
- *Minimum rule*
  For $\theta, \phi \in L_{Lit}$, $\kappa \in \mathcal{K}_\Omega$, $\eta \in (0,1]$ and $\zeta \in [0,1]$,
  $$\frac{\mathcal{T} \vdash P(\theta|\kappa) = \eta \qquad P(\phi|\theta) = \zeta \ \in \Phi}{\mathcal{T} \vdash P(\phi|\kappa) = min(\eta, \zeta)}$$
- *Right conjunction rule*
  For $p \in L$, $K_1, K_2 \subseteq \mathcal{K}_\Omega$ and $\eta, \zeta \in [0,1]$,
  $$\frac{\mathcal{T} \vdash P(p|\bigwedge K_1) = \eta \qquad \mathcal{T} \vdash P(p|\bigwedge K_2) = \zeta}{\mathcal{T} \vdash P(p|\bigwedge\{K_1 \cup K_2\}) = max^*(\eta, \zeta)}$$
- *Exhaustivity rule*
  For $p \in L$, $\kappa \in \mathcal{K}_\Omega$, $K \subseteq \mathcal{K}_\Omega$ and $\eta \in [0,1]$,
  $$\frac{\mathcal{T} \vdash P(p|\bigwedge K) = \eta \qquad \forall\zeta \in [0,1] \ \mathcal{T} \nvdash P(p|\kappa) = \zeta}{\mathcal{T} \vdash P(p|\kappa \wedge \bigwedge K) = \eta}$$

Notice that the *Exhaustivity* rule does not have any bearing on the *decidability* of whether $P(p|\kappa) = \zeta$ is provable from $\mathcal{T}$ or not for $\zeta \in [0,1]$, $p \in L$ and $\kappa \in \mathcal{K}_\Omega$. The *Exhaustivity* rule can only be applied after its provability or non-provability has been decided.

Given a theory $\mathcal{T}$ of *CadPL* and a statement $\Theta \in \mathcal{F}\mathcal{L}_c^=$, a proof of Θ from $\mathcal{T}$ is defined as a finite sequence of *sequents* of the form

$$\mathcal{T} \vdash \Theta_1, \ldots, \mathcal{T} \vdash \Theta_n$$

with $\Theta_n = \Theta$ and where, for $i \in \{1, \ldots, n\}$, each Θ_i in $\mathcal{T} \vdash \Theta_i$ follows from $\mathcal{T}$ by the application of one of the rules above, from Θ_j in a previous sequent (with $j < i$) or from Θ_j, Θ_k in previous sequents (with $j, k < i$) by one of the rules above.

Let Θ be the statement $P(\theta|\phi) = \eta$, for some $\eta \in [0,1]$ and $\theta, \phi \in SL$. We say that there exists a *maximal* proof of Θ from $\mathcal{T}$ if there exists a proof of Θ from $\mathcal{T}$ and there is no proof from $\mathcal{T}$ of $P(\theta|\phi) = \zeta$ with $\eta \prec \zeta$.

We say that Θ follows *maximally* from $\mathcal{T}$ (denoted by $\mathcal{T} \vdash_{CadPL} \Theta$) if there exists a maximal proof of Θ from $\mathcal{T}$.

For the next proposition let $\mathcal{T} = \Omega \cup \Phi_{CadBin}$, with

$$\Omega = \{P(\phi_1|\kappa_1) = \eta_1, \ldots, P(\phi_m|\kappa_m) = \eta_m\},$$

$\mathcal{K}_\Omega = \{\kappa_1, \ldots, \kappa_m\} \subset L_{Lit}$ and $\Gamma = \{\phi_1, \ldots, \phi_m\}$ a subset of literals occurring in Φ_{CadBin}.

Proposition 4. *Let $p \in L$ and $\eta \in [0,1]$. We have that*

$$\mathcal{T} \vdash_{CadPL} P(p|\bigwedge \mathcal{K}_\Omega) = \eta$$

if and only if $(p, \eta)_{max^*}$ is the outcome of a run of Cadiag2's inference process on $\mathcal{T}$.

Proof.[4] In order to prove the left implication let us consider a run of *Cadiag*2's inference mechanism on $\mathcal{T}$. The inference starts from pairs of the form $(\phi, \eta)_0$ and $(\neg\phi, 1-\eta)_0$ for some $\eta \in [0,1]$ for all $\phi \in \Gamma$. In *CadPL* a pair of the form $(\phi, \eta)_0$, for $\phi \in \Gamma$, corresponds to a sequent of the form $\mathcal{T} \vdash P(\phi|\kappa) = \eta$, for $\kappa \in \mathcal{K}_\Omega$. The pair $(\neg\phi, 1-\eta)_0$ corresponds to the sequent $\mathcal{T} \vdash P(\neg\phi|\kappa) = 1 - \eta$. The former corresponds to an application of the *Reflexivity* rule. The latter follows from the first one by an application of the *Negation* rule.

Let us assume now that we are at the n^{th} step of the inference process and that a rule of the form $P(\theta|\psi) = \zeta$ is triggered, for some $\zeta \in [0,1]$ and $\theta, \psi \in L_{Lit}$. Let us suppose that we have $(\psi, \mu)_{n-t}$, the pair that triggers the rule at the n^{th} step of the process, for $\mu \in (0,1]$ and $t \leq n-1$. In *CadPL* that would correspond to a sequent of the form $\mathcal{T} \vdash P(\psi|\kappa) = \mu$ derived from a previous step in the inference, for $\kappa \in \mathcal{K}_\Omega$. The inference mechanism in *Cadiag*2 produces the pairs $(\theta, min(\zeta, \mu))_n$ and $(\neg\theta, 1 - min(\zeta, \mu))_n$ which, in *CadPL*, corresponds to the sequents $\mathcal{T} \vdash P(\theta|\kappa) = min(\zeta, \mu)$ and $\mathcal{T} \vdash P(\neg\theta|\kappa) = 1 - min(\zeta, \mu)$ respectively, which follow by an application of the *Minimum* rule and, for the latter, an application of the *Negation* rule on the former.

[4] For the sake of brevity we will deal with sentences as if they were equivalence classes. If anything applies to a sentence of the form $\neg\phi$, with $\phi \in L_{Lit}$, we also assume that it applies to any logical equivalent of ϕ without mentioning it.

At the end of the process *Cadiag*2 generates the pair $(p,\eta)_{max^*}$ for each sentence $p \in L$ involved in the inference, where η is the maximal value (with respect to the ordering $\preceq$) among those assigned to p along the inference. This maximization process is achieved in *CadPL* by means of repeated applications of the *Right conjunction* rule. Instances of the *Exhaustivity* rule (if necessary) complete the inferential counterpart of *Cadiag*2 in *CadPL*.

In order to prove the right implication let us suppose that we have a maximal proof of the form

$$\mathcal{T} \vdash \Theta_1, ..., \mathcal{T} \vdash \Theta_m,$$

where Θ_m is the statement $P(p|\bigwedge \mathcal{K}_\Omega) = \eta$, for some $\eta \in [0,1]$ and $p \in L$.

The first sequent of the proof needs to respond to an instance of the *Reflexivity* rule, $\mathcal{T} \vdash P(\phi|\kappa) = \eta$, for some $\phi \in \Gamma$, $\kappa \in \mathcal{K}_\Omega$ and $\eta \in [0,1]$. The corresponding counterpart of this sequent in *Cadiag*2 is the pair $(\phi,\eta)_0$.

Let us move now to the n^{th} sequent, with $n \leq m$. The n^{th} sequent can be an instance of the *Reflexivity* rule, $\mathcal{T} \vdash P(\phi|\kappa) = \eta$, for some $\phi \in \Gamma$, $\eta \in [0,1]$ and $\kappa \in \mathcal{K}_\Omega$. The counterpart for this sequent in *Cadiag*2 is the pair $(\phi,\eta)_0$.

The n^{th} sequent can follow from a previous one in the proof by an instance of the *Negation* rule. Let us suppose that the n^{th} sequent is $\mathcal{T} \vdash P(\neg\theta|\psi) = 1-\eta$ for some $\eta \in [0,1]$, $\theta \in L_{Lit}$ and $\psi \subseteq L_{Lit}$ and that there is a sequent $\mathcal{T} \vdash \Theta_i$, for some $i < n$, of the form $\mathcal{T} \vdash P(\theta|\psi) = \eta$. The latter corresponds to a pair of the form $(\theta,\eta)_t$ in *Cadiag*2 and the former to the pair $(\neg\theta, 1-\eta)_t$, where t is the step in the inference process at which such pairs have been generated.

The n^{th} sequent can follow from a previous one by an instance of the *Minimum* rule. Let us assume that the n^{th} sequent is $\mathcal{T} \vdash P(\theta|\kappa) = min(\eta,\zeta)$, for some $\theta \in L_{Lit}$, $\kappa \in \mathcal{K}_\Omega$, $\eta \in [0,1]$ and $\zeta \in (0,1]$, that $\mathcal{T} \vdash P(\psi|\kappa) = \zeta$ is a previous sequent in the proof and that $P(\theta|\psi) = \eta \ \in \Phi_{CadBin}$. The latter corresponds in *Cadiag*2 to the pair $(\psi,\zeta)_t$ and the former to the pair $(\theta, \min(\eta,\zeta))_{t+k}$, where $t, t+k$ indicate the steps at which the pairs have been generated by the inference process.

The n^{th} sequent can follow from previous sequents by an application of the *Right conjunction* rule. The counterpart in *Cadiag*2 of such an outcome consists of the maximization process at the end of the inference. Instances of the *Exhaustivity* rule are irrelevant to the inference in *Cadiag*2.

This completes the proof. ■

It is worth commenting on some features of the inference rules of *CadPL* in connection with probability theory.

Soundness with respect to probabilistic semantics of the *Reflexivity*, *Negation* and *Equivalence* rules is clear. The *Minimum* rule is certainly not sound with respect to such semantics. The *Right conjunction rule* is not sound and it can generate probabilistic consequences that are *inconsistent* with its premises and the theory $\mathcal{T}$ (in the sense that such consequences along with the premises and the theory are not simultaneously satisfiable by a probability function). The *Exhaustivity* rule assumes some probabilistic *independence* among sentences that may not actually be independent. Overall, *CadPL* does not score well within probability theory. This is no surprise. The computation of conditional probabilistic statements in a compositional way, as done by *Cadiag*2 primarily by means of the *min* and *max** operators, is clearly bound to be probabilistically unsound. One may wonder though what could be done in order to improve the inference on probabilistic grounds from a knowledge base like Φ_{CadBin}. The answer seems to be 'not much'. Certainly a Φ_{CadBin}-like knowledge base (i.e., a knowledge base given by some binary probabilistic conditional statements) is not the most convenient for inferential purposes in probability theory for medical applications like *Cadiag*2. As is well known, there are other knowledge-base structures better suited for that purpose, Bayesian networks being the most celebrated among them, see (Castillo et al., 1997) or (Pearl, 1988).

In terms of consistency, it is worth noting that *CadPL* satisfies what we can call *weak consistency* – called *weak soundness* in (Hajek, 1988) –, defined as follows: if there is a maximal proof in *CadPL* of a statement of the form $P(\phi|\bigwedge\Delta) = 1$ (or $P(\phi|\bigwedge\Delta) = 0$) from a certain theory $\mathcal{T}$, with $\phi \in SL$ and $\Delta \subseteq SL$ then, if there is a maximal proof in *CadPL* of a statement of the form $P(\phi|\bigwedge\Delta^*) = \eta$, with $\Delta \subset \Delta^*$, then $\eta = 1$ (or $\eta = 0$ respectively). That is to say, if *CadPL* concludes certainty about the occurrence of some event or about the truth or falsity of some sentence then adding new evidence does not alter this certainty. Weak consistency is provided in *CadPL* and so in *Cadiag*2's inference mechanism by the operator *max** defined over the ordering $\preceq$.

It is also worth noting that one could guarantee consistency (i.e., satisfiability) by considering Φ_{CadBin} a subset of $\mathcal{FL}_s^{\geq}$ (in place of $\mathcal{FL}_s^{=}$, regarding the values of the conditional statements as *lower probability bounds* rather than as *exact* probabilities) and by restricting the system to a *positive fragment* of L_{Lit} (i.e., only one of p, $\neg p$ can occur in Φ_{CadBin}). This way consistency is trivially guaranteed for Φ_{CadBin} together with any outcomes produced by the system during the inference process.

In terms of soundness there does not seem to be much that one can do in order to improve the inference mechanism for knowledge bases like Φ_{CadBin}, or at least not much that one can do that does not come at the price of generating probabilistic statements with very low probabilistic bounds (when working in $\mathcal{FL}^{\geq}$), which would make *Cadiag*2 potentially useless for practical purposes. There is some room for improvement for some steps in the inference that come by the addition of some independence assumptions among some of the medical entities in Φ_{CadBin}. Under such independence assumptions the *product* operator in place of the *min* operator could yield soundness for the inference steps referred.

5 CONCLUSIONS

*Cadiag*2 is a reasonably well-performing medical expert system (Adlassnig et al., 1986), but how it is so is far from clear. The inference engine of *Cadiag*2 was built with methods of approximate reasoning in fuzzy set theory but, as such, it was not based on any logical formalism or theory embedded with a clear semantics. This fact motivated the main aim of this paper, which was no other than the *understanding* of *Cadiag*2 in a *logical* way.

The natural interpretation of the inference rules of *Cadiag*2 (i.e., probabilistic) placed us upon the attempt of interpreting the inference itself probabilistically. We formalised this interpretation by means of the system *CadPL*, the logical (probabilistic) counterpart of the inference engine of *Cadiag*2. The unsoundness of some of the rules of *CadPL* (and thus of some inference steps in *Cadiag*2) and the inconsistency of the calculus (and thus of the inference process in *Cadiag*2) was made clear. Apart from these drawbacks, otherwise expected, some other aspects of *CadPL* were also stressed and analysed. At the end of the paper some possibilities for an improvement of *Cadiag*2 in terms of soundness and consistency were also mentioned.

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