STRUCTURE-PRESERVING ALGORITHMS FOR DISCRETE-TIME ALGEBRAIC MATRIX RICCATI EQUATIONS

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Abstract: Structure-preserving algorithms for solving discrete-time algebraic matrix Riccati equations are presented. The proposed techniques extract the stable deflating subspaces for extended, inverse-free symplectic matrix pencils. The algorithms are based on skew-Hamiltonian/Hamiltonian pencils derived by an extended Cayley transformation, which only involves matrix additions and subtractions. The structure-preserving approach has the potential to avoid the numerical difficulties which are encountered for a traditional, non-structured solution, returned by the currently available software tools.

1 INTRODUCTION

Consider the *continuous-time algebraic Riccati equation (CARE)*,

$$0 = Q + A^H XE + E^H XA - E^H XWXE$$

and the *discrete-time algebraic Riccati equation (DARE)*,

$$E^H XE = Q + A^H XA - A^H XB(R + B^H XB)^{-1} B^H XA,$$

where $A, E, W, Q \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $R \in \mathbb{C}^{m\times m}$, $W = W^H$, $Q = Q^H$, $R = R^H$, E is nonsingular, and $W := BR^{-1}B^H$. More general equations are obtained by replacing $E^H XB$ and $A^H XB$ above by $L + E^H XB$ and $L + A^H XB$, respectively, where $L \in \mathbb{C}^{n\times m}$. The real case is obtained by replacing $\mathbb{C}$ by $\mathbb{R}$, and the conjugate-transpose operator H by the transpose operator T.

In applications, usually the *stabilizing solution* X_* is required, hence, e.g., for DARE, $\lambda E - (A - B(R + B^H X_* B)^{-1} B^H X_* A)$ is a (Schur) stable matrix pencil, i.e., $\Lambda(A - B(R + B^H X_* B)^{-1} B^H X_* A, E) \subset \mathbb{C}^- := \{z \in \mathbb{C} : |z| < 1\}$, where $\Lambda(M)$ denotes the spectrum of a matrix or pencil M.

CAREs and DAREs arise in many applications, such as, stabilization and linear-quadratic regulator problems, Kalman filtering, linear-quadratic Gaussian (H_2-) optimal control problems, computation of (sub)optimal H_∞ controllers, model reduction techniques based on stochastic, positive or bounded real LQG balancing, factorization procedures for transfer functions (here, usually $E \neq I_n$).

There are several basic approaches for solving algebraic Riccati equations (AREs):

1. Treat an ARE as a nonlinear system of equations using *Newton's method (with line search)*.
2. Use the connection to *Hamiltonian eigenproblem*.

The second approach for CARE, with $E = I_n$, is based on the identity

$$\begin{bmatrix} A & -W \\ -Q & -A^H \end{bmatrix} \begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix} (A - WX),$$

hence, if X is stabilizing, then $\Lambda(A - WX) = \Lambda(H) \cap \mathbb{C}^-$, where H is the first matrix in the above formula, and $\mathbb{C}^- := \{z \in \mathbb{C} : \Re(z) < 0\}$. Consequently, the columns of $[I_n\ X^T]^T$ span the *stable* invariant subspace of the Hamiltonian matrix H. Therefore, it is possible to compute the stable H-invariant subspace via eigendecomposition or block-Schur factorization,

$$U^{-1}HU = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix},$$

and the solution is given by $X = U_{21}U_{11}^{-1}$.

If R is ill-conditioned, it is advisable to use extended matrix pencils, for better accuracy (Bender and Laub, 1987a; Bender and Laub, 1987b; Lancaster and Rodman, 1995; Mehrmann, 1991; Van Dooren, 1981):
– *extended pencil for CARE*:

$$N - \lambda M = \begin{bmatrix} A & 0 & B \\ Q & A^H & L \\ L^H & B^H & R \end{bmatrix} - \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & -E^H & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

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– *extended pencil for DARE*:

$$N-\lambda M=\begin{bmatrix} A & 0 & B \\ Q & -E^H & L \\ L^H & 0 & R \end{bmatrix} - \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & -A^H & 0 \\ 0 & -B^H & 0 \end{bmatrix}.$$

If $\begin{bmatrix} U_1^T & U_2^T & U_3^T \end{bmatrix}^T$ spans the stable deflating subspace of $N-\lambda M$, then $X_* = U_2(EU_1)^{-1}$. The feedback gain matrix for the linear-quadratic optimal regulator can be computed directly via $G = U_3U_1^{-1}$.

If R is nonsingular, $E = I_n$, and $L = 0$, the above pencils can be reduced to $2n \times 2n$ Hamiltonian and symplectic pencils, respectively, by removing the subpencils with infinite eigenvalues (Paige and Van Loan, 1981; Pappas et al., 1980; Mehrmann, 1991). A pencil $N-\lambda M$ is *Hamiltonian* if $N\mathcal{J}M^H = -M\mathcal{J}N^H$, and it is *symplectic* if $N\mathcal{J}N^H = M\mathcal{J}M^H$, where

$$\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

The general pencils inherit most of the spectral properties of the corresponding reduced Hamiltonian or symplectic pencils.

The pencils above have much structure, which should be exploited in order to improve the numerical properties of the Riccati solvers. The approach we follow is to transform the discrete-time problem to an equivalent continuous-time problem, and use the newly developed skew-Hamiltonian/Hamiltonian eigensolvers for the latter problem, suitably extended.

2 EQUIVALENCE OF PENCILS IN CONTINUOUS-TIME AND DISCRETE-TIME PROBLEMS

A block column permutation (and sign change) gives, equivalently:

– extended pencil for CARE:

$$\lambda \begin{bmatrix} 0 & E & 0 \\ -E^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^H & Q & L \\ B^H & L^H & R \end{bmatrix};$$

– extended pencil for DARE:

$$\lambda \begin{bmatrix} 0 & E & 0 \\ -A^H & 0 & 0 \\ -B^H & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ -E^H & Q & L \\ 0 & L^H & R \end{bmatrix}.$$

These pencils are special cases of the following *block structured C-type* and *D-type pencils* (Xu, 2006):

$$\lambda\mathcal{E}_C - \mathcal{A}_C = \lambda \begin{bmatrix} 0 & \widetilde{F} \\ -\widetilde{F}^H & 0 \end{bmatrix} - \begin{bmatrix} 0 & \widetilde{G} \\ \widetilde{G}^H & \widetilde{D} \end{bmatrix}, \quad (1)$$

and

$$\lambda\mathcal{E}_D - \mathcal{A}_D = \lambda \begin{bmatrix} 0 & F \\ -G^H & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^H & D \end{bmatrix}, \quad (2)$$

respectively, where $F, G, \widetilde{F}, \widetilde{G} \in \mathbb{C}^{n,q}$, $q = n+m$, and $D, \widetilde{D} \in \mathbb{C}^{q,q}$ are *Hermitian*, i.e., $D = D^H$, $\widetilde{D} = \widetilde{D}^H$.

These pencils have important *spectral properties*:
C-type: symmetry about $\Re(z) = 0$, i.e., pairs $(\lambda, -\bar{\lambda})$;
D-type: symmetry about $|z| = 1$, i.e., pairs $(\lambda, \bar{\lambda}^{-1})$.

An equivalence transformation between the C-type and D-type pencils can be established starting from the *Cayley transformation*, $\mathbf{c} : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$, defined by

$$\mu = \mathbf{c}(\lambda) = (\lambda-1)(\lambda+1)^{-1};\ \mathbf{c}(-1) = \infty,\ \mathbf{c}(\infty) = 1.$$

Specifically, the *generalized Cayley transformation* for matrix pairs is given by

$$(\mathcal{F}, \mathcal{B}) = \mathbf{c}(\mathcal{E}, \mathcal{A}) = (\mathcal{A}+\mathcal{E}, \mathcal{A}-\mathcal{E}). \quad (3)$$

Let

$$(\widetilde{\mathcal{E}}, \widetilde{\mathcal{A}}) := \mathbf{c}(\mathcal{E}_D, \mathcal{A}_D).$$

Then, an eigenvalue pair $(\lambda, \bar{\lambda}^{-1})$ of $\lambda\mathcal{E}_D - \mathcal{A}_D$ is transformed to an eigenvalue pair $(\mu, -\bar{\mu})$ of $\lambda\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$, with $\mu = \mathbf{c}(\lambda)$, $-\bar{\mu} = \mathbf{c}(\bar{\lambda}^{-1})$.

Unfortunately, $\lambda\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ has not the same block structure as $\lambda\mathcal{E}_C - \mathcal{A}_C$, and it cannot be put into the continuous-time setting. This inconvenience can be removed using the Cayley transformation followed by a *drop/add transformation* (Xu, 2006):

$$(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}(\mathcal{E}_D, \mathcal{A}_D),$$

where $\mathbf{t}(\cdot) = \mathbf{d}(\mathbf{c}(\cdot))$, and $\mathbf{d}$ corresponds to dropping/adding D in the $\mathcal{E}$ part.

The Cayley plus drop/add transformation is suggestively represented by the following $\mathbf{t}$ *transformation diagram*:

$$\lambda\mathcal{E}_D - \mathcal{A}_D = \lambda \begin{bmatrix} 0 & F \\ -G^H & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^H & D \end{bmatrix}$$

$$\mathbf{c} \downarrow\uparrow \mathbf{c}^{-1}$$

$$\lambda\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}} = \lambda \begin{bmatrix} 0 & \widetilde{F} \\ -\widetilde{F}^H & D \end{bmatrix} - \begin{bmatrix} 0 & \widetilde{G} \\ \widetilde{G}^H & D \end{bmatrix}$$

$$\text{drop } D \text{ from } \widetilde{\mathcal{E}} \downarrow\uparrow \text{add } D \text{ to } \widetilde{\mathcal{E}}$$

$$\lambda\mathcal{E}_C - \mathcal{A}_C = \lambda \begin{bmatrix} 0 & \widetilde{F} \\ -\widetilde{F}^H & 0 \end{bmatrix} - \begin{bmatrix} 0 & \widetilde{G} \\ \widetilde{G}^H & \widetilde{D} \end{bmatrix}$$

where $\widetilde{F} := G+F$, $\widetilde{G} := G-F$, $\widetilde{D} = D$. It is worth mentioning that the $\mathbf{t}$ transformation involves matrix additions and subtractions only.

Only regular pencils are considered in the sequel. A pencil $\lambda\mathcal{E} - \mathcal{A}$ is *regular* if $\mathcal{E}$ and $\mathcal{A}$ are square and $\det(\gamma\mathcal{E} - \mathcal{A}) \neq 0$ for some $\gamma \in \mathbb{C}$. A *necessary regularity condition* is: if the C-type and D-type pencils of order $n+q$ are regular, then

$$q - \operatorname{rank}\widehat{D} \leq n \leq q,$$

where $\widehat{D} = \widetilde{D}$ and $\widehat{D} = D$, for C-type and D-type pencils, respectively.

The relation between the eigen-structure of $\lambda\mathcal{E} - \mathcal{A}$ and $\lambda\mathcal{F} - \mathcal{B}$, $(\mathcal{F},\mathcal{B}) = \mathbf{t}(\mathcal{E},\mathcal{A})$, can be summarized as follows (see, e.g., (Mehrmann, 1991; Xu, 2006)):

(i) $\lambda\mathcal{E} - \mathcal{A}$ is regular if and only if (iff) $\lambda\mathcal{F} - \mathcal{B}$ is regular.

(ii) $\lambda \in \Lambda(\mathcal{E},\mathcal{A})$ iff $\mu = \mathbf{c}(\lambda) \in \Lambda(\mathcal{F},\mathcal{B})$, and λ and μ have the same geometric, partial, and algebraic multiplicities.

(iii) If $\lambda\mathcal{E} - \mathcal{A}$ is regular, then, $\mathcal{R}_\lambda = \mathcal{R}_\mu$, $\mathcal{L}_\lambda = \mathcal{L}_\mu$, $\mu = \mathbf{c}(\lambda)$, where $\mathcal{R}_x$ and $\mathcal{L}_x$ are the *right* and *left deflating subspaces* corresponding to eigenvalue(s) x.

The C-type pencil (1) is *skew-Hermitian/Hermitian*, i.e., $\mathcal{E}_C^H = -\mathcal{E}_C, \mathcal{A}_C^H = \mathcal{A}_C$, and it has the following main eigen-structure properties:

(i) $\lambda \in \Lambda(\mathcal{E}_C,\mathcal{A}_C)$ iff $-\bar{\lambda} \in \Lambda(\mathcal{E}_C,\mathcal{A}_C)$, and λ and $-\bar{\lambda}$ have the same geometric, partial, and algebraic multiplicities.

(ii) $\mathcal{R}_\lambda = \mathcal{L}_{-\bar{\lambda}}$ and $\mathcal{L}_\lambda = \mathcal{R}_{-\bar{\lambda}}$.

(iii) U is a basis matrix of a right deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ corresponding to $\lambda S - T$ iff U is a basis matrix of a left deflating subspace corresponding to $\lambda(-S^H) - T^H$.

The eigenvalue pairing $(\lambda, -\bar{\lambda})$ **does not hold** for λ with $\Re(\lambda) = 0$, since then $\lambda = -\bar{\lambda}$. But for such an eigenvalue, $\mathcal{R}_\lambda = \mathcal{L}_\lambda$. This also holds for $\lambda = \infty$.

The regular D-type pencil (2) has the following main eigen-structure properties (Mehrmann, 1991; Xu, 2006):

(i) Nonzero finite eigenvalues come in pairs $(\lambda, \bar{\lambda}^{-1})$, and λ, $\bar{\lambda}^{-1}$ have the same geometric, partial, and algebraic multiplicities.

(ii) span $U = \mathcal{R}_\lambda$, span $V = \mathcal{L}_\lambda$ iff span $\widehat{V} = \mathcal{R}_{\bar{\lambda}^{-1}}$, span $\widehat{U} = \mathcal{L}_{\bar{\lambda}^{-1}}$, where span X denotes the subspace spanned by the columns of the matrix X, and

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{C}^{n+q,\ell},$$

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad S^H V^H \mathcal{E}_D = V^H \mathcal{A}_D\,, \quad (4)$$

for $T, S \in \mathbb{C}^{\ell,\ell}$ with $\Lambda(T) = \Lambda(S^H) = \{\lambda\}$, $\lambda \neq 0$ (with algebraic multiplicity ℓ), and

$$\widehat{U} = \begin{bmatrix} U_1 T \\ U_2 \end{bmatrix}, \quad \widehat{V} = \begin{bmatrix} V_1 S \\ V_2 \end{bmatrix},$$

$$\mathcal{E}_D \widehat{V} S^{-1} = \mathcal{A}_D \widehat{V}, \quad T^{-H} \widehat{U}^H \mathcal{E}_D = \widehat{U}^H \mathcal{A}_D\,. \quad (5)$$

Moreover, $\det V^H \mathcal{E}_D U \neq 0$ iff $\det \widehat{U}^H \mathcal{E}_D \widehat{V} \neq 0$.

(iii) $U = \begin{bmatrix} U_1^T & U_2^T \end{bmatrix}^T$ is a basis matrix of a right deflating subspace (left deflating subspace) of $\lambda\mathcal{E}_D - \mathcal{A}_D$ corresponding to $T \in \mathbb{C}^{p,p}$ nonsingular, iff $\widehat{U} = \begin{bmatrix} (U_1 T)^T & U_2^T \end{bmatrix}^T$ is a basis matrix of a left deflating subspace (right deflating subspace) of $\lambda\mathcal{E}_D - \mathcal{A}_D$ corresponding to T^{-H}.

(iv) If $0 \in \Lambda(\mathcal{E}_D,\mathcal{A}_D)$ with algebraic multiplicity ℓ_0, then $\infty \in \Lambda(\mathcal{E}_D,\mathcal{A}_D)$ with algebraic multiplicity greater than or equal to ℓ_0.

(v) The formulas for the relations between the basis matrices of right/left deflating subspace for $\lambda = 0$ or ∞ are more complicated than for the case $\lambda \neq 0, \infty$. The sizes of the submatrices depend on the algebraic multiplicities of $0 \in \Lambda(G^H, F^H)$ and $0 \in \Lambda(F, G)$.

The eigenvalue pairing $(\lambda, \bar{\lambda}^{-1})$ **does not hold** for $|\lambda| = 1$, since then $\lambda = \bar{\lambda}^{-1}$. But for such an eigenvalue, U in (4) is a basis matrix of $\mathcal{R}_\lambda$ iff $\widehat{U}$ in (5) is a basis matrix of $\mathcal{L}_\lambda$.

Eigenvalues 0 and ∞ are **paired in a weak sense**, since the algebraic multiplicity of ∞ may be greater than or equal to the algebraic multiplicity of 0, and $\mathcal{R}_0$ and $\mathcal{L}_0$ are only related to certain subspaces of $\mathcal{L}_\infty$ and $\mathcal{R}_\infty$, respectively.

The *equivalence relation between D-type and C-type pencils* is shown below.

Assume $(\mathcal{E}_C,\mathcal{A}_C) = \mathbf{t}(\mathcal{E}_D,\mathcal{A}_D)$ and that $\lambda\mathcal{E}_D - \mathcal{A}_D$ (or $\lambda\mathcal{E}_C - \mathcal{A}_C$) is regular. Then,

(i) $\lambda \in \Lambda(\mathcal{E}_D,\mathcal{A}_D)$, $\lambda \neq -1, \infty$, iff $\mu = \mathbf{c}(\lambda) \in \Lambda(\mathcal{E}_C,\mathcal{A}_C)$, $\mu \neq \infty, 1$, and λ and μ have the same geometric, partial, and algebraic multiplicities.

(ii) span $U = \mathcal{R}_\lambda^D$, span $V = \mathcal{L}_\lambda^D$ iff span $\widetilde{U} = \mathcal{R}_\mu^C$, span $\widetilde{V} = \mathcal{L}_\mu^C$, where the superscript C or D refers to (1) or (2), respectively, U and V satisfy (4) for $T, S \in \mathbb{C}^{\ell,\ell}$ with $\Lambda(T) = \Lambda(S^H) = \{\lambda\}$ (with algebraic multiplicity ℓ), and

$$\widetilde{U} = \begin{bmatrix} U_1(I+T) \\ 2U_2 \end{bmatrix}, \quad \widetilde{V} = \begin{bmatrix} V_1(I+S) \\ 2V_2 \end{bmatrix},$$

$$\mathcal{E}_C \widetilde{U} \widetilde{T} = \mathcal{A}_C \widetilde{U}, \quad \widetilde{S}^H \widetilde{V}^H \mathcal{E}_C = \widetilde{V}^H \mathcal{A}_C\,,$$

where $\widetilde{T} = \mathbf{c}(T)$, $\widetilde{S} = \mathbf{c}(S)$, $\Lambda(\widetilde{T}) = \Lambda(\widetilde{S}^H) = \{\mu\}$, $\mu = \mathbf{c}(\lambda)$. Moreover, $\det V^H \mathcal{E}_D U \neq 0$ iff $\det \widetilde{V}^H \mathcal{E}_C \widetilde{U} \neq 0$.

(iii) If $-1 \in \Lambda(\mathcal{E}_D,\mathcal{A}_D)$, with algebraic multiplicity ℓ_{-1}, then $\infty \in \Lambda(\mathcal{E}_C,\mathcal{A}_C)$, with algebraic multiplicity greater than or equal to ℓ_{-1}. Suppose also $-1 \in \Lambda(G^H, F^H)$, with algebraic multiplicity r_1. Let

$$U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \in \mathbb{C}^{n+q,\ell_{-1}}, \quad U_{11} \in \mathbb{C}^{n,r_1},$$

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad \operatorname{rank} E_D U = \ell_{-1},$$

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \in \mathbb{C}^{\ell_{-1},\ell_{-1}}, \quad T_{11} \in \mathbb{C}^{r_1,r_1}\,,$$

with $\Lambda(T) = \{-1\}$. If U is a basis matrix of $\mathcal{R}_{-1}^D$, then the columns of

$$\widetilde{U} = \begin{bmatrix} 2U_{11} & U_{12}(T_{22}+I) \\ 0 & 2U_{22} \end{bmatrix}$$

span an ℓ_{-1}-dimensional (right and left) deflating subspace of $\lambda\mathcal{E}_C - \mathcal{A}_C$ corresponding to eigenvalue ∞.

(iv) Let ℓ_{-1}, ℓ_0, and ℓ_∞ be the algebraic multiplicities of the eigenvalues $-1, 0, \infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and $\widetilde{\ell}_1$, $\widetilde{\ell}_\infty$ the algebraic multiplicities of the eigenvalues $1, \infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. Then, $\ell_0 = \widetilde{\ell}_1$ and

$$\widetilde{\ell}_\infty = \ell_\infty - \ell_0 + \ell_{-1}, \qquad \ell_\infty = \widetilde{\ell}_\infty - \ell_{-1} + \widetilde{\ell}_1.$$

Specifically, with **t**, $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ comes from $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ (with multiplicity ℓ_{-1}) and $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ (with multiplicity $\ell_\infty - \ell_0$).

If $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ (i.e., $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$), then $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$, and it comes from $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. Only part of $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ is transformed into 1, to match $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$.

3 DEFLATING SUBSPACES FOR SKEW-HAMILTONIAN/ HAMILTONIAN PENCILS

The *structure-preserving algorithms and software* are more advanced for CAREs, based on deflating subspaces for skew-Hamiltonian/Hamiltonian pencils. Extensions of the *HAPACK approach* are currently under development. In the sequel, the pencils $\lambda M - N$ will be represented in the numerically better form $\alpha M - \beta N$, with $\lambda = \alpha/\beta$ (possibly ∞).

Since the structured algorithms for skew-Hamiltonian/Hamiltonian pencils work on problems with even size, a basic idea is to embed the matrix pencil, adding $k \geq 0$ fictitious controls, so that $m + k$ is even. The solution of the optimal control problem corresponding to CARE, hence to

$$\alpha\mathcal{E}_c - \beta\mathcal{A}_c = \alpha \begin{bmatrix} E & 0 & 0 \\ 0 & -E^H & 0 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A & 0 & B \\ Q & A^H & L \\ L^H & B^H & R \end{bmatrix},$$

is unchanged for k new controls, with $\widetilde{B} = 0_{n\times k}$, $\widetilde{R} = I_k$, and D replaced by block-diag$(D, \widetilde{R})$, with

$$D := \begin{bmatrix} Q & L \\ L^H & R \end{bmatrix}.$$

Partition, with $\ell = (m+k)/2$, $B_i \in \mathbb{C}^{n\times\ell}$, $L_i \in \mathbb{C}^{n\times\ell}$, $R_{ij} \in \mathbb{C}^{\ell\times\ell}$, $i, j = 1, 2$,

$$\begin{aligned} \begin{bmatrix} B & \widetilde{B} \end{bmatrix} &= \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \\ \begin{bmatrix} Q & L & 0 \\ L^H & R & 0 \\ 0 & 0 & \widetilde{R} \end{bmatrix} &= \begin{bmatrix} Q & L_1 & L_2 \\ L_1^H & R_{11} & R_{12} \\ L_2^H & R_{21} & R_{22} \end{bmatrix}. \end{aligned}$$

Reordering the variables and equations, the following *skew-Hamiltonian/Hamiltonian pencil* is obtained

$$\alpha\mathcal{E}_c^e - \beta\mathcal{A}_c^e = \alpha \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E^H & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A & B_1 & 0 & B_2 \\ L_2^H & R_{12}^H & B_2^H & R_{22} \\ -Q & -L_1 & -A^H & -L_2 \\ -L_1^H & -R_{11} & -B_1^H & -R_{12} \end{bmatrix}. \quad (6)$$

Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be skew-Hamiltonian/Hamiltonian, i.e., $(\mathcal{S}\mathcal{J})^H = -\mathcal{S}\mathcal{J}$, $(\mathcal{H}\mathcal{J})^H = \mathcal{H}\mathcal{J}$. For some cases, including in linear-quadratic optimization applications, $\mathcal{S}$ is given in a factored form, the so-called *skew-Hamiltonian Cholesky factorization*, defined by $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. For instance, in (6) with $\mathcal{S} = \mathcal{E}_c^e$,

$$\mathcal{Z} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_\ell & 0 & 0 \\ 0 & 0 & E^H & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A skew-Hamiltonian matrix having such a factorization is said to be *$\mathcal{J}$-semidefinite*.

Some properties of skew-Hamiltonian/Hamiltonian pencils are summarized below (Benner et al., 2002):

(i) Real skew-Hamiltonian matrices are $\mathcal{J}$-semidefinite.

(ii) The structure of skew-Hamiltonian/Hamiltonian matrix pencils is preserved under *$\mathcal{J}$-congruence*:

$$\alpha\mathcal{S} - \beta\mathcal{H} \rightarrow \mathcal{J}\mathcal{Y}^H\mathcal{J}^T(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{Y},$$

for $\mathcal{Y}$ nonsingular.

(iii) A skew-Hamiltonian matrix $\mathcal{S}$ of order $2n$ is $\mathcal{J}$-semidefinite ($\mathcal{J}$-definite) iff $\imath\mathcal{J}\mathcal{S}$ has at most (exactly) n positive and at most (exactly) n negative eigenvalues, where $\imath := (-1)^{1/2}$.

(iv) If $\mathcal{S}$ is skew-Hamiltonian ($\mathcal{H}$ is Hamiltonian) and there is $\mathcal{Y}$ nonsingular, such that

$$\begin{aligned} \mathcal{J}\mathcal{Y}^H\mathcal{J}^T\mathcal{S}\mathcal{Y} &= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^H \end{bmatrix} \\ (\mathcal{J}\mathcal{Y}^H\mathcal{J}^T\mathcal{H}\mathcal{Y} &= \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^H \end{bmatrix}) \end{aligned}$$

with $S_{11}, S_{12}\,(H_{11}, H_{12}) \in \mathbb{C}^{n\times n}$, then $\mathcal{S}$ $(\imath\mathcal{H})$ is $\mathcal{J}$-semidefinite.

(v) Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be regular skew-Hamiltonian/ Hamiltonian with ν pairwise distinct, nonzero finite eigenvalues $\imath\alpha_i$, of algebraic multiplicity p_i, and associated right deflating subspace $\mathcal{Q}_i$, $i = 1:\nu$; let p_∞, $\mathcal{Q}_\infty$, be defined similarly for eigenvalue ∞. The following statements are equivalent:

(a) There exists a nonsingular matrix $\mathcal{Y}$, such that

$$\mathcal{J}\mathcal{Y}^H\mathcal{J}^T(\alpha\mathcal{S}-\beta\mathcal{H})\mathcal{Y} = \alpha\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^H \end{bmatrix} - \beta\begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^H \end{bmatrix}, \quad (7)$$

where S_{11} and H_{11} are upper triangular, S_{12} skew-Hermitian, and H_{12} Hermitian.
(b) There exists a unitary matrix $\mathcal{Q}$, such that (7) holds for $\mathcal{Y}$ replaced by $\mathcal{Q}$.
(c) $\mathcal{Q}_k^H\mathcal{J}\mathcal{S}\mathcal{Q}_k$ is congruent to a $p_k \times p_k$ copy of $\mathcal{J}$, $k = 1,2,\ldots,\nu$; $\mathcal{Q}_\infty^H\mathcal{J}\mathcal{H}\mathcal{Q}_\infty$ is congruent to a $p_\infty \times p_\infty$ copy of $\imath\mathcal{J}$.

(vi) *Factored version:* Let $\alpha\mathcal{S}-\beta\mathcal{H}$ be a skew-Hamiltonian/Hamiltonian pencil with nonsingular $\mathcal{J}$-semidefinite skew-Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. If any of the equivalent statements above holds, then there is a unitary matrix $\mathcal{Q}$ and a unitary symplectic matrix $\mathcal{U}$, such that

$$\mathcal{U}^H\mathcal{Z}\mathcal{Q} = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$

$$\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{H}\mathcal{Q} = \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^H \end{bmatrix},$$

where Z_{11}, Z_{22}^T, and H_{11} are $n \times n$ upper triangular.

(vii) If $\imath\mathcal{H}$ is also nonsingular $\mathcal{J}$-semidefinite, i.e., $\imath\mathcal{H} = \mathcal{J}\mathcal{W}^H\mathcal{J}^T\mathcal{W}$, then

$$\mathcal{U}^H\mathcal{Z}\mathcal{Q} = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$

$$\mathcal{U}^H\mathcal{W}\mathcal{Q} = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix},$$

where Z_{11}, Z_{22}^T, W_{11}, and W_{22}^T are $n \times n$ upper triangular.

(viii) *Factored version, real skew-Hamiltonian/skew-Hamiltonian case:* Let $\alpha\mathcal{S}-\beta\mathcal{N}$ be a real regular skew-Hamiltonian/skew-Hamiltonian pencil with $\mathcal{S} = \mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{Z}$. Then, there is an orthogonal matrix $\mathcal{Q}$ and an orthogonal symplectic matrix $\mathcal{U}$, such that

$$\mathcal{U}^T\mathcal{Z}\mathcal{Q} = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{N}\mathcal{Q} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix},$$

where Z_{11}, Z_{22}^T are upper triangular, N_{11} is upper quasi-triangular, and $N_{12} = -N_{12}^T$. Moreover,

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\alpha\mathcal{S}-\beta\mathcal{N})\mathcal{Q} = \alpha\begin{bmatrix} Z_{22}^TZ_{11} & Z_{22}^TZ_{12}-Z_{12}^TZ_{22} \\ 0 & Z_{11}^TZ_{22} \end{bmatrix} - \beta\begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix}$$

is a $\mathcal{J}$-congruent skew-Hamiltonian/skew-Hamiltonian matrix pencil.

Comments:

1. The result (viii) above is used to compute the *structured Schur form* of order $4n$ for a **complex** skew-Hamiltonian/Hamiltonian pencil.
2. The periodic QZ algorithm is used.
3. Algorithms for eigenvalue reordering and deflating subspace computation are available.

Below is a summary about the related software:

- Fortran and MATLAB software for eigenvalues and deflating subspaces are under development.
- Both real and complex cases are considered.
- Factored or unfactored versions are covered.
- Auxiliary routines for problems (of even order) with (quasi-)triangular structure are included.
- Optimized kernels for problems of order 2, 3, or 4, called by the general solvers, are available.

To compute or reorder the eigenvalues, the computations begin with an initial reduction, called *generalized symplectic URV decomposition*, whose *factored version* is defined as follows (Benner et al., 2007):
Given a real skew-Hamiltonian/Hamiltonian $2n \times 2n$ pencil $\alpha\mathcal{T}\mathcal{Z}-\beta\mathcal{H}$, orthogonal matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ and orthogonal symplectic matrices $\mathcal{U}_1$, $\mathcal{U}_2$ are determined, such that

$$\mathcal{Q}_1^T\mathcal{T}\mathcal{U}_1 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

$$\mathcal{U}_2^T\mathcal{Z}\mathcal{Q}_2 = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$

$$\mathcal{Q}_1^T\mathcal{H}\mathcal{Q}_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

where T_{11}, T_{22}^T, Z_{11}, Z_{22}^T, and H_{11} are upper triangular, and H_{22}^T is upper quasi-triangular. The matrices $\mathcal{U}_1$ and $\mathcal{U}_2$ are stored compactly (the first n rows only), since, for $i = 1,2$,

$$\mathcal{U}_i = \begin{bmatrix} U_{i1} & U_{i2} \\ -U_{i2} & U_{i1} \end{bmatrix}.$$

4 NUMERICAL RESULTS

This section presents some preliminary numerical results. These results have been obtained on a portable Intel Dual Core computer at 2 GHz, with 2 GB RAM, and relative machine precision $\varepsilon \approx 1.11 \times 10^{-16}$, using Windows XP (Service Pack 2) operating system, Intel Visual Fortran 11.1 compiler, and MATLAB 7.8.0.347 (R2009a). The matrices

$$\mathcal{S} = \begin{bmatrix} A & B \\ C & A^H \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} D & E \\ F & -D^H \end{bmatrix},$$

where A, B, C, D, E, $F \in \mathbb{C}^{n\times n}$, have been initially generated with MATLAB commands of the form

```
list
A = 10*rand(n)-5 + (10*rand(n)-5)*1i;
```

where `rand` is the uniform (0,1) random generator, and `1i` is the MATLAB notation for the purely imaginary unit, $\imath$. Then, the B, C, E, and F matrices have been transformed using the formulas

$$B := B - B^H, \quad B := B/2; \quad C := C - C^H,$$
$$E := E + E^H, \quad E := E/2; \quad F := F + F^H,$$

to become skew-Hermitian, and Hermitian, respectively. Therefore, the pencil $\lambda\mathcal{S} - \mathcal{H}$ is skew-Hamiltonian/Hamiltonian.

The order n took the values $n = 100, 200, \ldots, 800$. For each order $n \leq 500$, 10 problems have been solved, and the means of the results are reported. For larger n values, one problem has been solved for each n. The generalized eigenvalues computed by a structure-preserving algorithm have been compared with those delivered by the standard QZ algorithm, optimally implemented in the MATLAB function `eig`.

Fig. 1 presents the ratios of the mean CPU times, in seconds, i.e., the speed-up factor of the structured algorithm, in comparison with the standard algorithm.

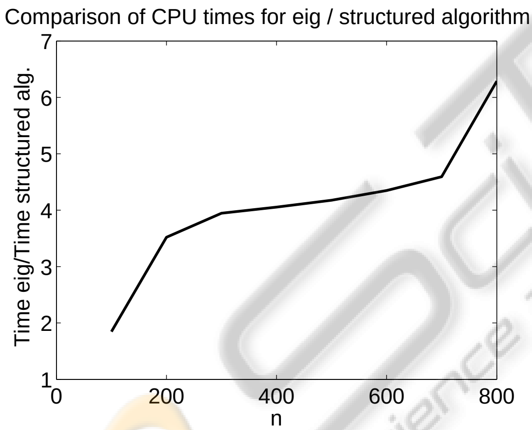

Figure 1: Ratios between the CPU times needed by the MATLAB function `eig` and the structure-preserving algorithm for randomly generated complex skew-Hamiltonian/Hamiltonian pencils of order 2n.

The deviation from symmetry of the eigenvalues computed by `eig` has also been computed as the difference between the vector of eigenvalues $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{2n}]^T$ and a permutation of the elements of the vector $-\lambda$, chosen so that the elements with the same indices in the two vectors be as close as possible. The largest norm has been $4 \cdot 10^{-10}$, and the smallest norm has been $1.90 \cdot 10^{-12}$. The norms should theoretically be 0.

5 CONCLUSIONS

Main issues related to the structure-preserving algorithms for solving discrete-time algebraic matrix Riccati equations are summarized. Stable deflating subspaces for extended, inverse-free symplectic matrix pencils, are computed. Algorithms based on skew-Hamiltonian/Hamiltonian pencils derived by an extended Cayley transformation, which only involves matrix additions and subtractions, are considered. The preliminary results are encouraging.

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