DESIGN OF ALLPASS FILTERS WITH SPECIFIED DEGREES OF FLATNESS AND EQUIRIPPLE PHASE RESPONSES

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Abstract: This paper proposes a new method for designing allpass filters having the specified degrees of flatness at the specified frequency point(s) and equiripple phase responses in the approximation band(s). First, a system of linear equations are derived from the flatness conditions. Then, the Remez exchange algorithm is used to approximate the equiripple phase responses in the approximation band(s). By incorporating the linear equations from the flatness conditions into the equiripple approximation, the design problem is formulated as a generalized eigenvalue problem. Therefore, we can solve the eigenvalue problem to obtain the filter coefficients, which have the equiripple phase response and satisfy the specified degrees of flatness simultaneously. Furthermore, a class of IIR filters composed of allpass filters are introduced as one of its applications, and it is shown that IIR filters with flat passband (or stopband) and equiripple stopband (or passband) can be designed by using the proposed method. Finally, some examples are presented to demonstrate the effectiveness of the proposed design method.

1 INTRODUCTION

Allpass filters possess constant magnitude response at all frequencies and are a basic scalar lossless building block (Mitra and Kaiser, 1993), (Regalia et al., 1988). Interconnections of allpass filters have found numerous applications in many practical filtering problems such as low-sensitivity filter structures, wavelet filter banks, and so on (Mitra and Kaiser, 1993), (Shenoi, 1999), (Regalia et al., 1988), (Laakso et al., 1996), (Lang, 1998), (Selesnick and Burrus, 1998), (Selesnick, 1999), (Zhang and Iwakura, 1999). In many applications, it is necessary to design an allpass filter both satisfying the specified degrees of flatness at the specified frequency point(s) and having equiripple phase response in the approximation band(s). For example, in the allpass-sum structure (Selesnick, 1999), the phase response of the allpass sub-filter is required to be flat in the band(s) where the corresponding filter has the flat magnitude response, and is equiripple in other band(s) to get the equiripple magnitude response. Many methods have been proposed for the phase design of allpass filters: the maximally flat design (Thiran, 1971), least squares design (Laakso et al., 1996), (Lang, 1998), and equiripple design (Zhang and Iwakura, 1999), (Tseng, 2003).

However, the approximation of allpass filters with both the specified degrees of flatness and equiripple phase responses in the approximation band(s) is still open.

In this paper, we propose a new method for designing allpass filters which have both the specified degrees of flatness at the specified frequency point(s) and equiripple phase responses in the approximation band(s). First, we derive a system of linear equations from the flatness conditions of the phase response at the specified frequency point(s). Then, we apply the Remez exchange algorithm to obtain the equiripple reponse in the approximation band(s). By incorporating the linear equations from the flatness conditions into the equiripple approximation, we formulate the design problem as a generalized eigenvalue problem (Zhang and Iwakura, 1996), (Zhang and Iwakura, 1999). Therefore, we can obtain the filter coefficients by iteratively solving the eigenvalue problem. The resulting allpass filters have the equiripple phase responses and satisfy the specified degrees of flatness simultaneously. Furthermore, as one of the applications of allpass filters, we introduce a class of IIR filters composed of allpass filters (Regalia et al., 1988), (Selesnick, 1999), whose design problem can be reduced to the phase approximation of the allpass sub-filter.

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Thus, we can design the filters with flat passband (or stopband) and equiripple stopband (or passband) by using the proposed method. Finally, some design examples are presented to demonstrate the effectiveness of the proposed design method.

2 ALLPASS FILTERS

It is well-known that the transfer function of an allpass filter $A(z)$ is defined by

$$A(z) = z^{-N} \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n z^{-n}}, \tag{1}$$

where N ($\in Z$) is filter degree, and a_n ($\in R$) are real coefficients and $a_0 = 1$.

It can be seen that $A(z)$ in Eq.(1) has unit magnitude response at all frequencies, and its phase response $\theta(\omega)$ is given by

$$\theta(\omega) = -N\omega + 2\tan^{-1} \frac{\sum_{n=0}^{N} a_n \sin n\omega}{\sum_{n=0}^{N} a_n \cos n\omega}. \tag{2}$$

Let $\theta_d(\omega)$ be the desired phase response. The difference $\theta_e(\omega)$ between $\theta(\omega)$ and $\theta_d(\omega)$ is

$$e^{j\theta_e(\omega)} = e^{j\{\theta(\omega)-\theta_d(\omega)\}} = \frac{\sum_{n=0}^{N} a_n e^{j\{(n-\frac{N}{2})\omega - \frac{\theta_d(\omega)}{2}\}}}{\sum_{n=0}^{N} a_n e^{-j\{(n-\frac{N}{2})\omega - \frac{\theta_d(\omega)}{2}\}}}, \tag{3}$$

and

$$\theta_e(\omega) = 2\tan^{-1} \frac{\sum_{n=0}^{N} a_n \sin\{(n-\frac{N}{2})\omega - \frac{\theta_d(\omega)}{2}\}}{\sum_{n=0}^{N} a_n \cos\{(n-\frac{N}{2})\omega - \frac{\theta_d(\omega)}{2}\}}. \tag{4}$$

Therefore, the design problem of allpass filters is the phase approximation of $\theta(\omega)$ to $\theta_d(\omega)$ in the approximation band(s), that is, the minimization of the phase error $\theta_e(\omega)$ in Eq.(4) in the specified criterion, e.g., in the least squares, and/or Chebyshev (minimax), and/or maximally flat sense. In the following, we discuss the design of allpass filters having equiripple phase responses in the approximation band(s) while satisfying the specified degrees of flatness at the specified frequency point(s).

3 ALLPASS FILTER DESIGN

In this section, we describe the design method of allpass filters with both the specified degrees of flatness and equiripple phase responses in the approximation band(s). Firstly, we consider the flatness condition of the phase response at the frequency point ω_p. It is required that the derivatives of $\theta(\omega)$ in Eq.(2) are equal to that of $\theta_d(\omega)$ at $\omega = \omega_p$, that is,

$$\left.\frac{\partial^r \theta(\omega)}{\partial \omega^r}\right|_{\omega=\omega_p} = \left.\frac{\partial^r \theta_d(\omega)}{\partial \omega^r}\right|_{\omega=\omega_p} (r = 0, 1, \cdots, K-1), \tag{5}$$

where K ($\in Z$) is a parameter that controls the degree of flatness. It is seen that to satisfy the specified degrees of flatness, the flatness conditions in Eq.(5) become

$$\left.\frac{\partial^r \theta_e(\omega)}{\partial \omega^r}\right|_{\omega=\omega_p} = 0 \qquad (r = 0, 1, \cdots, K-1). \tag{6}$$

From Eq.(4), we have

$$\tan \frac{\theta_e(\omega)}{2} = \frac{N(\omega)}{D(\omega)}, \tag{7}$$

where

$$\begin{cases} N(\omega) = \sum_{n=0}^{N} a_n \sin\{(n-\frac{N}{2})\omega - \frac{\theta_d(\omega)}{2}\} \\ D(\omega) = \sum_{n=0}^{N} a_n \cos\{(n-\frac{N}{2})\omega - \frac{\theta_d(\omega)}{2}\} \end{cases}. \tag{8}$$

Therefore, it is proven that the condition in Eq.(6) is equivalent to

$$\left.\frac{\partial^r N(\omega)}{\partial \omega^r}\right|_{\omega=\omega_p} = 0 \qquad (r = 0, 1, \cdots, K-1). \tag{9}$$

By substituting $N(\omega)$ in Eq.(8) into Eq.(9), we can derive a system of linear equations as follows,

$$\sum_{n=0}^{N} \left.\frac{\partial^r \sin\{(n-\frac{N}{2})\omega - \frac{\theta_d(\omega)}{2}\}}{\partial \omega^r}\right|_{\omega=\omega_p} a_n = 0, \tag{10}$$

for $r = 0, 1, \cdots, K-1$. For example, if a linear phase is required, that is, $\theta_d(\omega) = -\tau\omega$, then Eq.(10) is reduced to

$$\begin{cases} \sum_{n=0}^{N}(n-\frac{N-\tau}{2})^r \sin\{(n-\frac{N-\tau}{2})\omega_p\}a_n = 0 & \text{(even } r\text{)} \\ \sum_{n=0}^{N}(n-\frac{N-\tau}{2})^r \cos\{(n-\frac{N-\tau}{2})\omega_p\}a_n = 0 & \text{(odd } r\text{)} \end{cases}. \tag{11}$$

It is known that the phase response $\theta(\omega)$ is an odd function with respect to $\omega = 0$ and π. If $\omega_p = 0$ or π, then the equations with even r are satisfied without any conditions, and thus the number of the conditions reduces about a half, that is, $L = \lfloor \frac{K}{2} \rfloor$, where $\lfloor x \rfloor$ means the largest integer not greater than x. When $\omega_p \neq 0$ and π, then $L = K$.

When the above-mentioned conditions are imposed at several frequency points ω_{pi} $(i = 1,2,\cdots,M)$, the total number of the conditions is $L = \sum_{i=1}^{M} L_i$, where $L_i = \lfloor \frac{K_i}{2} \rfloor$ if $\omega_{pi} = 0$ or π, and $L_i = K_i$ if $\omega_{pi} \neq 0$ and π. Note that K_i is a parameter that controls the degree of flatness at ω_{pi}. Therefore, if $L = N$, we can solve a system of linear equations as shown in Eq.(10) to obtain a set of filter coefficients, which has the maximally flat phase response and satisfies the specified degrees of flatness at the specified frequency point(s) ω_{pi}.

Next, we consider the case of $L < N$. Besides satisfying the flatness conditions in Eq.(5), we want to obtain an equiripple phase response in the approximation band(s) by using the remaining degree of freedom. We apply the Remez exchange algorithm in the approximation band(s). Let ω_i $(i = 0,1,\cdots,N-L)$ are the extremal frequencies in the approximation band(s), we formulate $\theta_e(\omega)$ as

$$\tan\frac{\theta_e(\omega_i)}{2} = \frac{\sum_{n=0}^{N} a_n \sin\{(n-\frac{N}{2})\omega_i - \frac{\theta_d(\omega_i)}{2}\}}{\sum_{n=0}^{N} a_n \cos\{(n-\frac{N}{2})\omega_i - \frac{\theta_d(\omega_i)}{2}\}} = (-1)^i\delta, \tag{12}$$

where δ $(\in R)$ is an error. We incorporate Eq.(10) into Eq.(12), and formulate the design problem as a generalized eigenvalue problem. Then we rewrite Eqs.(10) and (12) in the matrix form as

$$\mathbf{PA} = \delta\mathbf{QA}, \tag{13}$$

where $\mathbf{A} = [a_0, a_1, \cdots, a_N]^T$, and the elements of the matrices $\mathbf{P}$ and $\mathbf{Q}$, for example, when the flatness condition in Eq.(5) is imposed at only one frequency point ω_p $(\neq 0$ and $\pi)$, are given by

$$P_{ij} = \begin{cases} \left.\dfrac{\partial^i \sin\{(j-\frac{N}{2})\omega - \frac{\theta_d(\omega)}{2}\}}{\partial\omega^i}\right|_{\omega=\omega_p} & (i = 0,1,\cdots,L-1) \\ \sin\{(j-\frac{N}{2})\omega_{(i-L)} - \frac{\theta_d(\omega_{(i-L)})}{2}\} & (i = L, L+1,\cdots,N) \end{cases}, \tag{14}$$

$$Q_{ij} = \begin{cases} 0 & (i = 0,1,\cdots,L-1) \\ (-1)^{(i-L)}\cos\{(j-\frac{N}{2})\omega_{(i-L)} - \frac{\theta_d(\omega_{(i-L)})}{2}\} & (i = L, L+1,\cdots,N) \end{cases}. \tag{15}$$

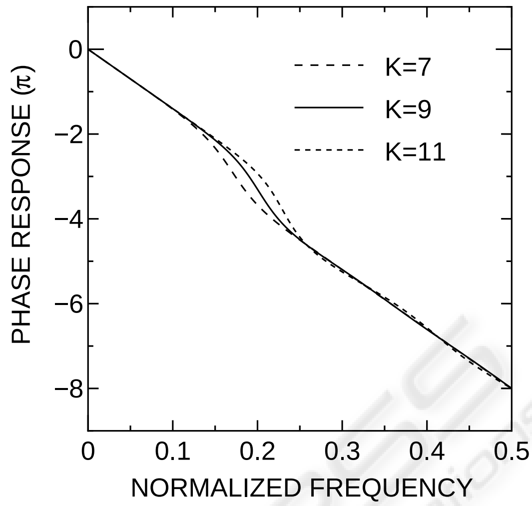

Figure 1: Phase responses of allpass filters.

Once the design specification: the filter degree N, the desired phase response $\theta_d(\omega)$, the degree of flatness K_i, the specified frequency point(s) ω_{pi}, and the extremal frequencies ω_i in the approximation band(s) are given, the elements P_{ij} and Q_{ij} of the matrices $\mathbf{P}$ and $\mathbf{Q}$ can be computed by Eqs.(14) and (15). Therefore, it should be noted that Eq.(13) corresponds to a generalized eigenvalue problem, i.e., δ is an eigenvalue, and $\mathbf{A}$ is a corresponding eigenvector. In order to minimize the error δ, we must find the absolute minimum eigenvalue by solving the eigenvalue problem, so that the corresponding eigenvector gives a set of filter coefficients a_n. To obtain an equiripple phase response, we make use of an iteration procedure so that the optimal filter coefficients is easily obtained. The design algorithm is shown as follows.

4 DESIGN ALGORITHM

Procedure. {Allpass Filter Design Algorithm.}

Begin

1) Read N, $\theta_d(\omega)$, K_i, and ω_{pi}.
2) Select initial extremal frequencies Ω_i ($i = 0, 1, \cdots, N-L$) equally spaced in approximation band(s).

Repeat

3) Set $\omega_i = \Omega_i$ ($i = 0, 1, \cdots, N-L$).
4) Compute **P** and **Q** by using Eqs.(14) and (15), and find the absolute minimum eigenvalue δ to obtain a set of filter coefficients a_n.
5) Search the peak frequencies Ω_i ($i = 0, 1, \cdots, N-L$) of $\theta_e(\omega)$ in approximation band(s).

Until

Satisfy the following condition for a prescribed small constant ε (for example, $\varepsilon = 10^{-8}$):

$$|\omega_i - \Omega_i| < \varepsilon \qquad \text{(for all } i\text{)}$$

End.

5 IIR FILTERS COMPOSED OF ALLPASS FILTERS

Many methods for designing IIR filters have been proposed in (Mitra and Kaiser, 1993), (Regalia et al., 1988), (Zhang and Iwakura, 1996), (Lang, 1998), (Hegde and Shenoi, 1998), (Selesnick and Burrus, 1998), (Selesnick, 1999). These design methods have considered the maximally flat and/or equiripple magnitude responses. It is required in some applications that the magnitude response of the filters is flat in passband(s) and equiripple in stopband(s) (Darlington, 1978), (Vaidyanathan, 1985), (Selesnick and Burrus, 1996), (Hegde and Shenoi, 1998). In this section, we discuss the design of IIR filters with flat passband(s) and equiripple stopband(s), which are composed of two allpass filters.

It is known in (Regalia et al., 1988), (Lang, 1998), (Selesnick and Burrus, 1998) and (Selesnick, 1999) that a parallel interconnection of two allpass filters (allpass-sum) has many advantages, such as low-sensitivity structures, low-complexity structures with low roundoff noise behavior, and so on. The classical digital (Butterworth, Chebyshev, and elliptic) filters can be realized as an allpass-sum structure. In addition, the allpass-sum structure can realize a more general class of transfer functions. Here, we consider this class of IIR filters whose transfer function is given by

$$H(z) = \frac{1}{2}[z^{-J}A_1(z) + A_2(z)], \tag{16}$$

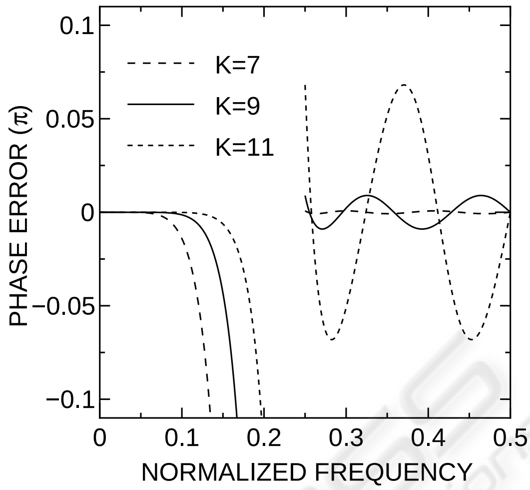

Figure 2: Phase errors of allpass filters.

where $A_1(z), A_2(z)$ are two causal stable allpass filters of degree N_1, N_2, and J ($\in Z$) is a nonnegative integer.

Eq.(16) can be rewritten to

$$H(z) = \frac{1}{2}A_1(z)[z^{-J} + A(z)], \tag{17}$$

where

$$A(z) = \frac{A_2(z)}{A_1(z)}, \tag{18}$$

whose degree is $N = N_1 + N_2$. Note that $A(z)$ needs not be causal stable. The magnitude response of $H(z)$ is given by

$$|H(e^{j\omega})| = |\cos\frac{\theta(\omega) + J\omega}{2}|, \tag{19}$$

where $\theta(\omega)$ is the phase response of $A(z)$. It is clear that the phase difference between $A(z)$ and z^{-J} must be $2n\pi$ in the passband(s) of $H(z)$, and $(2n+1)\pi$ in the stopband(s), where $n \in Z$. Therefore, the desired phase response of $A(z)$ is

$$\theta_d(\omega) = \begin{cases} -J\omega + 2n\pi & \text{(in passband)} \\ -J\omega + (2n+1)\pi & \text{(in stopband)} \end{cases}, \tag{20}$$

then the design problem of $H(z)$ becomes the phase approximation of $A(z)$. The conventional design methods, for example, the maximally flat design (Thiran, 1971), equiripple design (Zhang and Iwakura, 1999), (Tseng, 2003) and so on, can be used in the design. However, these methods cannot design allpass filters with flat and equiripple phase response in passband(s) and stopband(s), respectively. By using the design method proposed in the preceding section, we can obtain easily the flat passband(s) and equiripple stopband(s) of $H(z)$.

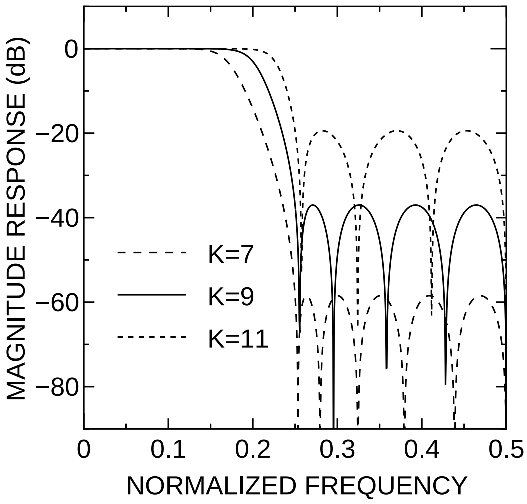

Figure 3: Magnitude responses of IIR lowpass filters.

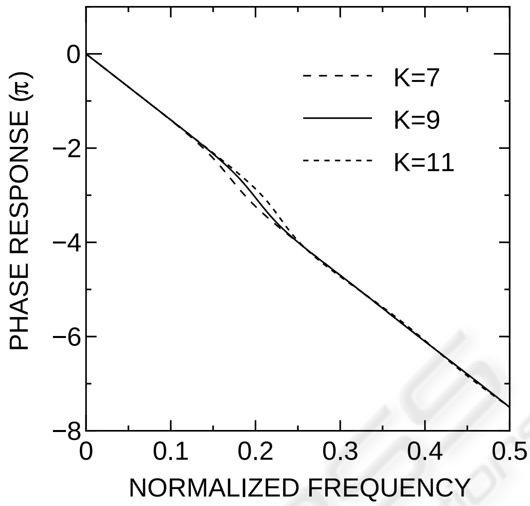

Figure 4: Phase responses of IIR lowpass filters.

For example, if we want $H(z)$ to be a lowpass filter, the desired phase response is given by

$$\theta_d(\omega) = \begin{cases} -J\omega & (0 \le \omega \le \omega_1) \\ -J\omega \pm \pi & (\omega_2 \le \omega \le \pi) \end{cases}, \qquad (21)$$

where ω_1 and ω_2 are the cutoff frequencies of the passband and stopband, respectively. Note that in this case, the filter degrees N_1 and N_2 must satisfy $N_2 - N_1 = J \mp 1$. If we set $N_1 = 0$ and $N = N_2 = J \mp 1$, then the filter will have an approximately linear phase response also (Laakso et al., 1996), (Zhang and Iwakura, 1999).

We use the proposed method to design the allpass filter $A(z)$, whose phase response $\theta(\omega)$ satisfies Eq.(5) at $\omega_p = 0$. Note that K should be an odd number, because $\theta(\omega)$ is an odd function with respect to $\omega = 0$. Thus, the resulting lowpass filter $H(z)$ has a flat magnitude response at $\omega_p = 0$, and the degree of flatness is $2K$.

6 DESIGN EXAMPLES

In this section, we present some examples to demonstrate the effectiveness of the proposed design method.

First, we consider the design of allpass filter of degree $N = 8$ with the desired phase response $\theta_d(\omega) = -7\omega$ in $[0, 0.3\pi]$ and $\theta_d(\omega) = -7\omega - \pi$ in $[0.5\pi, \pi]$. The degree of flatness is required to be $K = 9$ at $\omega_p = 0$, then $L = 4$. Since the remaining degree of freedom is $N - L = 4$, we have selected initial extremal frequencies $0.5\pi = \omega_0 < \omega_1 < \cdots < \omega_4 < \pi$ equally spaced in $[0.5\pi, \pi]$, and obtained the optimal filter coefficients a_n by using the design algorithm described in the section IV. The resulting phase response and phase error are shown in the solid line in Fig.1 and Fig.2, respectively. It is clear in Fig.2 that the phase response is flat at $\omega = 0$ and equiripple in $[0.5\pi, \pi]$. In Fig.1 and Fig.2, the phase responses of two allpass filters with $K = 7$ and $K = 11$ are shown also. It is seen that the degree of flatness K can be arbitrarily specified. It is found that these allpass filters are causal stable since all poles are within the unit circle (Zhang and Iwakura, 1999).

Next, we use the obtained allpass filters to construct IIR lowpass filters: $H(z) = \frac{1}{2}[z^{-7} + A(z)]$. The magnitude and phase responses of the IIR filters are shown in Fig.3 and Fig.4, respectively. It is seen in Fig.3 and Fig.4 that these lowpass filters have the flat passband and equiripple stopband responses, while the phase responses are approximately linear.

7 CONCLUSIONS

In this paper, we have proposed a new method for designing allpass filters which have both the specified degrees of flatness and equiripple phase responses in the approximation band(s). Firstly, a system of linear equations have been derived from the flatness conditions of the phase responses, then the Remez exchange algorithm is used to get the equiripple responses in the approximation band(s). The design problem has been formulated as a generalized eigenvalue problem by incorporating the flatness conditions into the equiripple approximation, thus, a set of filter coefficients can be easily obtained by solving the eigenvalue problem. Furthermore, as one application of allpass filters, a class of IIR filters composed of two allpass filters has been discussed also. Finally, some examples have been presented to demonstrate the effectiveness of the proposed design method.

REFERENCES

Darlington, S. (Dec. 1978). Filters with chebyshev stopbands, flat passbands, and impulse responses of finite duration. In *IEEE Trans. Circuits and Systems*. Vol.CAS-25, No.12, pp.966–975.

Hegde, R. and Shenoi, B. A. (Nov. 1998). Magnitude approximation of digital filters with specified degrees of flatness and constant group delay characteristics. In *IEEE Trans. Circuits and Systems II*. Vol.45, No.11, pp.1476–1486.

Laakso, T. I., Valimaki, V., Karjalainen, M., and Laine, U. K. (Jan. 1996). Splitting the unit delay: Tools for fractional delay filter design. In *IEEE Signal Processing Mag.* Vol.13, No.1, pp.30–60.

Lang, M. (Sep. 1998). Allpass filter design and applications. In *IEEE Trans. Signal Processing*. Vol.46, No.9, pp.2505-2514.

Mitra, S. K. and Kaiser, J. F. (1993). *Handbook for Digital Signal Processing*. Wiley, New York.

Regalia, P. A., Mitra, S. K., and Vaidyanathan, P. P. (Jan. 1988). The digital allpass filter: A versatile signal processing building block. In *Proc. IEEE*. Vol.76, No.1, pp.19–37.

Selesnick, I. W. (Jan. 1999). Lowpass filters realizable as allpass sums: design via a new flat delay filter. In *IEEE Trans. Circuits and Systems II*. Vol.46, No.1, pp.40–50.

Selesnick, I. W. and Burrus, C. S. (June 1998). Generalized digital butterworth filter design. In *IEEE Trans. Signal Processing*. Vol.46, No.6, pp.1688–1694.

Selesnick, I. W. and Burrus, C. S. (Sep. 1996). Exchange algorithms for the design of linear phase fir filters and differentiators having flat monotonic passbands and equiripple stopbands. In *IEEE Trans. Circuits and Systems II*. Vol.43, No.9, pp.671–675.

Shenoi, B. A. (1999). *Magnitude and Delay Approximation of 1-D and 2-D Digital Filters*. Springer-Verlag, Berlin, Germany.

Thiran, J. P. (Nov. 1971). Recursive digital filters with maximally flat group delay. In *IEEE Trans. Circuit Theory*. Vol.CT-18, No.6, pp.659–664.

Tseng, C. C. (Sep. 2003). Design of iir digital allpass filters using least *p*th phase error criterion. In *IEEE Trans. Circuits and Systems II*. Vol.50, No.9, pp.653–656.

Vaidyanathan, P. P. (Sep. 1985). Optimal design of linear phase fir digital filters with very flat passbands and equiripple stopbands. In *IEEE Trans. Circuits and Systems*. VVol.CAS-32, No.9, pp.904–917.

Zhang, X. and Iwakura, H. (Feb. 1999). Design of iir digital allpass filters based on eigenvalue problem. In *IEEE Trans. Signal Processing*. Vol.47, No.2, pp.554-559.

Zhang, X. and Iwakura, H. (June 1996). Design of iir digital filters based on eigenvalue problem. In *IEEE Trans. Signal Processing*. Vol.44, No.6, pp.1325–1333.