SIMPLE DERIVATION OF A STATE OBSERVER OF LINEAR TIME-VARYING DISCRETE SYSTEMS

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Abstract: In this paper, a simple calculation method to derive the Luenberger observer for linear time-varying discrete systems is presented. For this purpose, the simple design method of the pole placement for linear time-varying discrete systems is proposed. It is shown that the pole placement controller can be derived simply by finding some particular "output signal" such that the relative degree from the input to this new output is equal to the order of the system. Using this fact, the feedback gain vector can be calculated directly from plant parameters without transforming the system into any standard form.
Then, this method is applied to the design of the observer, i.e., because of the duality of linear time-varying discrete system, the state observer can be derived by simple calculations.

1 INTRODUCTION

The design of the state observer for linear time-varying discrete systems is well established. As for the continuous case, the condition for a system to be a state observer is very simple. However, different from the time-invariant case, calculation procedure to obtain the observer gain is not straightforward. This paper gives a simple calculation method to design the state observer for linear time-varying discrete systems.

Since the design of the observer is based on the pole placement technique, simplified calculation method to derive the pole placement feedback gain vector for linear time-varying discrete systems is considered first. We define the pole placement of linear time-varying discrete systems as follows. The problem is to find a time-varying state feedback gain for linear discrete time-varying discrete system, so that the closed loop system is equivalent to the time-invariant system with desired poles.

Usually, the pole placement design procedure needs the change of variable to the Flobenius standard form, and hence, is very complicated. To simplify this procedure, it will be shown that the pole placement controller can be derived simply by finding some particular "output signals" such that the relative degree from the input to this output is equal to the order of the system [4]. Using this fact, the feedback gain vector can be calculated directly from plant parameters without transforming the system into any standard form.

Because of the duality of the linear discrete time-varying system, the simplified pole placement technique can be applied to the design of the state observer for linear discrete time-varying discrete systems.

In the sequel, the simple pole placement technique is proposed in Section 2, and then, this method is used to the observer design problem in Section 3.

2 POLE PLACEMENT OF LINEAR DISCRETE TIME-VARYING SYSTEMS

Consider the following linear time-varying discrete system with a single input.

$$x(k+1) = A(k)x(k) + b(k)u(k) \qquad (1)$$

Here, $x \in R^n$ and $u \in R^1$ are the state variable and the input signal respectively. $A(k) \in R^{n\times n}$ and $b(k) \in R^n$ are time-varying parameter matrices. The problem is to find the state feedback

$$u = h^T(k)x(k) \qquad (2)$$

which makes the closed loop system equivalent to the time invariant linear system with arbitrarily stable poles.

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Definition 1. The system (1) is called completely reachable in step n from the origin, if for any $x_1 \in R^n$, there exists a finite input $u(m)$ $(m = k, \cdots, k+n-1)$ such that $x(k) = 0$ and $x(k+n) = x_1$.

Lemma 1. The system (1) is completely reachable in step n from the origin, if and only if

$$\begin{aligned} & rank\ [\ b(k+n-1), \Phi(k+n,k+n-1)b(k+n-2), \\ & \quad \cdots, \Phi(k+n,k+1)b(k)\ ] \\ & \qquad = rank\ U_R(k) = n, \qquad \forall k \end{aligned} \tag{3}$$

where $\Phi(i,j)$ is the transition matrix from $k = j$ to $k = i$, i.e.,

$$\Phi(i,j) = A(i-1)A(i-2)\cdots A(j) \quad i > j \tag{4}$$

$\nabla\nabla$

Now, consider the problem of finding a new output signal $y(k)$ such that the relative degree from $u(k)$ to $y(k)$ is n. Here, $y(k)$ has the following form.

$$y(k) = c^T(k)x(k) \tag{5}$$

Then, the problem is to find a vector $c(k) \in R^n$ that satisfies this condition.

Lemma 2. The relative degree from u to y defined by (5) is n, if and only if

$$\begin{aligned} & c^T(k+1)b(k) = 0 \\ & c^T(k+2)\Phi(k+2,k+1)b(k) = 0 \\ & \quad \vdots \\ & c^T(k+n-1)\Phi(k+n-1,k+1)b(k) = 0 \\ & c^T(k+n)\Phi(k+n,k+1)b(k) = 1 \end{aligned} \tag{6}$$

(Here, $c^T(k+n)\Phi(k+n,k+1)b(k) = 1$ without loss of generality.) $\nabla\nabla$

Proof : This is obvious by checking $y(k+1)$, $\cdots$, $y(k+n)$.

If the system (1) is completely reachable in step n, there exists a vector $c(k)$ such that the relative degree from $u(k)$ to $y(k) = c^T(k)x(k)$ is n. And, from (6), such a vector, $c(k)$, is obtained by

$$\begin{aligned} c^T(k) &= [0, \cdots, 0,\ 1]\ [\ b(k-1),\ \Phi(k,k-1)b(k-2), \\ & \qquad \cdots, \Phi(k,k+1-n)b(k-n)\ ]^{-1} \\ &= [0,\ 0,\ \cdots,\ 1]U_R^{-1}(k-n) \end{aligned} \tag{7}$$

The next step is to derive the state feedback for the arbitrary pole placement.

The new output, $y(k) = c^T(k)x(k)$, with $c(k)$ obtained by (7), satisfies the following equations.

$$\begin{aligned} y(k) &= c^T(k)x(k) \\ y(k+1) &= c^T(k+1)\Phi(k+1,k)x(k) \\ &\ \vdots \\ y(k+n-1) &= c^T(k+n-1)\Phi(k+n-2,k)x(k) \\ y(k+n) &= c^T(k+n)\Phi(k+n-1,k)x(k) + u(k) \end{aligned} \tag{8}$$

Let $q(z)$ be a desired stable polynomial of z-operator, i.e.,

$$q(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_0 \tag{9}$$

By multiplying $y(k+i)$ by α_i $(i = 0, \cdots, n-1)$ and then summing them up, the following equation is obtained from (8).

$$q(p)y(k) = d^T(k)x(k) + u(k) \tag{10}$$

where $d(k) \in R^n$ is defined by the following.

$$d^T(k) = [\ \alpha_0, \alpha_1, \cdots, \alpha_{n-1}, 1\ ] \times \begin{bmatrix} c^T(k) \\ c^T(k+1)\Phi(k+1,k) \\ \vdots \\ c^T(k+n)\Phi(k+n,k) \end{bmatrix} \tag{11}$$

Hence, the state feedback,

$$u = -d^T(k)x(k) + r(k) \tag{12}$$

makes the closed loop system as follows.

$$q(z)y(k) = r(k) \tag{13}$$

where $r(k)$ is an external input signal.

This control system can be summarized as follows. The given system is

$$x(k+1) = A(k)x(k) + b(k)u(k) \tag{14}$$

and, using (4), (9), and (11) the state feedback for the pole placement is given by

$$u(k) = -d^T(k)x(k). \tag{15}$$

Then, the closed loop system becomes

$$x(k+1) = (A(k) - b(k)d^T(k))x(k). \tag{16}$$

Let $T(k)$ be the time varying matrix defined by

$$T(k) = \begin{bmatrix} c^T(k) \\ c^T(k+1)\Phi(k+1,k) \\ \vdots \\ c^T(k+n-1)\Phi(k+n-1,k) \end{bmatrix} \tag{17}$$

and define the new state variable $w(k)$ by the following equations.

$$x(k) = T(k)w(k), \qquad w = \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+n-1) \end{bmatrix} \tag{18}$$

Using the above, (16) is transformed into

$$\begin{aligned} & w(k+1) \\ &= T^{-1}(k+1)(A(k)-b(k)d^T(k))T(k)w(k) \\ &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 1 \\ -\alpha_0 & \cdots & \cdots & -\alpha_{n-1} \end{bmatrix} w(k) \\ &= A^* w(k) \end{aligned} \tag{19}$$

This implies that the closed loop system is equivalent to the time invariant linear system which has the desired closed loop poles ($\det(zI - A^*) = q(z)$).

Theorem 2. If the system (1) is completely reachable in step n, then, the matrix for the change of variable, $T(k)$, given by (17) is nonsingular for all k. $\nabla\nabla$

Example 1.
Consider the following unstable system.

$$x(k+1) = A(k)x(k) + b(k)u(k) \tag{20}$$

where

$$\begin{aligned} A(k) &= \begin{bmatrix} 1 & 2+cos0.1k \\ 2+sin0.2k & 2 \end{bmatrix} \\ b(k) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{21}$$

From (7), $c^T(k)$ is obtained as follows.

$$\begin{aligned} c^T(k) &= [\,0,\ 1\,][\,b(k-1),\ A(k-1)b(k-2)\,]^{-1} \\ &= \begin{bmatrix} \frac{1}{2+cos0.1(k-1)} & 0 \end{bmatrix} \end{aligned} \tag{22}$$

The purpose is to design the state feedback so that the closed loop system is equivalent to the linear time invariant system with $\lambda_1 = 0.4$ and $\lambda_2 = 0.5$ as its closed loop poles. This implies that the desired closed loop characteristic polynomial is

$$q(z) = z^2 + 0.9z + 0.2.$$

From (11),

$$\begin{aligned} d^T(k) &= [\,0.2,\ 0.9,\ 1\,] \\ & \quad \times \begin{bmatrix} c^T(k) \\ c^T(k+1)A(k) \\ c^T(k+2)A(k+1)A(k) \end{bmatrix} \\ &= \begin{bmatrix} d_1(k) & d_2(k) \end{bmatrix} \end{aligned} \tag{23}$$

In the above, $d_1(k)$ and $d_2(k)$ are given by

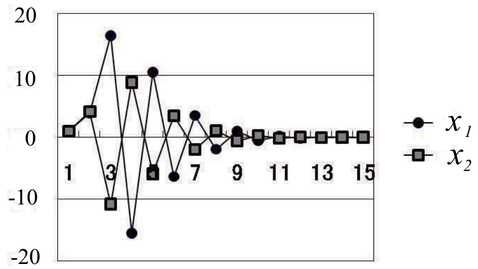

Figure 1: Responce of the state variable (x) of the system.

$$\begin{aligned} d_1(k) &= \frac{0.2}{\gamma(k-1)} + \frac{0.9}{\gamma(k)} + \frac{1}{\gamma(k+1)} + 2 + sin0.2k \\ d_2(k) &= 0.9 + \frac{\gamma(k)}{\gamma(k+1)} + 2 \end{aligned}$$

where

$$\gamma(k) = 2 + cos0.1k$$

Fig.1 shows the simulation results.

3 STATE OBSERVER

In this section, we consider the design of the observer for the following linear time-varying system.

$$\begin{aligned} x(k+1) &= A(k)x(k) + b(k)u(k) \\ y(k) &= g^T(k)x(k) \end{aligned} \tag{24}$$

Here, $y(k) \in R$ is the output signal of this system. The problem is to design the full order state observer for (24). Consider the following system as a candidate of the observer.

$$\begin{aligned} \hat{x}(k+1) &= F(k)\hat{x}(k) + b(k)u(k) + h(k)y(k) \\ &= F(k)\hat{x}(k) + b(k)u(k) + h(k)g^T(k)x(k) \end{aligned} \tag{25}$$

where $F(k) \in R^{n\times n}$, and $h(k) \in R^n$. Define the state error $e(k) \in R^n$ by

$$e = x(k) - \hat{x}(k) \tag{26}$$

Then, $e(k)$ satisfies the following error equation.

$$\begin{aligned} e(k+1) &= F(k)e(k) + (A(k) - F(k) \\ & \quad -h(k)g^T(k))x(k) \end{aligned} \tag{27}$$

Hence, (25) is a state observer of (24) if $F(k)$ and $h(k)$ satisfy the following condition.

$$\begin{aligned} F(k) &= A(k) - h(k)g^T(k) \\ F(k) &: \text{ arbitrarily stable matrix} \end{aligned} \tag{28}$$

Then, the problem is to find $h(k)$ such that F(k) is equevalent to a constant matrix F^* with arbitrarily stable poles. Consider the pole placement control problem of the following system.

$$w(k+1) = A^T(-k)w(k) + g(-k)v(k) \tag{29}$$

where $w(k) \in R^n$ and $v(k) \in R^1$ are the state variable and an input signal.

Let $\Psi(i,j)$ be the state transient matrix of the system (29). Then, we have the following relation.

$$\Phi^T(i,j) = \Psi(-j,-i) \tag{30}$$

Definition 2. The system (24) is called completely obsermable in step n, if from $y(k), y(k+1), \cdots, y(k+n-1)$, the state, $x(k)$, can be determined uniquely for any k.

Lemma 3. The system (24) is completely observable in step n, if and only if

$$rank \begin{bmatrix} g^T(k) \\ g^T(k+1)\Phi(k+1,k) \\ \vdots \\ g^T(k+n-1)\Phi(k+n-1,k) \end{bmatrix} = rank\, U_o(k) = n, \quad \forall k \tag{31}$$

From the property of the duality of the time varying discrete system, if the pair $(A(k), g^T(k))$ is completely observable in step n, the pair $(A^T(-k), g(-k))$ is completely reachable in step n. Then, if the pair $(A(k), g^T(k))$ is completely observable in step n, the system (29) has a state feedback

$$v(k) = h^T(-k)w(k) \tag{32}$$

such that the closed loop system is equevalent to the linear time invariant system with arbitrarily stable poles.

This implies that for some state transformation matrix, $P(-k) \in R^n$,

$$P^{-1}((k+1))(A^T(-k) - g(-k)h^T(-k))P(k) = F^{*T} \tag{33}$$

where, F^{*T} is a constant matrix with arbitrarily stable poles. From this and the duality, we have the following equation.

$$P^{-1}(-k)(A(k) - h(k)g^T(k))P(-(k+1)) = F^* \tag{34}$$

Hence, using this $h(k)$, the state observer for the system (24) is obtained.

Example 2.
Consider the following system.

$$\begin{aligned} x(k+1) &= A(k)x(k) + b(k)u(k) \\ y(k) &= g^T(k)x(k) \end{aligned} \tag{35}$$

where

$$\begin{aligned} A(k) &= \begin{bmatrix} 0 & 1 \\ -0.7 & -(1.2+0.5cos0.4k) \end{bmatrix} \\ b(k) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g^T(k) = [\,2,\ 1\,] \end{aligned} \tag{36}$$

The dual system matrices are as follows.

$$\begin{aligned} A^T(-k) &= \begin{bmatrix} 0 & -0.7 \\ 1 & -(1.2+0.5cos0.4(-k)) \end{bmatrix} \\ g(-k) &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned} \tag{37}$$

From (7), $c^T(-k)$ for the new output matrix is obtained as

$$\begin{aligned} c^T(-k) &= [\,0,\ 1\,][\,g(-(k-1)), \\ &\qquad A^T(-(k-1))g(-(k-2))\,]^{-1} \\ &= \frac{1}{\gamma(-(k-1))}\begin{bmatrix} -1 & 2 \end{bmatrix} \end{aligned} \tag{38}$$

where,

$$\begin{aligned} \gamma(-k) &= 4.7 - 2\lambda(k) \\ \lambda(k) &= 1.2 + 0.5cos0.4k. \end{aligned} \tag{39}$$

The purpose is to design the state feedback so that the closed loop system is equivalent to the linear time invariant system with $\lambda_1 = 0.3$ and $\lambda_2 = 0.4$ as its closed loop poles. This implies that the desired closed loop characteristic polynomial is

$$q(z) = z^2 + 0.7z + 0.12.$$

From (11), $d^T(-k)$ is calculated as follows.

$$\begin{aligned} d^T(-k) &= [\,0.12,\ 0.7,\ 1\,] \\ &\quad \times \begin{bmatrix} c^T(-k) \\ c^T(-(k+1))A(-k) \\ c^T(-(k+2))A(-(k+1))A(-k) \end{bmatrix} \\ &= \begin{bmatrix} d_1(-k) & d_2(-k) \end{bmatrix} \end{aligned} \tag{40}$$

Here, $d_1(k)$ and $d_2(k)$ are

$$\begin{aligned} d_1(k) &= -\frac{0.12}{\gamma(k-1)} + \frac{0.7}{\gamma(k)} \\ &\quad + \frac{1}{\gamma(k+1)}(0.2 - \lambda(k+1)) \\ d_2(k) &= \frac{0.12}{\gamma(k-1)} + \frac{0.7}{\gamma(k)}(0.2 - \lambda(k)) \\ &\quad + \frac{1}{\gamma(k+1)}\{-0.2 - (0.2 + \lambda(k+1))\lambda(k)\} \end{aligned}$$

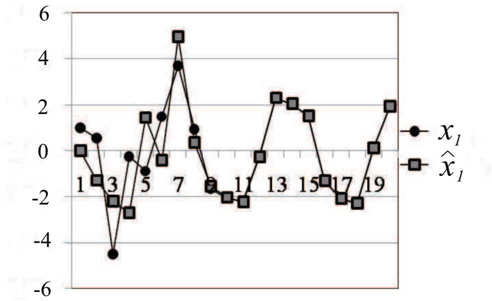

Figure 2: Responce of $x_1(k)$ and $\hat{x}_1(k)$.

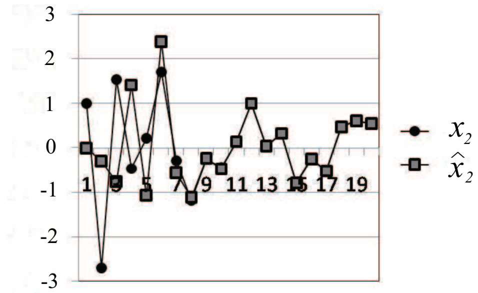

Figure 3: Responce of $x_2(k)$ and $\hat{x}_2(k)$.

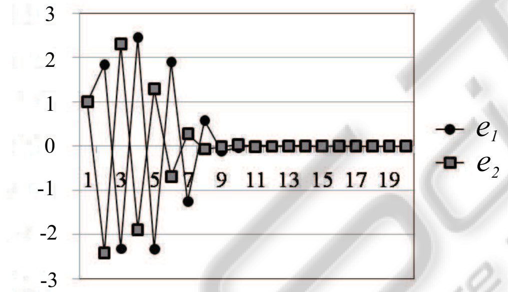

Figure 4: Responce of the estimation error ($e_1(k) = x_1(k) - \hat{x}_1(k)$, $e_2(k) = x_2(k) - \hat{x}_2(k)$).

Hence, the observer gain vector, $h(k)$, is obtained as

$$h(k) = -d(k) \tag{41}$$

and, using this $h(k)$, the observer is

$$\hat{x}(k+1) = \{A(k) - h(k)g^T(k)\}\hat{x}(k) + b(k)u(k) + h(k)y(k) \tag{42}$$

Fig.2 $\sim$ 4 show the simulation results with $u(k) = 2cos(0.9k)$. The initial condition of the plant is $x_1(1) = x_2(1) = 1$.

4 CONCLUSIONS

In this paper, a simple design method for the state observer for linear time-varying discrete systems is proposed. We first proposed the simple derivation method of the pole placement state feedback gain for liner time-varying discrete system. Feedback gain can be calculated directly from the plant parameters without the transformation of the system into any standard form, which makes the design procedure very simple. This technique is applied to the observer design procedure using the duality of the linear time-varying system. The author appreciates the helpful comments of the anonymous reviewers.

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