REASONING ABOUT BOUNDED TIME DOMAIN
An Alternative to NP-Complete Fragments of LTL

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Abstract: It is known that linear-time temporal logic (LTL) is one of the most useful logics for reasoning about time and for verifying concurrent systems. It is also known that the satisfiability problem for LTL is PSPACE-complete and that finding NP-complete fragments of LTL is an important issue for constructing efficiently executable temporal logics. In this paper, an alternative NP-complete logic called bounded linear-time temporal logic is obtained from LTL by restricting the time domain of temporal operators.

1 INTRODUCTION

It is known that *linear-time temporal logic* (LTL) (Pnueli, 1977) is one of the most useful logics for reasoning about time and for verifying concurrent systems by *model checking* (Clarke et al., 1999; Holzmann, 2006). It is also known that in almost all cases, the model checking problems for LTL and its fragments are equivalent to the satisfiability problems for them. For this reason, the satisfiability problems for LTL fragments are known to be an important issue for constructing efficiently executable temporal logics. The satisfiability problem for LTL is PSPACE-complete (Sistla and Clarke, 1985) and finding NP-complete fragments of LTL has been well-studied (Demri and Schnoebelen, 2002; Etessami et al., 1997; Muscholl and Walukiewicz, 2005; Walukiewicz, 1998). This paper tries to construct an alternative to such an NP-complete fragment by restricting the time domain of temporal operators. Although the standard temporal operators of LTL have an infinite (unbounded) time domain, i.e., the set ω of natural numbers, the bounded operators which are presented in this paper have a *bounded time domain* which is restricted by a fixed positive integer l, i.e., the set $\omega_l := \{x \in \omega \mid x \leq l\}$. Despite this restriction, the proposed bounded operators can derive almost all the typical LTL axioms including the temporal induction axiom.

To restrict the time domain of temporal operators is not a new idea. Such an idea has been discussed (Biere et al., 2003; Cerrito et al., 1999; Cerrito and Mayer, 1998; Hodkinson et al., 2000; Kamide, 2008). It is known that to restrict the time domain is a technique to obtain a decidable or efficient fragment of first-order LTL (Hodkinson et al., 2000). Restricting the time domain implies not only some purely theoretical merits, but also some practical merits for describing temporal databases and planning specifications (Cerrito et al., 1999; Cerrito and Mayer, 1998), and for implementing an efficient model checking algorithm called *bounded model checking* (Biere et al., 2003). Such practical merits are due to the fact that there are problems in computer science and artificial intelligence where only a finite fragment of the time sequence is of interest (Cerrito et al., 1999).

The contents of this paper are then summarized as follows. In Section 2, a logic called *bounded linear-time temporal logic* (BLTL) is obtained from LTL by restricting the time domain of temporal operators. In order to obtain a theorem for embedding BLTL into classical propositional logic (CL), a semantics for CL is also defined. In Section 3, the NP-completeness of the satisfiability problem for BLTL is shown using the embedding theorem of BLTL into CL. In Section 4, conclusions and related works are briefly addressed.

2 BOUNDED LINEAR-TIME TEMPORAL LOGIC

Formulas of BLTL are constructed from (countably many) propositional variables, $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), X (next), G (globally) and F (eventually) where X, G and F are

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bounded versions of the standard operators of LTL. Lower-case letters $p,q,\ldots$ are used to denote propositional variables, and Greek lower-case letters $\alpha,\beta,\ldots$ are used to denote formulas. We write $A \equiv B$ to indicate the syntactical identity between A and B. The symbol ω is used to represent the set of natural numbers. Lower-case letters i,j and k are used to denote any natural numbers. The symbol $\geq$ or $\leq$ is used to represent a linear order on ω. Let l be a fixed positive integer. Then, the symbol ω_l is used to denote the set $\{i \in \omega \mid i \leq l\}$. In the following discussion, l is fixed as a certain positive integer.

Definition 2.1 (BLTL). *Let S be a non-empty set of states. A structure $M := (\sigma, I)$ is a* model *if*

1. *σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S,*
2. *I is a mapping from the set Φ of propositional variables to the power set of S.*

A satisfaction relation $(M,i) \models \alpha$ for any formula α, where M is a model (σ, I) and i ($\in \omega$) represents some position within σ, is defined inductively by

1. *for any $p \in \Phi$, $(M,i) \models p$ iff $s_i \in I(p)$,*
2. *$(M,i) \models \alpha \wedge \beta$ iff $(M,i) \models \alpha$ and $(M,i) \models \beta$,*
3. *$(M,i) \models \alpha \vee \beta$ iff $(M,i) \models \alpha$ or $(M,i) \models \beta$,*
4. *$(M,i) \models \alpha \to \beta$ iff $(M,i) \models \alpha$ implies $(M,i) \models \beta$,*
5. *$(M,i) \models \neg\alpha$ iff not-$[(M,i) \models \alpha]$,*
6. *for any $i \leq l-1$, $(M,i) \models \mathrm{X}\alpha$ iff $(M,i+1) \models \alpha$,*
7. *for any $i \geq l$, $(M,i) \models \mathrm{X}\alpha$ iff $(M,l) \models \alpha$,*
8. *$(M,i) \models \mathrm{G}\alpha$ iff $\forall j \geq i$ with $j \in \omega_l$ $[(M,j) \models \alpha]$,*
9. *$(M,i) \models \mathrm{F}\alpha$ iff $\exists j \geq i$ with $j \in \omega_l$ $[(M,j) \models \alpha]$,*
10. *for any $m \in \omega$, $(M,l+m) \models \alpha$ iff $(M,l) \models \alpha$.*

A formula α is valid *in* BLTL *if $(M,0) \models \alpha$ for any model $M := (\sigma, I)$. A formula α is* satisfiable *in* LTL *if $(M,0) \models \alpha$ for some model $M := (\sigma, I)$.*

Since BLTL depents on l, it should precisely be named e.g., BLTL[l], but, for the sake of simplicity, we use the name BLTL in the following.

An expression $\alpha \leftrightarrow \beta$ means $(\alpha \to \beta) \wedge (\beta \to \alpha)$. Expressions $\bigwedge C$ and $\bigvee C$ are used to represent the finite conjunction and disjunction of the formulas in C, respectively.

Proposition 2.2. *The following formulas are valid in* BLTL*:*

1. $\mathrm{X}(\alpha \circ \beta) \leftrightarrow \mathrm{X}\alpha \circ \mathrm{X}\beta$ *where* $\circ \in \{\to, \wedge, \vee\}$,
2. $\mathrm{X}(\neg\alpha) \leftrightarrow \neg(\mathrm{X}\alpha)$,
3. $\mathrm{G}\alpha \to \alpha$,
4. $\mathrm{G}\alpha \to \mathrm{X}\alpha$,
5. $\mathrm{G}\alpha \to \mathrm{XG}\alpha$,
6. $\mathrm{G}\alpha \to \mathrm{GG}\alpha$,
7. $\alpha \wedge \mathrm{G}(\alpha \to \mathrm{X}\alpha) \to \mathrm{G}\alpha$ *(temporal induction),*
8. *for any $m \in \omega$,* $\mathrm{X}^{l+m}\alpha \leftrightarrow \mathrm{X}^l\alpha$,
9. $\mathrm{G}\alpha \leftrightarrow \bigwedge\{\mathrm{X}^i\alpha \mid i \in \omega_l\}$,
10. $\mathrm{F}\alpha \leftrightarrow \bigvee\{\mathrm{X}^i\alpha \mid i \in \omega_l\}$.

Proof. We show some critical cases. Let M be an arbitrary model and $\models$ be an arbitrary satisfaction relation on M.

(7): We show $(M,0) \models \alpha \wedge \mathrm{G}(\alpha \to \mathrm{X}\alpha) \to \mathrm{G}\alpha$. Suppose $(M,0) \models \alpha \wedge \mathrm{G}(\alpha \to \mathrm{X}\alpha)$, i.e., (a): $(M,0) \models \alpha$ and (b): $(M,0) \models \mathrm{G}(\alpha \to \mathrm{X}\alpha)$. We will show $(M,0) \models \mathrm{G}\alpha$, i.e., $\forall j \in \omega_l$ $[(M,i) \models \alpha]$. From (b), we obtain:

$(M,0) \models \mathrm{G}(\alpha \to \mathrm{X}\alpha)$

iff $\forall j \in \omega_l$ $[(M,j) \models \alpha \to \mathrm{X}\alpha]$

iff $\forall j \in \omega_l$ $[(M,j) \models \alpha \implies (M,j) \models \mathrm{X}\alpha]$

iff (c): $\forall j \in \omega_l$ $[(M,j) \models \alpha \implies (M,j+1) \models \alpha$ (if $j \leq l-1$) or $(M,l) \models \alpha$ (if $j \geq l$)$]$.

We now show the required fact $\forall i \in \omega_l$ $[(M,i) \models \alpha]$ by mathematical induction on i. Base step: We have $(M,0) \models \alpha$ by (a). Induction step: Suppose $(M,k) \models \alpha$ with $k \leq l-1$. Then, we obtain $(M,k+1) \models \alpha$ by (c). Suppose $(M,k) \models \alpha$ with $k \geq l$. Then, we obtain $(M,l) \models \alpha$ by (c), and hence obtain $(M,k+1) \models \alpha$ where $k+1 = l+m$ with $m \in \omega$.

(8): We obtain: $(M,0) \models \mathrm{X}^{l+n}\alpha$ iff $(M,l) \models \alpha$ iff $(M,0) \models \mathrm{X}^l\alpha$.

(9): We obtain: $(M,0) \models \mathrm{G}\alpha$ iff $\forall j \in \omega_l$ $[(M,j) \models \alpha]$ iff $\forall j \in \omega_l$ $[(M,0) \models \mathrm{X}^j\alpha]$ iff $(M,0) \models \bigwedge\{\mathrm{X}^j\alpha \mid j \in \omega_l\}$. ∎

Remark that 8, 9 and 10 in Proposition 2.2 are regarded as characteristic axioms concerning the time bound l. Note that 9 and 10 in Proposition 2.2 become the axioms of LTL if ω_l is replaced by ω. Thus, BLTL is quite natural as a bounded time formalism.

Formulas of classical logic (CL) are constructed from (countably many) propositional variables, $\to$, $\neg$, $\bigwedge$ (finite conjunction) and $\bigvee$ (finite disjunction).

Definition 2.3 (CL). *Let Θ be a finite (non-empty) set of formulas. V is a mapping from the set Φ of propositional variables to the set $\{t, f\}$ of truth values. V is called a* valuation*. A satisfaction relation $V \models \alpha$ for any formula α is defined inductively by*

1. *$V \models p$ iff $V(p) = t$ for any $p \in \Phi$,*
2. *$V \models \neg\alpha$ iff not-$(V \models \alpha)$,*
3. *$V \models \alpha \to \beta$ iff $V \models \alpha$ implies $V \models \beta$,*
4. *$V \models \bigwedge\Theta$ iff $V \models \alpha$ for any $\alpha \in \Theta$,*
5. *$V \models \bigvee\Theta$ iff $V \models \alpha$ for some $\alpha \in \Theta$.*

A formula α is valid *(*satisfiable*) in* CL *if $V \models \alpha$ for any (some) valuation V.*

3 NP-COMPLETENESS

Definition 3.1. *Fix a countable non-empty set Φ of propositional variables. Define the sets $\Phi_i := \{p_i \mid p \in \Phi\}$ $(i \in \omega)$ of propositional variables where $p_0 = p$ (i.e., $\Phi_0 := \Phi$). The language $\mathcal{L}^b$ of* BLTL *is defined using Φ, $\rightarrow, \wedge, \vee, \neg$,* X, G *and* F. *The language $\mathcal{L}$ of* CL *is defined using $\bigcup_{i\in\omega} \Phi_i$, $\rightarrow, \neg$, $\bigwedge$ and $\bigvee$. The binary versions of $\bigwedge$ and $\bigvee$ are also denoted as $\wedge$ and $\vee$, respectively, and these binary symbols are included in the definition of $\mathcal{L}$.*

A mapping f from $\mathcal{L}^b$ to $\mathcal{L}$ is defined by

1. *for any $p \in \Phi$, $f(\mathrm{X}^i p) := p_i \in \Phi_i$, especially, $f(p) := p \in \Phi$,*
2. *$f(\mathrm{X}^i(\alpha \circ \beta)) := f(\mathrm{X}^i\alpha) \circ f(\mathrm{X}^i\beta)$ where $\circ \in \{\rightarrow, \wedge, \vee\}$,*
3. *$f(\mathrm{X}^i\neg\alpha) := \neg f(\mathrm{X}^i\alpha)$,*
4. *for any $m \geq l$, $f(\mathrm{X}^m\mathrm{X}\alpha) := f(\mathrm{X}^l\alpha)$,*
5. *$f(\mathrm{X}^i\mathrm{G}\alpha) := \bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega_l\}$,*
6. *$f(\mathrm{X}^i\mathrm{F}\alpha) := \bigvee\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega_l\}$.*

Remark that the mapping f in Definition 3.1 is a polynomial-time reduction since $f(\alpha)$ can be computed by subformulas of α.

Lemma 3.2. *Let f be the mapping defined in Definition 3.1, and S be a non-empty set of states. For any model $M := (\sigma, I)$ of* BLTL, *any satisfaction relation $\models$ on M and any state s_i in σ, we can construct a valuation V of* CL *and a satisfaction relation $\models$ of* CL *such that for any formula α in $\mathcal{L}^b$,*

$$(M,i) \models \alpha \text{ iff } V \models f(\mathrm{X}^i\alpha).$$

Proof. Let Φ be a non-empty set of propositional variables and Φ_i be the set $\{p_i \mid p \in \Phi\}$. Suppose that M is a model (σ, I) where

I is a mapping from Φ to the power set of S.

Suppose that

V is a mapping from $\bigcup_{i\in\omega} \Phi_i$ to $\{t, f\}$.

Suppose moreover that I and V satisfy the following condition:

$$\forall i \in \omega, \forall p \in \Phi \;\; [s_i \in I(p) \text{ iff } V(p_i) = t].$$

Then, the lemma is proved by induction on the complexity of α. For the sake of simplicity, V of $V \models$ is omitted in the following.

• Base step: $\alpha \equiv p \in \Phi$. $(M,i) \models p$ iff $s_i \in I(p)$ iff $V(p_i) = t$ iff $\models p_i$ iff $\models f(\mathrm{X}^i p)$ (by the definition of f).

• Induction step.

(Case $\alpha \equiv \beta \wedge \gamma$): $(M,i) \models \beta \wedge \gamma$ iff $(M,i) \models \beta$ and $(M,i) \models \gamma$ iff $\models f(\mathrm{X}^i\beta)$ and $\models f(\mathrm{X}^i\gamma)$ (by induction hypothesis) iff $\models f(\mathrm{X}^i\beta) \wedge f(\mathrm{X}^i\gamma)$ iff $\models f(\mathrm{X}^i(\beta \wedge \gamma))$ (by the definition of f).

(Cases $\alpha \equiv \beta \vee \gamma$ and $\alpha \equiv \beta \rightarrow \gamma$): Similar to the above case.

(Case $\alpha \equiv \neg\beta$): $(M,i) \models \neg\beta$ iff not-$[(M,i) \models \beta]$ iff not-$[\models f(\mathrm{X}^i\beta)]$ (by induction hypothesis) iff $\models \neg f(\mathrm{X}^i\beta)$ iff $\models f(\mathrm{X}^i\neg\beta)$ (by the definition of f).

(Case $\alpha \equiv \mathrm{X}\beta$):

Subcase $(i \leq l-1)$: $(M,i) \models \mathrm{X}\beta$ iff $(M,i+1) \models \beta$ iff $\models f(\mathrm{X}^{i+1}\beta)$ (by induction hypothesis) iff $\models f(\mathrm{X}^i(\mathrm{X}\beta))$.

Subcase $(i \geq l)$: $(M,i) \models \mathrm{X}\beta$ iff $(M,l) \models \beta$ iff $\models f(\mathrm{X}^l\beta)$ (by induction hypothesis) iff $f(\mathrm{X}^i\mathrm{X}\beta)$ (by the definition of f).

(Case $\alpha \equiv \mathrm{G}\beta$): $(M,i) \models \mathrm{G}\beta$ iff $\forall j \geq i$ with $j \in \omega_l$ $[(M,j) \models \beta]$ iff $\forall j \geq i$ with $j \in \omega_l$ $[\models f(\mathrm{X}^j\beta)]$ (by induction hypothesis) iff $\forall k \in \omega_l$ $[\models f(\mathrm{X}^{i+k}\beta)]$ iff $\models \gamma$ for all $\gamma \in \{f(\mathrm{X}^{i+k}\beta) \mid k \in \omega_l\}$ iff $\models \bigwedge\{f(\mathrm{X}^{i+k}\beta) \mid k \in \omega_l\}$ iff $\models f(\mathrm{X}^i\mathrm{G}\beta)$ (by the definition of f).

(Case $\alpha \equiv \mathrm{F}\beta$): Similar to the above case. ∎

Lemma 3.3. *Let f be the mapping defined in Definition 3.1, and S be a non-empty set of states. For any valuation V of* CL *and any satisfaction relation $\models$ of* CL, *we can construct a model $M := (\sigma, I)$ of* BLTL *and a satisfaction relation $\models$ on M such that for any formula α in $\mathcal{L}^b$,*

$$V \models f(\mathrm{X}^i\alpha) \text{ iff } (M,i) \models \alpha.$$

Proof. Similar to the proof of Lemma 3.2. ∎

Theorem 3.4 (Embedding). *Let f be the mapping defined in Definition 3.1. For any formula α, α is valid (satisfiable) in* BLTL *iff $f(\alpha)$ is valid (satisfiable) in* CL.

Proof. By Lemmas 3.2 and 3.3. ∎

We then obtain the main theorem of this paper as follows.

Theorem 3.5 (Complexity). *The validity and satisfiability problems of* BLTL *are Co-NP-complete and NP-complete, respectively.*

Proof. The validity and satisfiability problems of CL are known to be Co-NP-complete and NP-complete, respectively. By decidability of CL, for each α, it is possible to decide if $f(\alpha)$ is valid (satisfiable) in BLTL. Then, by Theorem 3.4, the validity and satisfiability problems of BLTL are decidable. Since f is a polynomial-time reduction, the validity and satisfiability problems of BLTL are also Co-NP-complete and NP-complete, respectively. ∎

4 CONCLUSIONS AND RELATED WORKS

In this paper, BLTL, which is obtained from LTL by restricting the time domain of temporal operators, was introduced, and the satisfiability problem for BLTL was shown to be NP-complete by using a theorem for embedding BLTL into CL. The embedding theorem had a central role for showing the NP-completeness of BLTL. The embedding theorem may also be justified by the usefulness of the bounded model checking technique (Cerrito and Mayer, 1998), which uses a propositional satisfiability checking technique. It was thus shown in this paper that the existing satisfiability checking techniques of CL are available for BLTL. This is an advantage of BLTL.

In the following, some related works are briefly reviewed. It is known (Sistla and Clarke, 1985) that the LTL fragment endowed with the standard operators X, G and F are PSPACE-complete and that the fragment endowed with either X or (F and G) has NP-complete satisfiability problems. Some NP-complete fragments of LTL have been well-studied (Demri and Schnoebelen, 2002; Etessami et al., 1997; Muscholl and Walukiewicz, 2005; Walukiewicz, 1998). Some restrictions on the nesting of operators and on the number of propositions were proposed by Demri and Schnoebelen (Demri and Schnoebelen, 2002). Restricting X to operators X_a ($a \in \Sigma$) that enforce the current letter to be a was proposed by Muscholl and Walukiewicz (Muscholl and Walukiewicz, 2005). The formula $X\alpha$ is expressed as $\bigvee_{a\in\Sigma} X_a\alpha$ where Σ is the alphabet. They proved that the satisfiability problem for the LTL fragment with X_a ($a \in SI$), F and G is NP-complete. Finally it is mentioned that a Gentzen-type sequent calculus for a modification of BLTL was proposed by Kamide (Kamide, 2008).

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