CHARACTERISTICS OF DEFINING HYPERPLANES OF CONSTANT RETURNS TO SCALE TECHNOLOGY IN DEA

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Abstract: In this paper characteristics of defining hyperplanes of constant returns to scale technology in DEA have been investigated. A defining hyperplane namely *H* is a type of hyperplane that with the elimination of *H,* the production possibility set (PPS) will be enlarged (In this paper a defining hyperplane exactly is the full dimensional efficient facet (FDEF) and may be found in Olesen and Peterson (1996, 2003)). The point of view of some of the characteristics is conceptual and the interpretation of defining hyperplanes of constant returns to scale technology can be achieved by these conceptual characteristics. However, some of the characteristics are practical and one can easily utilize them in practice. Some parts of topology and convex analysis have been considered to show the truth of characteristics.

1 INTRODUCTION

Data envelopment analysis (DEA) is a non-parametric approach which was suggested by Charnes et al. (1978) to measure the relative efficiency of a decision making unit (DMU) and provide DMUs with relative performance assessment on multiple inputs and outputs. Based on different essential properties and corresponding to different characteristics of the production possibility set (PPS) and production frontiers, different DEA models, such as the CCR model, the BCC model and the FDH model, have been introduced.

An important task of DEA is to identify the returns to scale (RTS) of DMUs based on the position of the supporting hyperplanes of efficient frontier. Therefore, the investigation of different types of hyperplanes of efficient frontier or PPS is an important part of DEA.

No many papers in DEA have been written on the subject of "investigation of efficient frontier" and "characteristics of different types of hyperplanes". Finding of the piecewise linear frontier of production function which identifies the efficient frontier and efficient DMUs in DEA has been investigated by Jahanshahloo et al. (2005), in particular the aim of their study was to develop a way to obtain efficient frontier by using 0-1 integer programming, then by means of it, identification of efficient DMUs and their returns to scale characteristics. Also, searching of efficient frontier in DEA, has been considered by Korhonen (1997). Korhonen tried to provide the decision maker (DM) an interactive method which allows him or her to incorporate performance information in to the efficient frontier analysis by enabling him or her to make a free search on efficient frontier, furthermore, Korhonen provided the DM all references of an inefficient DMU, enabling him or her to choose the most preferable unit as reference. Furthermore, Jahanshahloo et al. (2007) suggested a way of finding strong defining hyperplanes of production possibility set in DEA, particularly their method is based on the relation between efficient surfaces and strong defining hyperplanes of production possibility set. Also, Cooper et al. (2007) make it possible to select the weights, obtained by the multiplier model in DEA, associated with the facets of higher dimension that a DMU generates, in particular their method supplies model for locating facets of the maximum possible dimension of the efficient frontier. Furthermore, the construction of all DEA efficient frontiers in generalized data envelopment analysis (GDEA) has been discussed by Yu. et al. (1996).

Almost in all of the abovementioned researches, there is no investigation about the characteristics of defining hyperplanes of production possibility set that is so essential in DEA. In this paper, we have

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presented some essential theorems in order to identify the defining hyperplanes of constant returns to scale (CRS) technology. These theorems enable us to recognize whether a hyperplane obtained by the optimal solution of the multiplier form of CCR model is a defining hyperplane.

Furthermore, one of the most important task of defining hyperplanes of production possibility set is sensitivity analysis that enable us to determine the amounts of perturbations of data that can be tolerated by a DMU on efficient frontier before becoming inefficient. Also, we can utilize the concept defined in this paper in order to evaluate the efficiency of DMUs by using the defining hyperplanes of PPS, which efficient DMUs are on them.

Some of the characteristics presented in this paper are more conceptual, however others are more practical. Furthermore, the conceptual point of view of theorems presented in this paper, enable us to interpret the characteristics of defining hyperplanes of CRS technology. Although some of the theorems are so practical and one can easily utilize them in practice. Not only, the conceptual point of view of theorems is essential and is so useful to interpretation of defining hyperplanes of CRS technology, but also the practical point of view of theorems is a necessity and enable us to utilize the characteristics in practice.

The aim of this paper is to use the conceptual point of view of some parts of topology and convex analysis and a combination of them with DEA to present some conceptual and practical characteristics in order to determine when a hyperplane of PPS is a defining hyperplane. The main idea of this paper is based on the geometrical interpretation of efficient facets of the highest dimension of the frontier that the DMU under assessment contributes to span. In particular a defining hyperplane is a full dimensional efficient facet (FDEF) and may be found in Olesen and Peterson (2003). These geometrical interpretations enable us to establish the presented characteristics. Some of these characteristics are conceptual that we will not be able to utilize them in practice. Although, we use these conceptual characteristics in order to establish some practical characteristics that one may easily utilize them in practice.

The sections of this paper are organized as follows. In the next section, Section 2, we provide additional background of our paper. In Section 3, we give basic concepts of some parts of topology, convex analysis and DEA models. Section 4 investigates the characteristics of defining hyperplanes of constant returns to scale (CRS) technology. In Section 5, we present an example to illustrate the characteristics.

2 BACK GROUND

As previously noted, this paper is dealt with the characteristics of defining hyperplanes of CRS technology in DEA. These defining hyperplanes play an important role in DEA as previously mentioned.

In this paper, we restrict attention to geometrical differences between defining hyperplanes of CRS technology and those supporting hyperplanes of CRS technology that are not defining. As we know, these two kinds of hyperplanes play a crucial role in DEA, since they are generally utilized to determine different types of concepts such as efficiency, bench mark DMUs, rates of substitution and transformation, returns to scale, sensitivity analysis and etc.

The main idea of this paper is based on geometrical interpretation of defining hyperplanes of CRS technology. In order to state a geometrical characteristics of defining hyperplanes of CRS technology, we use a combination of different kinds of concepts such as interior points of a set, an ε-neighborhood around a point and geometrical interpretation of CRS technology efficient frontier to state a specific relation between the dimension of intersection of each defining hyperplanes with the production possibility set (PPS) of CRS technology that we use this characteristics to show the truth of others stated characteristics.

Secondly, we utilize a model proposed by Cooper et al. (2007) to determine a hyperplane that is binding at the maximum number of extreme efficient units. With utilizing the abovementioned hyperplane namely H^*, we define a created DMU obtained by center of gravity of extreme efficient units that the abovementioned hyperplane H^* is binding at them. Eventually, a set of feasible directions obtained by connecting the created DMU to each extreme efficient unit that the hyperplane H^* is binding at them has been defined to present a practical characteristic.

3 THEORETICAL CONSIDERATIONS

3.1 Some Basic Concepts of Topology

In this subsection we review some topological properties of sets and some basic results from convex analysis.

Definition 1. Given a point $x \in R^n$, a $\varepsilon-$*neighborhood* around it is the set

$$N_\varepsilon(x) = \{y \mid \|y - x\| < \varepsilon\} \tag{1}$$

Definition 2. Let X be an arbitrary set in R^n. x is said to be in the *interior* of X, denoted by $\text{int}\, X$, if $N_\varepsilon(x) \subset X$ for some $\varepsilon > 0$.

Definition 3. Let X be an arbitrary set in R^n. x is said to be in the *boundary* of X, denoted by ∂X, if $N_\varepsilon(x)$ contains at least one point in X and one point not in X for every $\varepsilon > 0$.

Definition 4. A set X in R^n is called a *convex set* if given any two points x_1 and x_2 in X then $\lambda x_1 + (1-\lambda)x_2 \in X$ for each $\lambda \in [0,1]$.

Definition 5. A point x in a convex set X is called an *extreme point* of X, if x can not be represented as a strict convex combination of two distinct points in X.

Definition 6. A *hyperplane* H in R^n is a set of the form

$$\{x \mid px = k\} \tag{2}$$

where p is a non-zero vector in R^n and k is a scalar. Also, p is usually called the *normal* or the *gradient* to the hyperplane.

Definition 7. A hyperplane divides R^n into two regions, called half spaces. Hence two *half spaces* H^+ and H^- may be defined in the following manner:

$$H^+ = \{x \mid px \geq k\} \tag{3}$$

$$H^- = \{x \mid px \leq k\} \tag{4}$$

where p is a non-zero vector in R^n and k is a scalar. Also,

$$H = H^+ \cap H^- \tag{5}$$

Definition 8. A *polyhedral set* or *polyhedron* is the intersection of a finite number of halfspaces. A bounded polyhedral set is called a *polytope*.

Suppose that the polyhedral set under discussion in the following definitions has the form

$$X = \{x \mid Ax \leq b, x \geq 0\} \tag{6}$$

where A is $m \times n$ and b is an m-vector. The hyperplanes associated with the $(m+n)$ defining halfspaces

$$\{x \mid a^i x \leq b^i\}, i = 1,\ldots,m \tag{7}$$

And

$$\{x \mid e_j x \geq 0\}, j = 1,\ldots,n \tag{8}$$

are called *defining hyperplanes* of X.

Definition 9. Let $\bar{x} \in X$. A constraint $a^l x \leq b^l$ is *binding, or tight, or active,* at $\bar{x} \in X$, if

$$a^l \bar{x} = b^l \tag{9}$$

Definition 10. A hyperplane H is a *supporting hyperplane* of X, if

$$H \cap X \neq \Phi \;\&\; (X \subseteq H^+ \text{ or } X \subseteq H^-) \tag{10}$$

Definition 11. The set of points in X that correspond to some non-empty of binding defining hyperplanes of X are called *faces* of X. Given any face F of X if $r(F)$ is the maximum number of linearly independent defining hyperplanes binding at all points feasible to F, then the *dimension* of F, denoted by $\dim(F)$, is equal to $n - r(F)$.

Also, the highest dimensional face of X is of dimension $\dim(X) - 1$ and it is called a *facet* of X.

3.2 DEA Background

Assume that we have n DMUs each consuming m inputs and producing s outputs. Let X be an

$(m \times n)$-matrix and Y be a $(s \times n)$-matrix consisting of non-negative elements, containing observed input and output measures for the DMUs, respectively. We denote by $X_j \geq 0, X_j \neq 0, j = 1,...,n$ (the jth column of X) the vector of inputs consumed by DMU_j. A similar notation is used for outputs.

The traditional CCR models, as introduced by Charnes et al. (1978) are fractional linear programs, which can easily be formulated and as linear programs. Those models are so-called constant returns to scale (CRS) models. Later Banker et al. (1984) developed the so-called BCC models with variable returns to scale (VRS).

The CCR and BCC models are the basic model types in DEA. Those basic models can be presented in a primal or dual form. The usage of primal and dual varies in the literature, and it is more straightforward to call them multiplier and envelopment models, respectively. The multiplier model provides information on the weights of inputs and outputs. The weights are interpreted as prices in many applications. The envelopment models provide the user with information on the lacks of outputs and the surplus of inputs of a unit. Also, the envelopment model characterizes the reference set for the units. Moreover, the production possibility set (PPS) of CCR and BCC models can be interpreted from the structure of envelopment models. Since, we are interested in CCR models in this paper, we represent the PPS of constant returns to scale (CRS) technology in the following manner:

$$T_c = \left\{ \begin{array}{l} Z = (X,Y)^T \,\middle|\, \mathbf{X}\lambda \leq X, \mathbf{Y}\lambda \geq Y, \\ \lambda \geq 0 \end{array} \right\} \quad (11)$$

Based on the PPS of CRS technology the envelopment form of CCR model is in the following manner:

$$\begin{array}{lll} Min & \theta & \\ S.t. & \mathbf{X}\lambda \leq \theta X_o & \\ & \mathbf{Y}\lambda \geq Y_o & \\ & \lambda \geq 0. & \end{array} \quad (12)$$

The multiplier form of model CCR based on the dual of model (12) is as follows:

$$Max \quad U^T Y_o \quad (13)$$

$$\begin{array}{ll} S.t. & V^T X_o = 1 \quad (13.1) \\ & U^T Y_j - V^T X_j \leq 0, \; j = 1,...,n \quad (13.2) \\ & U \geq 0, V \geq 0 \end{array}$$

We know that in the optimal solution (U^*, V^*) of model (13), at least one constraints of (13.2) is binding. Also, it is easy to show, this optimal solution (U^*, V^*) is the normal vector of a supporting hyperplane

$$H^* = \left\{ (X,Y)^T \,\middle|\, U^{*T} Y - V^{*T} X = 0 \right\} \quad (14)$$

which, supports T_c constructed by observed data.

Definition 12. DMU_o is an *extreme efficient unit* if in the evaluation of DMU_o, the optimal solution of model (12) is unique and

$$\lambda_o^* = 1, \lambda_{j \neq o}^* = 0 \quad (15)$$

Also, the indices of all extreme efficient units is denoted by

$$E = \left\{ j \,\middle|\, DMU_j \text{ is an extreme efficient unit} \right\} \quad (16)$$

We know that T_c is the intersection of some hyperplanes. We call some of these hyperplanes as *defining hyperplanes* if with the elimination of these hyperplanes, T_c will be enlarged.

Definition 13. A hyperplane H is a defining hyperplane of T_c if with the elimination of H, T_c will be enlarged (A defining hyperplane used in this paper exactly is FDEF defined by Olesen and Peterson (2003)).

4 CHARACTERISTICS OF DEFINING HYPERPLANES OF CRS TECHNOLOGY

In this section, we present some essential theorems in order to recognize all defining hyperplanes of T_c. In these theorems some important characteristics of defining hyperplanes of T_c have been identified. These theorems enable us to recognize when a hyperplane is a defining hyperplane of T_c. Therefore, using these theorems one will be able to recognize any defining hyperplanes of T_c which was not possible before. As mentioned in previous sections, if (U^*, V^*) is an optimal solution of the multiplier model of CRS technology (13), then

$$\{(X,Y) \mid U^{*T}Y - V^{*T}X = 0\} \quad (17)$$

will be a supporting hyperplane of T_c. Assume that

$$H_t = \left\{(X,Y)^T \,\middle|\, U_t^T Y - V_t^T X = 0\right\} \quad (18)$$
$$t = 1,\dots,K$$

are all defining hyperplanes of T_c which we are interested in. Also consider two defining half-spaces

$$H_t^+ = \left\{(X,Y)^T \,\middle|\, U_t^T Y - V_t^T X \geq 0\right\} \quad (19)$$

and

$$H_t^- = \left\{(X,Y)^T \,\middle|\, U_t^T Y - V_t^T X \leq 0\right\} \quad (20)$$

obtained by hyperplane H_t for each $t = 1,\dots,K$. With out loss of generality, we can assume that T_c is the intersection of all defining half-spaces H_t^-, $t = 1,\dots,K$ in the following manner:

$$T_c = \bigcap_{t=1}^K H_t^- \quad (21)$$

Theorem 1. *The hyperplane* H *is a defining hyperplane of* T_c *if and only if the dimension of* $(T_c \cap H)$ *equals* $m+s-1$.

Proof. Assume that the dimension of $(T_c \cap H)$ equals $m+s-1$. On one hand, since, $T_c \cap H \neq \Phi$, we can find a point such as $\bar{Z} = (\bar{X}, \bar{Y})^T \in T_c \cap H$ for which there exists a $(m+s-1)$ dimensional $\varepsilon-$neighborhood $N_\varepsilon(\bar{Z}) \subseteq T_c \cap H$. On the other hand, $\bar{Z} = (\bar{X}, \bar{Y})^T$ is a point contained in T_c for which, only one hyperplane such as H is binding. Therefore, with the elimination of half-space H^- obtained by hyperplane H from T_c (without loss of generality assume that $T_c \subseteq H^-$), the point $\bar{Z} = (\bar{X}, \bar{Y})^T$ will be an interior point of T_c' (T_c' is the set obtained by the elimination of half-space H^- from T_c). Note that $T_c \subseteq T_c'$. Since, $\bar{Z} = (\bar{X}, \bar{Y})^T$ is an interior point of T_c', therefore, there exists an $\bar{\varepsilon} > 0$, for which, $N_{\bar{\varepsilon}}(\bar{Z}) \subseteq T_c'$. Also, since $\bar{Z} = (\bar{X}, \bar{Y})^T \in \partial T_c$, each $N_\varepsilon(\bar{Z})$ contains at least one point in T_c and one point not in T_c for every $\varepsilon > 0$. Now, assume that this neighborhood is $N_{\bar{\varepsilon}}(\bar{Z})$. Thus, there exists a point such as Z_o for which $Z_o \in N_{\bar{\varepsilon}}(\bar{Z})$ and $Z_o \notin T_c$. This shows that $Z_o \in T_c'$ and $Z_o \notin T_c$. Consequently, $T_c \subset T_c'$ and therefore, it means that with the elimination of half-space H^- from T_c, T_c has been enlarged. Thus, H is a defining hyperplane of T_c.

To show the converse, assume that the hyperplane H is a defining hyperplane of T_c. It is obvious that the dimension of each hyperplane such as H in R^{m+s} such as T_c is equal to $(m+s-1)$. Since, H is a defining hyperplane of T_c therefore, there exists a point such as $\bar{Z} = (\bar{X}, \bar{Y})^T$ in the interior of $T_c \cap H$. Now, with the elimination of half-space H^- from T_c, we will encounter with a set called T_c' ($T_c \subset T_c'$). Since, $\bar{Z} = (\bar{X}, \bar{Y})^T \in \text{int}(T_c \cap H)$, thus, $\bar{Z} = (\bar{X}, \bar{Y})^T$ will be an interior point of T_c'. This implies that there exists an $\bar{\varepsilon} > 0$ for which the $(m+s)-$dimensional $N_{\bar{\varepsilon}}(\bar{Z}) \subset T_c'$. Now, it is trivial that $N_{\bar{\varepsilon}}(\bar{Z}) \cap H$ is $(m+s-1)-$dimensional and this implies that $T_c \cap H$ is $(m+s-1)-$dimensional. This completes the proof. □

Theorem 1 shows a characteristic of defining hyperplane of T_c. In order to simplify and find more simple methods for introduction of defining hyperplanes of T_c, we need to use the following model that has been introduced by Cooper et al. (2007) with some minor modification:

$$\begin{aligned}
Min \quad & \sum_{j=1}^{n} l_j \\
St. \quad & U^T Y_P = 1 \\
& V^T X_P = 1 \\
& U^T Y_j - V^T X_j + t_j = 0, j \in E \\
& t_j - l_j M \le 0, j \in E \\
& l_j \in \{0,1\}, j \in E \\
& U \ge 0, V \ge 0, t_j \ge 0, j \in E.
\end{aligned} \tag{22}$$

Where M is sufficiently large positive number and E is the set of indices of all extreme efficient DMU's defined in previous sections.

Since, $E \neq \Phi$, thus model (22) finds a hyperplane which, is binding, at the maximum number of extreme efficient units.

Assume that

$$(U^*, V^*, T^*, L^*),\ T^* = (t_1^*, \ldots, t_{|E|}^*),\ L^* = (l_1^*, \ldots, l_{|E|}^*) \tag{23}$$

is an optimal solution of model (22). We define the hyperplane H^* in the following manner:

$$H^* = \left\{ (X, Y)^T \,\middle|\, U^{*T} Y - V^{*T} X = 0 \right\} \tag{24}$$

The following theorem emphasizes the existence of a defining hyperplane of T_c at each extreme efficient unit.

Theorem 2. *There exists at least one defining hyperplane of* T_c *such as H for each* $j \in E$*, for which,* $Z_j = (X_j, Y_j)^T \in H$.

Proof. As we know, $T_c = \bigcap_{t=1}^{K} H_t^-$. To the contrary of the desired result, suppose that there is no defining hyperplane of T_c which is binding at $Z_j = (X_j, Y_j)^T, j \in E$. Therefore, $Z_j = (X_j, Y_j) \in \text{int}(H_t^-),\ t = 1, \ldots, K$. Consequently, $Z_j = (X_j, Y_j) \in \text{int}(T_c)$, which is in contradiction with $j \in E$. This completes the proof. □

In order to improve the conditions under which one can more easily identify the defining hyperplanes of T_c and present more practical characteristics of determining defining hyperplanes of T_c, we define a set based on the optimal solution of model (22) as follows:

$$E^* = \left\{ j \,\middle|\, t_j^* = 0, in\ (22) \right\} \tag{25}$$

The following theorem, shows that, E^* is not vacuous.

Theorem 3. $E^* \neq \Phi$.

Proof. The proof is obvious and omitted. □

The improvement of conditions and characteristics of determining defining hyperplanes of T_c made us define a created DMU in the following manner:

$$Z^* = (X^*, Y^*)^T = \frac{1}{|E^*|} \sum_{j \in E^*} (X_j, Y_j)^T \tag{26}$$

Particularly, $Z^* = (X^*, Y^*)^T$ is the center of gravity of extreme efficient units for which, the hyperplane H^* (defined based on the optimal solution of model (22)) is binding. The following theorem states that, $Z^* = (X^*, Y^*)^T$ is in boundary of T_c.

Theorem 4. $Z^* = (X^*, Y^*)^T \in \partial T_c$.

Proof. Noting theorem 3, we have $E^* \neq \Phi$.

Therefore, H^* defined in (24) is a supporting hyperplane of T_c. Since, T_c is a convex set therefore, $Z^* = (X^*, Y^*)^T \in T_c$. Also, we have

$$\begin{aligned}
& U^* Y^* - V^* X^* = \\
& U^* \Big(\frac{1}{|E^*|} \sum_{j \in E^*} Y_j \Big) - V^* \Big(\frac{1}{|E^*|} \sum_{j \in E^*} X_j \Big) = \\
& \frac{1}{|E^*|} \sum_{j \in E^*} (U^* Y_j - V^* X_j) = 0
\end{aligned} \tag{27}$$

This shows that the defining hyperplane H^* is binding at $Z^* = (X^*, Y^*)^T \in T_c$ and it means that $Z^* = (X^*, Y^*)^T \in \partial T_c$ and this completes the proof. □

Theorem 5. *If the optimal solution of model (13) in the evaluation of created unit* $Z^* = (X^*, Y^*)^T$ *is unique then the hyperplane* H^* *will be a defining hyperplane of* T_c.

Proof. Consider the following model, which is the multiplier form of CCR model based on the set E when unit $Z^* = (X^*, Y^*)^T$ is under evaluation:

$$\begin{aligned} Max \quad & U^T Y^* \\ S.t. \quad & U^T Y_j - V^T X_j \le 0,\ j \in E \\ & U^T Y^* - V^T X^* \le 0 \\ & V^T X^* = 1 \\ & U \ge 0, V \ge 0. \end{aligned} \tag{28}$$

Assume that model (28) has unique optimal solution $(\overline{U}, \overline{V})$. Define,

$$\overline{H} = \left\{ (X,Y)^T \middle| \overline{U}^T Y - \overline{V} X = 0 \right\} \tag{29}$$

It is obvious that $\overline{U}^T Y^* - \overline{V}^T X^* = 0$ and $\overline{H}$ is the only supporting hyperplane of T_c at $Z^* = (X^*, Y^*)^T$. Therefore, we can define a face of T_c for which, $Z^* = (X^*, Y^*)^T$ is on it as follows:

$$\overline{F} = \overline{H} \cap T_c$$

It is trivial that $\overline{F}$ is the only face contained $Z^* = (X^*, Y^*)^T$, therefore, the dimension of $\overline{F}$ equals to $m+s-1$ and this means that the dimension of $\overline{H} \cap T_c$ equals to $m+s-1$. Therefore, considering Theorem 1, $\overline{H}$ is a defining hyperplane of T_c. Note that $\overline{H}$ is equivalent to H^* and this means that H^* is a defining hyperplane of T_c, thus the proof is complete. □

To simplify and improve better recognition of defining hyperplanes of T_c, we utilize the following definition of feasible directions constructed by connecting $Z^* = (X^*, Y^*)^T$ to each extreme efficient unit that H^* is binding at them:

$$D = \left\{ d_j \middle| \begin{array}{l} d_j = (X_j, Y_j)^T - (X^*, Y^*)^T, \\ j \in E^* \end{array} \right\} \tag{30}$$

Theorem 6. *If the dimension of D equals to* $m+s-1$ *then* H^* *is a defining hyperplane of* T_c.

Proof. Noting that $Z^* = (X^*, Y^*)^T \in T_c$, $Z_j = (X_j, Y_j)^T \in T_c$ for each $j \in E^*$ and T_c is a convex set, we have $Z^* + \lambda(Z_j - Z^*) \in T_c$ for each $j \in E^*$ and $\lambda \in [0,1]$. Also, since $Z^* = (X^*, Y^*)^T \in H^*$, $Z_j = (X_j, Y_j)^T \in H^*$ for each $j \in E^*$ and H^* is a convex set, therefore $Z^* + \lambda(Z_j - Z^*) \in H^*$ for each $j \in E^*$ and $\lambda \in [0,1]$. Thus, these imply that $Z^* + \lambda(Z_j - Z^*) \in T_c \cap H^*$ for each $j \in E^*$ and $\lambda \in [0,1]$. Therefore, since, the dimension of D equals to $m+s-1$, thus we have $m+s-1$ independent feasible direction at Z^* in $T_c \cap H^*$. This implies that the dimension of $T_c \cap H^*$ equals to $m+s-1$ and by theorem 1, H^* is a defining hyperplane of T_c. Therefore the proof is complete. □

5 ILLUSTRATIVE EXAMPLE

In order to illustrate the characteristics of Theorems, we present a numerical example with the data set as in table 1. The CRS technology based on the data set in Table 1, has been illustrated in Fig. 1. This figure can be viewed as representing a section at a given output level, say $y = 1$, of the PPS generated two DMUs (A and B) that use two inputs and produce the same quantity of output ($y = 1$). The optimal solutions of (12) when assessing the efficiency of the extreme efficient DMU A or DMU B correspond to the coefficients of the supporting hyperplanes at A or B, which pass through origin. Model (22) then selects the hyperplane represented with a dark solid line conneting as distinct from the ones represented by the lighter dotted lines. The first one is obviously preferable to the latter because it is supported by two units (A and B) instead of by only one (A) or one (B). Moreover, in this particular case, this also means that it contains a FDEF of the frontier that DMU A and DMU B contribute to generate.

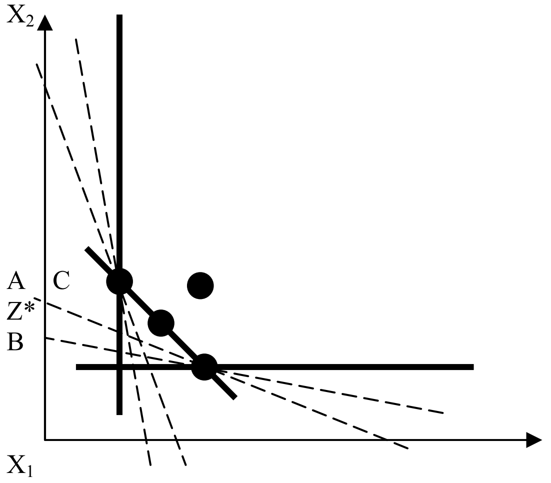

Figure 1: The graph of example.

Table 1: Data set.

DMU	Input 1	Input 2	Output
A	1	2	1
B	2	1	1
C	2	2	1

It is obvious that the defining hyperplanes of T_c are in the following manner:

$$H_1 = \{(x_1, x_2, y)^T \mid 2y - 2x_1 = 0\}$$
$$H_2 = \{(x_1, x_2, y)^T \mid 3y - x_1 - x_2 = 0\}$$
$$H_3 = \{(x_1, x_2, y)^T \mid 2y - 2x_2 = 0\}$$

We can see that $\dim(T_c \cap H_1) = 2$, $\dim(T_c \cap H_2) = 2$ and $\dim(T_c \cap H_3) = 2$ as it has been shown in Theorem 1. Therefore, the condition of Theorem 1, $\dim(T_c \cap H_t) = m + s - 1$ has been satisfied and this shows the truth of Theorem 1.

The optimal solution of model (3) shows that $E = \{1, 2\}$. As stated in Theorem 2, the hyperplanes H_1 and H_2 are two defining hyperplanes of T_c associated with $DMU\ A$ and the hyperplanes H_2 and H_3 are two defining hyperplanes of T_c associated with $DMU\ B$. These show the truth of Theorem 2.

If we solve the model (22) it will be obtained that $E^* = \{1, 2\}$ and this shows the truth of Theorem 3.

If we utilize the relation (26) we will encounter with a created DMU,

$$Z^* = (X^*, Y^*)^T = \frac{1}{2}(1, 2, 1)^T + \frac{1}{2}(2, 1, 1)^T = (1.5, 1.5, 1)^T \quad (31)$$

which has been shown in Fig. 1. It is trivial that the hyperplane H_2 is binding at $Z^* = (1.5, 1.5, 1)^T$. Consequently, $Z^* = (1.5, 1.5, 1)^T \in \partial T_c$ and this shows the truth of Theorem 4.

If we solve model (11) associated with created DMU, $Z^* = (1.5, 1.5, 1)^T$, we will obtain a unique optimal solution $(u^*, v_1^*, v_2^*) = \left(\frac{1}{3}, \frac{1}{3}, 1\right)$. Now, based on the optimal solution of model (28), the hyperplane H^* will be in the following manner:

$$H^* = \left\{(x_1, x_2, y)^T \,\middle|\, y - \frac{1}{3}x_1 - \frac{1}{3}x_2 = 0\right\} \quad (32)$$

that is exactly the hyperplane H_2. This shows the truth of Theorem 5.

The set D as stated in (13) is as follows:

$$D = \begin{Bmatrix} d_1 = (-0.5, 0.5, 0)^T, \\ d_2 = (0.5, -0.5, 0)^T \end{Bmatrix} \quad (33)$$

It is obvious that $\dim(D) = 1$. Since, H^* is a defining hyperplane of T_c, the converse of Theorem 6 does not hold and this shows that Theorem 6 is only a sufficient condition.

6 CONCLUSIONS

In this paper, some parts of topology and convex analysis have been utilized in order to state some characteristics of defining hyperplanes of CRS technology in DEA. These characteristics enable us to recognize whether a hyperplane obtained by the optimal solution of multiplier form of CCR model is a defining hyperplane. Some of the characteristics are conceptual and some of them can be easily utilized in practice. An illustrative example has been considered, in order to show the truth of characteristics stated in this paper.

We suggest as a future research, introduction of an algorithm to recognize all defining hyperplanes of

CRS technology based on characteristics presented in this paper. Also, we look for similar characteristics in the case of variable returns to scale technology as a future research.

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