EXPLICIT SOLUTION FOR THE MINIMUM DISTANCE BETWEEN TWO SOLID SEMI-INFINITE CIRCULAR CONES

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Keywords: Minimum Distance, Semi-infinite Cone, Geometric Reasoning, Explicit Solution.

Abstract: Multi-body kinematics and object rendering often involve minimum distance calculations. Explicit solutions exist for the distance between spheres, cylinders and other simple objects. Deriving the minimum distance between cones requires numerical minimization or geometrical approximations combined with analytical solutions for the simpler objects. This paper describes an explicit solution for the minimum distance between two solid semi-infinite circular cones. The method combines geometrical reasoning with analytical derivation. The solution also includes the location of the intersection points. Solution regions are identified and discussed. A numerical method based on minimizing the distance between two cone generators was used as part of the verification process. The exact solution was compared to results of approximation by regular polytopes. The explicit solution is robust, independent of coordinate system and invariant under rigid translation and rotation of the setup.

1 INTRODUCTION

Multi-body kinematics and object rendering often involve minimum distance calculations. Explicit solutions exist for simple objects, including points, lines, flat patches, spheres and cylinders. The minimum distance between two circular cones can be derived by numerical minimization or by polyhedral approximation, combined with explicit solutions for the simpler objects (The GJK Algorithm: Gilbert 1988, Jovanoski 2008, Manchem 2009. Polytopes: Chung 1996).

The geometrical approximations have inherent geometric inaccuracies and require iterative refinements. Numerical procedures based on exact parametric modeling require a good initial guess and some number crunching. Nearly tangent cones generators, steep slopes and discontinuities may cause convergence difficulties. The computational time of an iterative procedure may vary significantly depending on the parameters of the problem.

This paper describes an explicit solution for the minimum distance between two solid semi-infinite circular cones. The method is based on geometrical reasoning and vector algebra derivation. The solution also includes the location of the intersection points. Rendering and examples were implemented with Matlab®.

2 METHOD

2.1 Scope

The cones dealt with in this paper are solid, circular, semi-infinite, with positive generator angles smaller than $\pi/2$. Axes referred to in this paper are the axes of the cones. Each cone extends from apex to infinity in the positive direction of its axis.

2.2 Nomenclature

$\mathbf{a}$,$\mathbf{b}$ - cones' axes
a,b - distances to intersection points on **a** and **b**
$\mathbf{d}$ - minimum distance vector between the cones
d - minimum distance between the cones
f,g - apex shift along **a** and **b**
$\mathbf{p}$ - minimum distance vector between axes
p - distance between cones' axes
$\mathbf{r}$ - extended minimum distance vector
r - length of extended minimum distance vector
α,β - generator angles of the cones
γ - positioning angle (between cones' axes)
γ_{crit} - critical positioning angle
ω - rotational positioning angle of polytope

2.3 Statement of the Problem

Given the positioning of two known cones, the pro-

E. Karlin B. (2010).
EXPLICIT SOLUTION FOR THE MINIMUM DISTANCE BETWEEN TWO SOLID SEMI-INFINITE CIRCULAR CONES.
In *Proceedings of the International Conference on Computer Graphics Theory and Applications*, pages 154-159

blem is to find the minimum distance between their surfaces (see Figure 1). The parameters of the problem are the generator angles, the directions of the axes, the distance between the axes, location of the apexes, and the positioning angle (i.e., the angle between the axes).

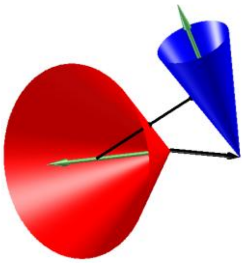

Figure 1: General view of two cones.

2.4 Geometric Setup

2.4.1 Non-Intersecting Axes

A-cone with axis **a** and generator angle α, and *B*-cone with axis **b** and generator angle β are positioned with angle γ and vector **p** between their axes (see Figure 2). For symmetry reasons, the angle between the axes is limited to $[0, \pi]$.

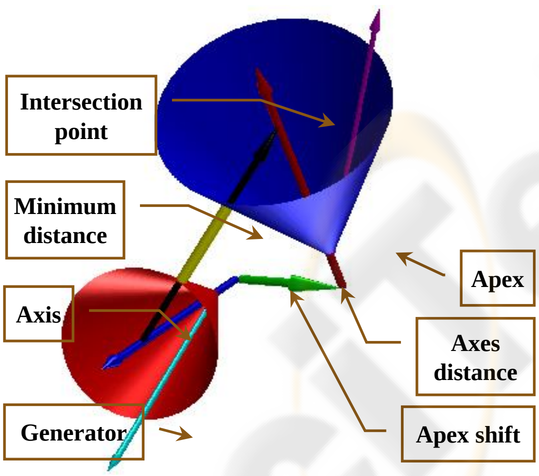

Figure 2: Geometric setup – non-intersecting axes.

2.4.2 Intersecting Axes

For intersecting axes, the problem is planar. The plane of reference contains the two axes. For each cone, the geometric components of interest are the axis, the apex, and the generator that lies in the reference plane and is nearest to the other cone.

2.5 Geometric Reasoning

Geometric reasoning includes identifying the different types of relative positioning, the geometric characteristics of each type, and defining the solution regions.

2.5.1 Geometric Types

There are three types of solutions: Surface-to-Surface, Apex-to-Surface, and Apex-to-Apex. The first two types have three regions: separation, tangency and intersection. By definition, the third type only has a separation region.

2.5.2 Surface to Surface

The minimum distance vector between the cones is external and normal to both surfaces. The extended vector intersects the *A*-axis at distance *a* from the *A*-apex and the *B*-axis at distance *b* from the *B*-apex. The normal to the cone is perpendicular to a specific generator in the plane defined by the generator and the axis (see Figure 2). The intersection point of the extended vector with the axis is invariant in space under translation of the cone along its axis.

2.5.3 Apex to Apex and Apex to Surface

For Apex-to-Surface, the minimum distance vector between the cones originates at the apex of one cone and is external and normal to the surface of the other cone.

In the case of Apex-to-Apex, the minimum distance vector between the cones is the vector between the apexes.

2.5.4 Intersecting Axes

For a setup with intersecting axes, there are three types of solutions: Apex-to-Generator, Apex-to-Apex, and Parallel-Generators. There are also three regions: separation, tangency and intersection. Tangency includes coincident apexes, apex on generator, and collinear generators.

2.6 Mathematical Formulation

2.6.1 Surface to Surface

The four vectors **a, b, p,** and **r** represented by their unit counterparts satisfy the following relationship,

$$r\hat{\mathbf{r}} + a\hat{\mathbf{a}} = p\hat{\mathbf{p}} + b\hat{\mathbf{b}} \tag{1}$$

The dot product of the equation with each of the unit vectors gives a set of four equations,

$$\begin{cases} r\hat{\mathbf{r}}\cdot\hat{\mathbf{a}} + a\hat{\mathbf{a}}\cdot\hat{\mathbf{a}} = p\hat{\mathbf{p}}\cdot\hat{\mathbf{a}} + b\hat{\mathbf{b}}\cdot\hat{\mathbf{a}} \\ r\hat{\mathbf{r}}\cdot\hat{\mathbf{b}} + a\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} = p\hat{\mathbf{p}}\cdot\hat{\mathbf{b}} + b\hat{\mathbf{b}}\cdot\hat{\mathbf{b}} \\ r\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} + a\hat{\mathbf{a}}\cdot\hat{\mathbf{p}} = p\hat{\mathbf{p}}\cdot\hat{\mathbf{p}} + b\hat{\mathbf{b}}\cdot\hat{\mathbf{p}} \\ r\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} + a\hat{\mathbf{a}}\cdot\hat{\mathbf{r}} = p\hat{\mathbf{p}}\cdot\hat{\mathbf{r}} + b\hat{\mathbf{b}}\cdot\hat{\mathbf{r}} \end{cases} \tag{2}$$

The distance vector **r** forms an angle of *α-π/2* with the *A*-axis and of *π/2-β* with the *B*-axis. Vector **p** is, by definition, perpendicular to the axes. So the dot products of **p, r, a** and **b** are,

$$\begin{cases} \hat{\mathbf{p}}\cdot\hat{\mathbf{a}} = 0;\ \ \hat{\mathbf{p}}\cdot\hat{\mathbf{b}} = 0;\ \ \hat{\mathbf{a}}\cdot\hat{\mathbf{b}} = \cos\gamma \\ \hat{\mathbf{r}}\cdot\hat{\mathbf{a}} = -\sin\alpha \quad \hat{\mathbf{r}}\cdot\hat{\mathbf{b}} = \sin\beta \end{cases} \tag{3}$$

By substituting these values into Equation 2 and rearranging the terms, the intersection distances satisfy the following set of linear equations,

$$\begin{cases} a - b\cos\gamma = r\sin\alpha \\ -a\cos\gamma + b = r\sin\beta \end{cases} \tag{4}$$

Solving the two equations gives the positions of the intersection points on the axes as a function of the angles and the distance between them,

$$\begin{cases} a = r\left(\sin\alpha + \sin\beta\cos\gamma\right)/\sin^2\gamma \\ b = r\left(\sin\alpha\cos\gamma + \sin\beta\right)/\sin^2\gamma \end{cases} \tag{5}$$

The distance between the intersection points satisfies the quadratic equation,

$$\begin{aligned} & r^2 - r(a\sin\alpha + b\sin\beta) - p^2 = 0 \\ \textbf{or}:\ \ & r\left[r - (a\sin\alpha + b\sin\beta)\right] = p^2 \end{aligned} \tag{6}$$

The intersection points on the axes are invariant under apex shifts. Substituting *a* and *b* into the equation gives the distance between the intersection points,

$$r = \frac{p}{\sqrt{1 - \dfrac{\sin^2\alpha + 2\sin\alpha\sin\beta\cos\gamma + \sin^2\beta}{\sin^2\gamma}}} \tag{7}$$

For zero apex shifts, the minimum distance d_0 is (see Figure 2),

$$\begin{aligned} d_0 &= r - a\sin\alpha - b\sin\beta \\ &= p\sqrt{1 - \frac{\sin^2\alpha + 2\sin\alpha\sin\beta\cos\gamma + \sin^2\beta}{\sin^2\gamma}} \end{aligned} \tag{8}$$

For negative values of the discriminant in Equation 8 the cones intersect (this is unconditional intersection). The discriminant is zero for *α+β=γ*, indicating tangency at infinity. For apex shifts of *f* and *g* (see Figure 2), the minimum distance is,

$$d = d_0 + f\sin\alpha + g\sin\beta \tag{9}$$

Substituting *r* from Equation 7 into Equation 5 gives the positioning of the extended minimum distance vector.

2.6.2 Critical Positioning Angle

For semi-infinite cones, *a* and *b* are non-negative. Then from Equation 5, since $sin^2\gamma$ and *r* are positive, for zero apex shifts,

$$\begin{cases} \sin\alpha + \cos\gamma\sin\beta \ge 0 \\ \sin\alpha\cos\gamma + \sin\beta \ge 0 \end{cases} \tag{10}$$

By geometric reasoning, for *α<β* the minimum distance vector starts at the *A*-apex with *a=0*, and for *α>β* the minimum distance vector ends at the *B*-apex with *b=0*. By setting the two cases of Equation 10 to zero, the critical value of the positioning angle for either of the two cases is,

$$\gamma_{crit} = \begin{cases} \arccos\left(-\sin\beta/\sin\alpha\right), & for\ \alpha \ge \beta \\ \arccos\left(-\sin\alpha/\sin\beta\right), & for\ \beta \ge \alpha \end{cases} \tag{11}$$

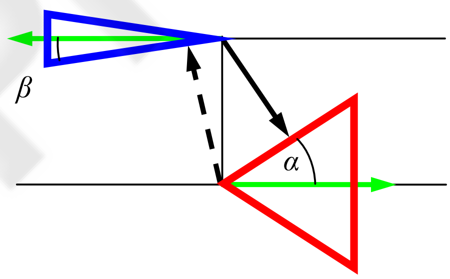

Figure 3: Minimum for *α>β* and *γ=π*.

For values of the positioning angle between critical value γ_{crit} and *π*, the problem reduces to apex-to-surface, and the minimum distance is constant (see Figure 3),

$$\begin{cases} d = p\cos\alpha,\ a = p\tan\alpha,\ b = 0\ \ for\ \ \alpha \ge \beta \\ d = p\cos\beta,\ a = 0,\ b = p\tan\beta\ \ for\ \ \beta \ge \alpha \end{cases} \tag{12}$$

For *α=β* (identical cones) and $\gamma_{crit}=\pi$, the two generators associated with the minima are parallel. Any vector that is parallel to the minimum distance vector between the two apexes (region marked in yellow in Figure 4) is also a solution,

$$d = p\cos\alpha,\ a = b = p\tan\alpha \tag{13}$$

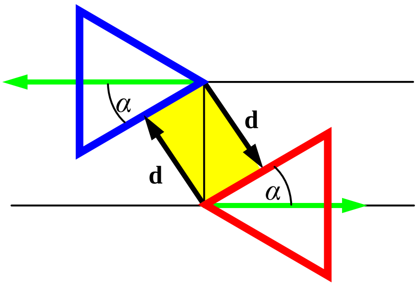

Figure 4: Minimum for $\alpha=\beta$ and $\gamma=\pi$.

2.6.3 Apex Positioning

From Equation 9, any combination of shifts of the cones along their axes that satisfies the equation $f\sin\alpha+g\sin\beta=-d_0$ brings the cones into tangency. In particular, it happens with translation of the *A*-cone by $-d_0/\sin\alpha$ or translation of the *B*-cone by $-d_0/\sin\beta$ along the appropriate axis. Shifts beyond the point of tangency give a negative minimum distance and cause intersection of the cones. These are conditional tangency and intersection (they depend on shift values).

For the special case $\alpha=\beta$ and $\gamma=\pi$ (Equation 13), the minimum reduces to a single line for a relative apexes shift that is equal to $p\cdot\mathbf{tan}\alpha$. For a larger shift, the minimum distance is the distance between the apexes. A relative shift of $-p\cdot\mathbf{cot}\alpha$ brings the cones into tangency along segments of the two generators. A larger shift in that direction causes intersection of the cones.

2.6.4 Apex-to-Apex

In the Apex-to-Apex case, for each of the two cones, define a cone with coinciding apex, axis in the opposite direction, and generator angle of $\pi/2-\alpha$ or $\pi/2-\beta$. These are the complementary cones. When any of the cones is included entirely in the other complementary cone, the minimum distance is the distance between the apexes.

2.6.5 Identical Cones

When the two generator angles are equal, $\alpha=\beta$, the cones are identical. From Equation 8, the minimum distance between the surfaces is then,

$$d_0=\sqrt{1-\sin^2\alpha/\sin^2(\gamma/2)} \tag{14}$$

This solution has three regions: (i) separation for $\gamma>2\alpha$; (ii) tangency of the surfaces at infinity for $\gamma=2\alpha$; (iii) intersection for $\gamma<2\alpha$. Hence, regular identical cones intersect when half the angle between their axes is smaller than the generator angle. Otherwise, there is a regular minimum distance solution for cones with un-shifted apexes.

2.7 Intersecting Axes

2.7.1 Coincident Apexes

Coincident apexes are situated at the intersecting point of the axes. In this case the distance between the cones is zero.

2.7.2 Parallel Generators

Parallel generators occur when the angles satisfy $\alpha+\beta=\gamma$. The minimum distance is then the distance between the two generators. Tangency occurs when the distance is zero, and intersection occurs when it is negative.

2.7.3 Apex-to-Generator

For the case of Apex-to-Generator, the minimum distance is from the apex to the nearest point on the inner generator of the other cone. The appropriate combination (*A*-apex to *B*-cone or *B*-apex to *A*-cone) is determined by the specific geometry.

Tangency occurs when an apex is situated on the inner generator of the other cone. Intersection occurs when an apex is situated between the two generators of the other cone.

3 ANALYSIS AND EXAMPLES

3.1 Verification

Verification of solution and implementation was carried out in part by comparing the explicit solution with the results of numerical minimization based on the distance between two cone generators: (a) Initial guess: the generator nearest to the other cone in the plane defined by the axis and the vector between the axes; (b) Variables of the problem: the rotation angle of the generator around the axis for each of the cones; (c). The three types of regions have known explicit solutions for the distance between two given generators. They are are Ray-to-Ray, Point-to-Ray, and Point-to-Point; (d). The cost function for the minimization is the distance between the two generators.

The algorithm was implemented in Matlab® using a general minimization function without gradient. The process converged to the value of the explicit solution within the required error bound (in most of the region). It did, however, take longer by four orders of magnitudes.

3.2 Comparison to Polytopes

Cones can be approximated by circumscribed regular polytopes (see Figure 5). For each polytope, additional parameters of the problem are the number of facets and the rotational positioning angle ω.

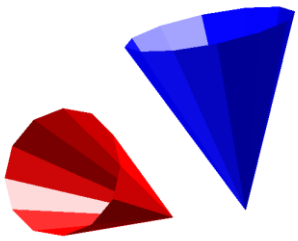

Figure 5: General view of two regular polytopes.

For non-intersecting axes, in the surface-to-surface region the problem reduces to finding the nearest pair of edges (one from each polytope). The result is then compared to the explicit solution for the cones. For simplification, the space metric was scaled by the distance between the axes and the apexes were set on the minimum vector between the axes. The maximum possible distance for surface-to-surface is then unity.

Figure 6 shows the approximation error versus the axes positioning angle for various values of facet numbers (color coded).

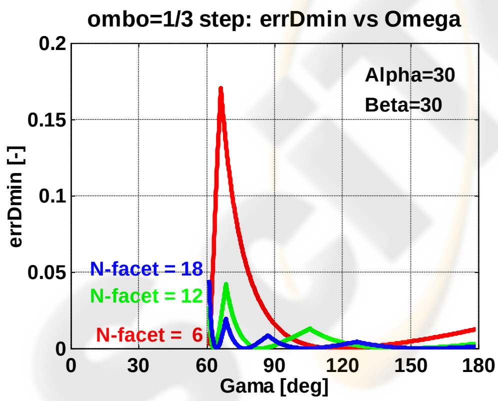

Figure 6: Approximation error vs axes angle.

Figure 7 shows the approximation error versus the rotational positioning angle of the first polytope for various values of facet numbers (color coded). In both Figures 6 and 7, the rotation the second polytope is ⅓ step where the step is 360° divided by the number of facets.

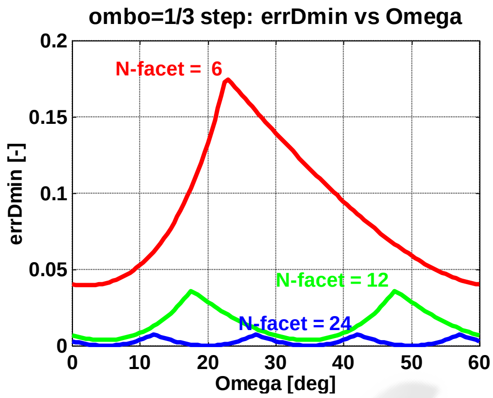

Figure 7: Approximation error vs rotation angle.

3.3 Examples

The explicit expressions were used for several cases. In all cases β was set to 30º. For simplification, the space metric was scaled by the distance between the axes and the apexes were set on the minimum vector between the axes. The maximum possible distance for surface-to-surface is then unity.

Figure 8 shows the minimum distance versus positioning angle for various values of α (color-coded).

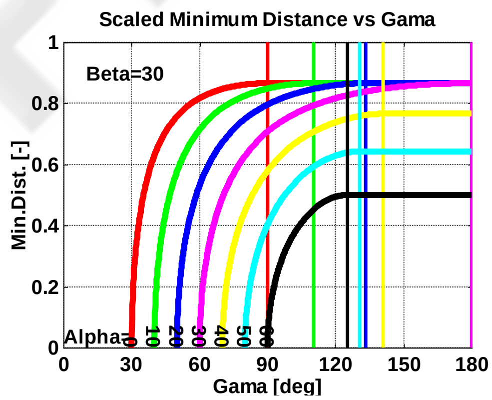

Figure 8: Scaled minimum distance vs axes angle.

Figure 9 shows the distance along the A-axis versus positioning angle for various values of α (color-coded). In both Figures 8 and 9, the transition to a constant value at the critical positioning angle γ_{crit} is marked with vertical lines with matching colors. For α=0, A-cone is a straight line.

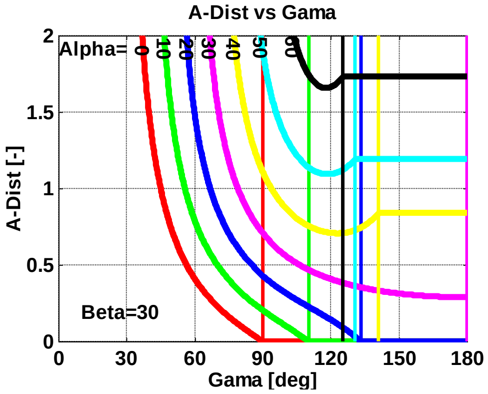

Figure 9: Intersection *A*-distance vs axes angle.

Figure 10 shows the minimum distance error due to a parametric error of 10 in α. From observing the shape and starting point of the minimum distance (see Figure 8), it is obvious there is a region with an indefinite error in the minimum distance (the cones intersect in this region). It should be noted that the solution itself is exact, and it is the parametric error that is propagated into the minimum distance.

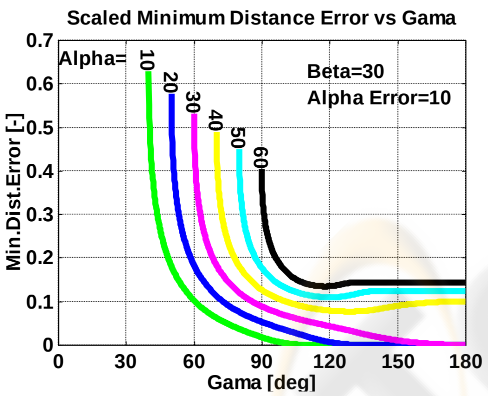

Figure 10: Scaled minimum distance error vs axes angle.

4 CONCLUSIONS

Explicit expressions were derived for the minimum distance between two solid semi-infinite circular cones. The derivation is based on geometric reasoning and vector algebra. Special regions and cases were identified and discussed. A numerical method based on minimizing the distance between two generators was used as part of the verification process. The exact solution was compared to results of approximation by regular polytopes. The explicit solution is robust, independent of coordinate system and invariant under rigid translation and rotation of the setup.

Future work will extend the scope of the problem to include shells of finite cones. Shells require a solution for a cone with generator angle larger than $\pi/2$. Finite cones enlarge the set of solution types to include the bases of the cones (contours and surfaces).

REFERENCES

Gilbert, E. G., Johnson, D.W. and Keerthi, S.S., 1988, "*A Fast Procedure for Computing the Distance Between Complex Objects in Three-Dimensional Space*," IEEE Journal of Robotics and Automation, Vol. 4, No. 2, pp. 193-203, April 1988.

Jovanoski, D., 2008, "*The Gilbert–Johnson–Keerthi (GJK) Algorithm*," Department of Computer Science University of Salzburg, February 2008.

Manchem, S. and Mukund, R., 2009, "*The Gilbert-Johnson-Keerthi Algorithm*," Indian Institute of Technology Guwahati, October 31, 2009.

Chung, K. and Wang, W., 1996, "*Quick Collision Detection of Polytopes in Virtual Environments*," ACM Symposium on Virtual Reality Software and Technology 1996, 1-4, July, 96, University of Hong Kong, Hong Kong.