A GEOMETRIC APPROACH TO CURVATURE ESTIMATION ON TRIANGULATED 3D SHAPES

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Abstract: We present a geometric approach to define discrete normal, principal, Gaussian and mean curvatures, that we call *Ccurvature*. Our approach is based on the notion of concentrated curvature of a polygonal line and a simulation of rotation of the normal plane of the surface at a point. The advantages of our approach is its simplicity and its natural meaning. A comparison with widely-used discrete methods is presented.

1 INTRODUCTION

Curvature is one of the main important notions used to study the geometry and the topology of a surface. In combinatorial geometry, many attempts to define a discrete equivalent of Gaussian and mean curvatures have been developed for polyhedral surfaces (Gatzke and Grimm, 2006; Surazhsky et al., 2003). Discrete approaches include smooth approximations of the surface using interpolation techniques (Hahmann et al., 2007), and approaches that deal directly with the mesh (Meyer et al., 2003; Taubin, 1995; Watanabe and Belyaev, 2001). All the methods are not satisfactory in what concerns approximation errors, control, and convergence when refining a mesh (Borrelli et al., 2003; Surazhsky et al., 2003; Xu, 2006).

In the fifties, Aleksandrov introduced *concentrated curvature* as an intrinsic curvature measure for polygonal surfaces (Aleksandrov, 1957). This technique has been used in the geometric modeling community under the name of *angle defect* method (Alboul et al., 2005; Akleman and Chen, 2006). Concentrated curvature does not suffer from the problems linked to errors and and their control, and satisfies a discrete analogous version of the well known Gauss-Bonnet theorem. However, Concentrated curvature depends weakly on the local geometric shape of the surface.

Here, we use Aleksandrov's idea to define concentrated curvature for polygonal lines. We prove in such a case that concentrated curvature is an intrinsic measure that is expressed using only the fracture angle of the polygonal line at its vertices. We then define a discrete normal curvature of a polygonal surface at a vertex. We simulate the rotation of normal planes to define principal curvatures and, thus, obtain new discrete estimators for Gaussian and mean curvatures. We call all such curvatures *Ccurvatures*, since they are obtained as generalization to surfaces of the concept of concentrated curvature for polygonal lines just introduced. The major advantage of this method is the use of intrinsic properties of a discrete mesh to define geometric features that have the same properties as the analytic methods. In this work, we also compare Gaussian and mean *C*curvatures with widely used discrete curvature estimators for analytic Gaussian and mean curvatures and with concentrated curvature (according to Aleksandrov's definition).

2 BACKGROUND NOTIONS

In this Section, we briefly review some fundamental notions on curvature in the analytic case and on concentrated curvature.

Let C be a smooth curve having parametric representation $(c(t))_{t\in R}$. The curvature $k(p)$ of C at a point $p = c(t_0)$ is given by

$$k(p) = \frac{1}{\rho} = \frac{|c'(t) \wedge c''(t)|}{|c'(t)|^3},$$

where ρ is called the *curvature radius*. Number ρ corresponds to the radius of the osculatory circle tangent to C at p.

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Let S be a smooth surface and Π be a plane which contains the unit normal vector $\overrightarrow{n_p}$ at a point $p \in S$. Plane Π intersects S through a smooth curve C containing p with curvature $k_C(p)$ at the point p called *normal curvature*. When Π turns around $\overrightarrow{n_p}$, curves C vary. There are two extremal curvature values $k_1(p) \leq k_2(p)$ which bound the curvature values of all curves C. The corresponding curves C_1 and C_2 are orthogonal at point p (Do Carno, 1976). These extremal curvatures are called *principal normal curvatures*. Note that, if the normal vector $\overrightarrow{n_p}$ is on the same side as the osculatory circle, then the curvature value k_i of curve C_i has a negative sign.

Definition 1. *The* Gaussian curvature K_p *and the* mean curvature H_p *at point* $p = \phi(x,y)$ *are defined, respectively, as*

$$K_p = k_1(p) * k_2(p), \qquad H_p = \frac{k_1(p)+k_2(p)}{2} \tag{1}$$

The formula defining H_p turns out to be the mean of all values of normal curvatures at point p. Gaussian and mean curvatures depend strongly on the (local) geometrical shape of the surface. We will see that this property is relaxed for concentrated curvature. A remarkable property of Gaussian curvature is given by the *Gauss-Bonnet Theorem*, which relates the geometry of a surface, given by the Gaussian curvature, to its topology, given by its Euler characteristic (see (Do Carno, 1976)).

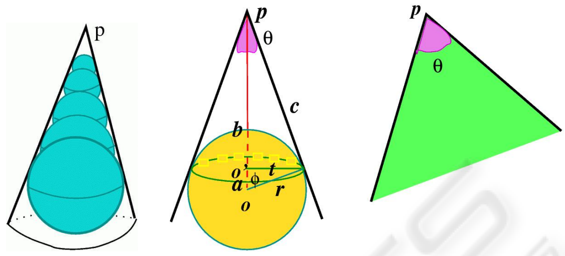
A *singular flat surface* is a surface endowed with a metric such that each point of the surface has a neighborhood which is either isometric to a Euclidean disc or a cone of angle $\Theta \neq 2\pi$ at its apex. Points satisfying this latter property are called *singular conical points*. Any piecewise linear triangulated surface has a structure of a singular flat surface. All vertices with a total angle different from 2π (or π for boundary vertices) are singular conical points. As shown below, the Gaussian curvature is accumulated at these points so that the Gauss-Bonnet formula holds.

Let Σ be a (piecewise linear) triangulated surface and let p be a vertex of the triangle mesh. Let $\Delta_1, \cdots, \Delta_n$ be the triangles incident at p such that Δ_i and Δ_{i+1} are edge-adjacent. If a_i, b_i are the vertices of triangle Δ_i different from p, then the total angle Θ_p at p is given by $\Theta_p = \sum_{i=1}^{n} \widehat{a_i P b_i}$

Definition 2. *(Aleksandrov, 1957; Troyanov, 1986) The* concentrated Gaussian curvature $K_{C2}(p)$, *at a vertex p of the triangulated surface, is the value*

$$K_{C2}(p) = \begin{cases} 2\pi - \Theta_p & \text{if } p \text{ is an interior vertex, and} \\ \pi - \Theta_p & \text{if } p \text{ is a boundary vertex.} \end{cases}$$

The above discrete definition of curvature can be justified as follows. The surface is assumed to have a conical shape at each of its vertices. Each cone is then approximated from its interior with a sphere S_r^2 of radius r, as shown in Figure 1(a). Following



(a) (b)

Figure 1: In (a), spheres tangent to a cone from it interior; in (b), parameters for computation.

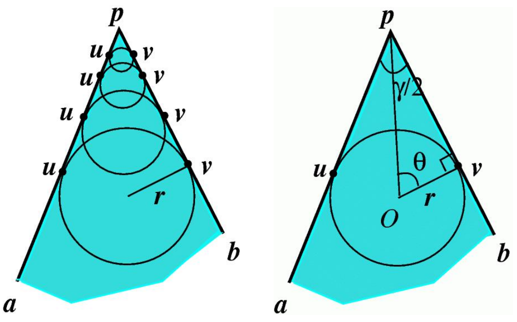
the Gauss-Bonnet theorem, the spherical cap approximating the cone has a total curvature which is equal to $2\pi - \Theta$, where Θ is the angle of the cone at its apex. This quantity does not depend on the radius of sphere S_r^2 by which we approach the cone and hence $2\pi - \Theta_p$ is an intrinsic value for the surface at vertex p. This remarkable property fully justifies the name *concentrated curvature*.

The local shape of the surface does not play a role here, unlike in the analytic case, where Gaussian curvature is strongly dependent of the local surface shape. Concentrated curvature satisfies a discrete equivalent of the Gauss-Bonnet theorem (Akleman and Chen, 2006).

3 *C*CURVATURE FOR POLYGONAL CURVES

In this Section, we use the concentrated curvature principle to define a concentrated curvature for polygonal curves. In this way, we can define principal concentrated curvatures for a triangulated surface and follow the same construction, used in Section 2 for analytic mean and Gaussian curvatures, to define similarly new discrete curvature estimators. We will call them *C*curvatures. The initial *C* is a shortcut for "concentrated". We will show also that *C*curvature does not suffer from convergence problems.

Let C be a simple polygonal curve in the three-dimensional Euclidean space. Let p be a vertex on C and a, b its two neighbors on C. Points a, b and p define a plane Π. If the angle $\gamma = \widehat{apb}$ is equal to π, then the curvature value $k(p)$ of C is 0. Otherwise, let $S_r \subset \Pi$ be a circle of any radius $r > 0$ tangent C at two points $u \in [p,a]$ and $v \in [p,b]$ (see Figure 2(a)).



(a) (b)

Figure 2: In (a), circles tangent to the sector from it interior. In (b), computing total curvature of arc (uv).

Let (uv) be the arc of S_r delimited by u and v and located in triangle $\Delta(upv)$. Then, the polygonal path apb can be smoothly approximated by the path $[au] \cup (uv) \cup [vb]$ for any value of r (see Figure 2(a)). The curvature value at any point of circle S_r is constant and equal to $1/r$. Let O be the center of S_r and θ be the angle $\widehat{uOp} = \widehat{vOp}$. The total curvature of (uv) is given by:

$$k = \int_{(uv)} \frac{1}{r} dl = \frac{1}{r} l(uv) = \frac{1}{r} 2r\theta = \pi - \gamma. \tag{2}$$

The above quantity does not depend on the radius of circle S_r through which we approximate the curve. This means that $\pi - \gamma$ is an intrinsic quantity of curve C at vertex p since it depends only on the fracture angle γ. Then, we can give the following definition:

Definition 3. Concentrated curvature *of C at p is the total curvature $\pi - \gamma$ of arcs (uv) approximating curve C around point p. We simply call it Ccurvature and we denote it by $k_C(p)$.*

Let C be a discrete piecewise-linear oriented planar curve. Suppose that C is parameterized, and hence oriented, by its natural arc length s and that the positive angle orientation is counterclockwise. At a vertex of the curve for which the next segment is not aligned with the previous one, the deviation angle γ at the vertex is computed in the positive sense from the previous segment to the new segment. With this convention the Ccurvature $\pi - \gamma$ of a parameterized curve may have negative or positive values.

4 *C*CURVATURE FOR TRIANGULATED 3D SURFACES

Let Σ be a piecewise linear triangulated surface and p be a vertex of Σ. Let $\vec{n}$ be the normal vector at p defined by the average of the normal vectors of the triangles incident in p. Let Π be a plane passing by p and containing the normal vector $\vec{n}$. This plane cuts surface Σ along a polygonal curve $C = \Sigma \cap \Pi$ containing point p. We compute the Ccurvature $k_C(p)$ at point p of curve C as described in Section 3. Note that the position of the normal vector $\vec{n}$ with respect to the polygonal curve C should be taken into account. If the normal vector $\vec{n}$ and the polygonal curve C lie in two different half planes (or equivalently they are separated by the "tangent" plane T_p whose normal vector is $\vec{n}$, see Figure 3), then the angle γ of C at p is smaller than π and the Ccurvature value $\pi - \gamma$ is positive. Otherwise, the angle γ of C at p is larger than π and the Ccurvature value $\pi - \gamma$ is negative. In this case, for simplicity of computation, we observe that if γ' is the geometric angle of C at p (i.e., γ' is the supplementary angle of γ: $\gamma + \gamma' = 2\pi$), then we have $\pi - \gamma = \gamma' - \pi$. This Ccurvature value corresponds to the normal curvature at vertex p.

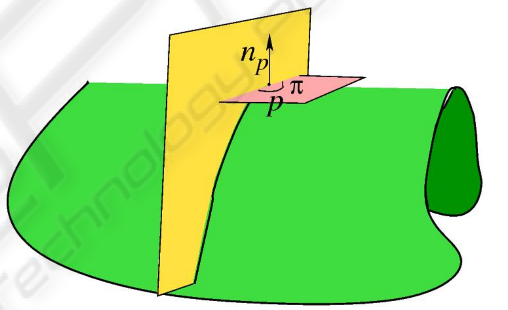


Figure 3: Intersection of plane Π with a smooth surface. Angle at p is divided into two equal angles.

When plane Π turns around $\vec{n}$, we obtain a set of Ccurvature values bounded (respectively from below and from above) by two extremal values $k_{C,1}(p) \leq k_{C,2}(p)$. Values $k_{C,1}(p)$ and $k_{C,2}(p)$ correspond to the *principal curvatures*. Based on them, we can define a mean and a Gaussian curvature, in a similar way as we do in the analytic case:

Definition 4. *The extremal values $k_{C,1}(p)$ and $k_{C,2}(p)$ bounding the set of $k_C(p)$'s are called the* principal *C*curvatures *at vertex p.*

Definition 5. *The* Gaussian *C*curvature $K_{C1}(p)$ *of Σ at vertex p is defined as the product $k_{C,1}(p) * k_{C,2}(p)$.*

Definition 6. *The* mean *C*curvature $H_C(p)$ *of surface Σ at vertex p is defined as the mean value of all normal Ccurvature values obtained by turning plane Π around the normal vector.*

Note that all these values are intrinsic values depending only on the local geometric shape of surface Σ.

In practice, we cannot compute all the normal Ccurvature values ($k_C(p)$) since the rotation of plane Π generates an infinite sequence of values. We can extract a subsequence for computing an approximation of principal Ccurvatures.

Following the way in which normal curvature is defined for smooth surfaces (see Section 2), we simulate a discrete rotation around the normal vector $\overrightarrow{n_p}$ at a vertex p of the plane Π containing $\overrightarrow{n_p}$, by considering one plane for each vertex v_i in the star of p. Given v_i, we take the curve where plane Π_i, containing p, $\overrightarrow{n_p}$ and v_i, intersects the star of p, and compute its curvature. This process gives a discrete rotation around p. Each intersection curve is a polygonal line $(v_i p w_i)$ where w_i is the intersection point between plane Π_i and the link of p. The angle $\gamma_i = \widehat{v_i p w_i}$, in the interior of the cone, between vectors $\overrightarrow{pv_i}$ and $\overrightarrow{pw_i}$ is defined by $\gamma_i := \arccos(\frac{<\overrightarrow{pv_i}, \overrightarrow{pw_i}>}{pv_i \cdot pw_i})$. The normal Ccurvature at p of the polygonal line $(v_i p w_i)$ is defined by $\mp(\pi - \gamma_i)$. The sign $+1$ or -1 is defined by following the position of the normal vector $\overrightarrow{n_p}$ with respect to the polygonal line $(v_i p w_i)$ as explained above. Following this construction, principal, mean and Gaussian Ccurvatures can be defined at p.

5 EXPERIMENTAL RESULTS

In this Section, we experimentally compare our Ccurvatures with other classic approaches to compute discrete curvatures.

We compare Gaussian Ccurvature with concentrated curvature (described in Section 2) and with Gaussian angle deficit. The *Gaussian angle-deficit* curvature estimator (Meyer et al., 2003) is defined at a vertex p by

$$K_p = \frac{1}{A}(2\pi - \Theta_p), \tag{3}$$

where Θ_j is the angle at p formed by the j-th triangle incident at p, and A is the area of the 1-ring neighborhood around p or the Voronoi region around p.

We compare mean Ccurvature with mean angle deficit. The *mean angle-deficit* curvature estimator (Meyer et al., 2003) is defined, at point p, as the magnitude of the following mean curvature vector normalized by the area of the surrounding (Voronoi or barycentric) neighborhood:

$$H_p = \frac{1}{4}\sum_{j=1}^{N}(\cot\alpha_j + \cot\beta_j)(p - x_j), \tag{4}$$

where angles α_j and β_j correspond to the remaining summits of the quadrilateral formed by the two triangles adjacent to edge px_j. Sign $\pm$ is assigned to $|H_p|$ following the direction of the mean curvature vector with respect to the normal vector of the surface at point p.

5.1 Behavior on the Sphere

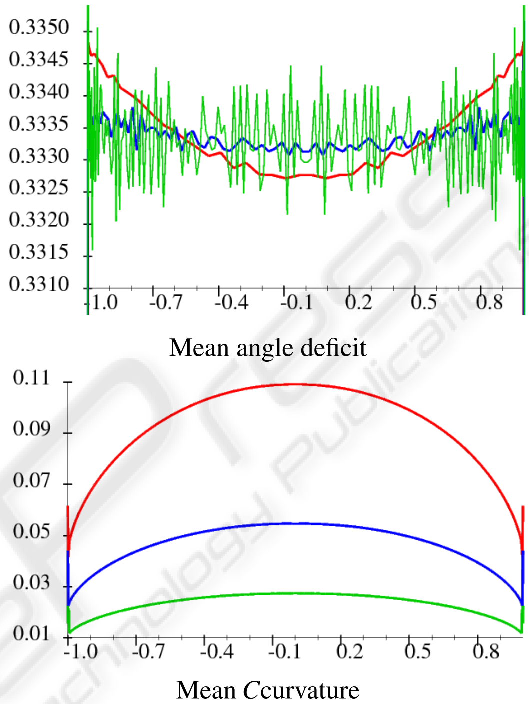


Figure 4: Curvature values along a meridian of the sphere, approximated with a polyhedron made up of 5k (red), 20k (blue), 80k (green) triangles.

The objective of discrete curvature estimators is not to produce discrete curvature values close to analytic ones, but to exhibit the same behavior as the analytic ones, although they may produce curvature values in their own range. This is sufficient for many applications (e.g., mesh segmentation at lines of minimal curvature).

On a sphere, both Gaussian and mean analytic curvatures are constant. Since discrete curvature estimators are mesh-dependent, we analyze their behavior on different triangle meshes approximating the sphere. If a sphere is approximated by a regular polyhedron (e.g., an icosahedron), then all discrete estimators give the same value at all vertices, as expected. Another way to approximate a sphere is drawing a number of tracks and sectors and triangulating the resulting net (the resolution is controlled by varying the number of tracks and sectors). Here, not all triangles and angles are equal, and thus the discrete estimators give variable results. We performed our tests with triangle meshes at increasing resolutions. Fig-

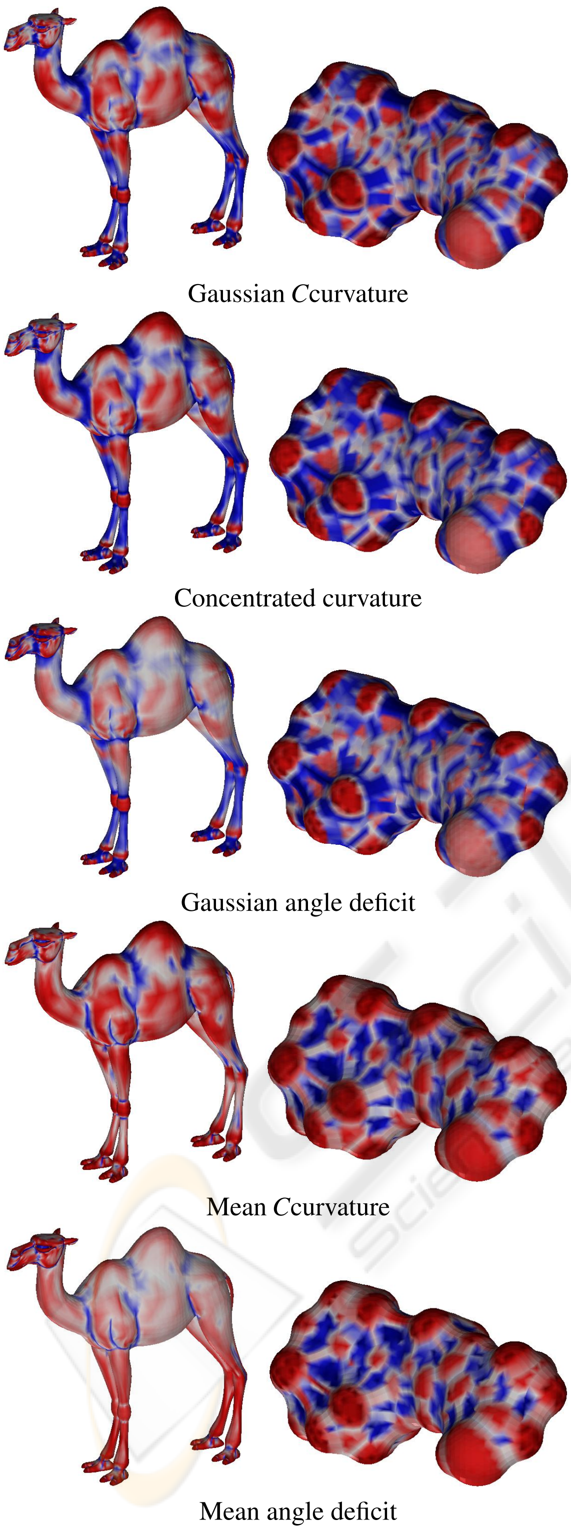


Figure 5: The various curvatures on the Camel and the Retinal meshes, in color scale.

ure 4 shows the curvature values along a meridian of the sphere, at various resolutions, for mean angle deficit and mean *C*curvature. Gaussian *C*curvature and concentrated curvature are very close, and similar to mean *C*curvature. Gaussian curvature behaves in a similar way as mean angle deficit. All estimators are affected by the different shape and size of triangles at the various latitudes, and give abnormal values at poles. Unlike all others, the two angle deficit measures provide higher values near the poles than near the equator of the sphere. Angle deficit estimators show a relevant noise, which becomes worse while increasing resolution, because of the division by area. The values of other measures vary smoothly along latitude, and tend to be closer to a constant value at higher resolutions.

5.2 Comparisons on Discrete 3D Shapes

We compare the curvature estimators on triangulated shapes of different kinds from the AIM@SHAPE repository (http://shapes.aim-at-shape.net/). The Camel mesh has 9770 vertices and 19536 triangles; the Retinal molecule mesh has 3643 vertices and 7282 triangles. Curvature values are plotted in a color scale in Figure 5. The color scale is from blue (minimum negative value) through white (zero) to red (maximum positive value), where the minimum and maximum values may be different in the specific curvature estimators.

Gaussian *C*curvature and concentrated curvature have a very similar behavior. The two angle deficit measures tend to give near-zero values (white) to wider areas, while the other estimators better follow the geometrical shape of the surface. This is especially evident on Camel mesh. On Retinal molecule, all mean estimators and all Gaussian estimators behave similarly. The mean and Gaussian *C*curvature values seem to provide a slightly better delimitation of positive and negative curvature areas.

In estimating *C*curvatures, we approximate the continuous rotation of the plane around the surface normal at a vertex p, in a discrete way, by using just the planes passing through the neighbors of p. To get more precision, we may refine the link (i.e., the boundary of the star) of p by adding new points on its edges, and consider a plane through each of them. Experiments show the resulting (mean and Gaussian) *C*curvature values are almost the same, and this confirm the validity of the approach.

To illustrate the application of curvature to 3D shape segmentation, we show in Figure 6 two segmentations based on mean *C*curvature, which highlight convex features.

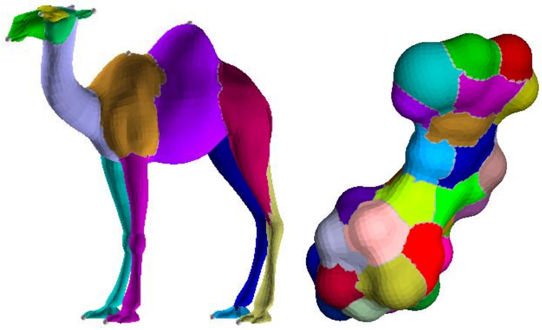


Figure 6: Segmentations of 3D shapes based on mean *C*curvature.

6 CONCLUDING REMARKS

We have presented a geometrical technique to estimate discretely normal, principal, mean and Gaussian curvatures on a triangulated piecewise linear surface. Our technique describes well the local geometric shape of the surface. The experimental results we presented validate the approach and show that *C*curvature behaves better than some mostly used techniques.

It would be interesting to further investigate the stability, or convergence of Ccurvature values to some intrinsic value, when adding more points to refine the discrete rotation of the cutting plane around a vertex normal.

Concentrated curvature of polygonal curves has interesting applications for spatial curves to characterize them via an additional discrete torsion notion, this will leads to interesting applications in GPS field and robotics (3D motion for example).

The principle of concentrated curvature has also been used to define discrete estimators for scalar curvature of 3-combinatorial manifolds.

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