EFFICIENT RECONSTRUCTION OF UNIFORM SAMPLES FROM BUNCHED NONUNIFORM SAMPLES

V. Harish, K. M. M. Prabhu
Department of Electrical Engineering, Indian Institute of Technology Madras, 600036 Chennai, India

Piet Sommen
Signal Processing Systems Group, Department of Electrical Engineering
Technical University Eindhoven, 5600 MB, Eindhoven, The Netherlands

Keywords: Nyquist sampling rate, Bunched sampling, Recurrent nonuniform sampling, Uniform discrete Fourier transform (DFT) filter bank.

Abstract: In this paper, we derive a mathematically equivalent frequency-domain relation between uniform and bunched nonuniform samples. This relation aids in the reconstruction of uniform samples, obtained from nonuniform samples, using a uniform discrete Fourier transform (DFT) modulated filter bank. We consider a general case of unequal spacing between the bunches of nonuniform samples. Simulation results demonstrate the practical utility of the theory proposed.

1 INTRODUCTION

Shannon sampling theorem states that a signal bandlimited to the frequencies $[-f_0, f_0]$, can be reconstructed perfectly from its samples taken uniformly at no less than the Nyquist rate $2f_0$ (Oppenheim et al., 1999). This theorem also states that there will be distortion due to aliasing if the above condition is not satisfied. In practice, there are situations in which the reconstruction of the signal is required from nonuniform samples, say, due to channel erasures. However, it has potential applications, which include data compression (Singh and Rajpal, 2007), speech coding, and error correcting codes (Marvasti, 2001). In (Ouderaa and Renneboog, 1988), an exact nonuniform sampling scheme is proposed based on Cauchy's residue theorem, while a method for nonuniform sampling based on amplitude of signals is proposed in (Wang et al., 2004). There are various nonuniform sampling techniques outlined in the literature (Jerri, 1977), (Marvasti, 2001), one of which is the periodic or recurrent nonuniform sampling (Papoulis, 1977). Recurrent nonuniform sampling finds an important application in time interleaved analog-to-digital converters (TI-ADCs) (Black and Hodges, 1980). The time skews within the TI-ADCs produce recurrent nonuniform samples. A digital signal processing approach is discussed in (Sommen and Janse, 2008), which relates uniform samples and recurrent nonuniform samples using a uniform discrete Fourier transform (DFT) modulated filter bank. In case of known time skews, the reconstruction of uniform samples is proposed in (Johansson and Löwenborg, 2006) using a synthesis system composed of fractional delay filters. However, in order to avoid the re-designing of fractional delay filters, a slight over sampling of bandlimited signal is considered in (Johansson and Löwenborg, 2006).

A near-perfect method of reconstructing uniform samples from bunched samples has been proposed in (Prendergast et al., 2004). In particular, the bunches of uniform samples in (Prendergast et al., 2004) are considered as equally spaced , which can be viewed as a special case of recurrent nonuniform samples. The proposed reconstruction technique in (Prendergast et al., 2004) uses least squares method. In this paper, we propose a linear relation between uniform samples and bunched nonuniform samples. But, unlike in (Prendergast et al., 2004), we consider a general case with unequal spacing between the bunches. The linear relation proposed in this paper aids in perfect reconstruction of uniform samples from bunched nonuniform samples.

One of the situations where bunched nonuniform sampling occurs is in the context of lithographic machines. A lithographic machine is a robotic machine

Harish V., M. M. Prabhu K. and Sommen P..
EFFICIENT RECONSTRUCTION OF UNIFORM SAMPLES FROM BUNCHED NONUNIFORM SAMPLES.
DOI: 10.5220/0003537003500356
In *Proceedings of the 8th International Conference on Informatics in Control, Automation and Robotics* (ICINCO-2011), pages 350-356
ISBN: 978-989-8425-74-4

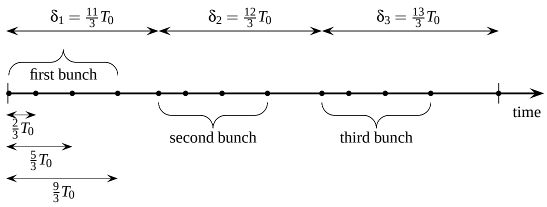

Figure 1: Timing diagram of bunched nonuniform samples.

which apply patterns on wafers at high speeds. During the measurement on wafers, nonuniform exposure parts of the movement profile produces bunched nonuniform sampling patterns. This happens due to missing data between the bunched samples. From these bunched nonuniform samples, we require to generate uniform samples. This paper is organized as follows : In Section 2, we define bunched nonuniform sampling scheme and present discrete time models for the same. In Section 3, a frequency domain relation between uniform and bunched nonuniform samples is described. This section also deals with the reconstruction of uniform samples from nonuniform samples. Section 4 presents the simulation results which demonstrates the theory proposed and Section 5 concludes the paper.

Notations Convention. In this paper, we follow the notation conventions similar to those used in (Sommen and Janse, 2008). Lower case letters for time domain, upper case letters for frequency domain representation of signals. Matrices, vectors are denoted by boldface, underlined boldface letters, respectively; $[\cdot]^T$ denotes transpose, while diag $\{\cdot\}$ represents diagonal matrix.

2 BUNCHED NONUNIFORM SAMPLING (BNU)

We consider recurrent frames of nonuniform samples, where the samples in each frame are grouped into bunches. Precisely, bunches within a frame contain equal number of samples, but these bunches are not equally spaced. However, the number of samples within a frame satisfies the Nyquist rate. Figure 1 depicts a typical bunched nonuniform sampling paradigm. In Fig. 1, we observe that one recurrent frame of duration $12T_0$ $[s]$ contains three bunches. Each bunch contains four nonuniform samples. In each bunch, the last three samples are separated by $\frac{2}{3}T_0,\ \frac{5}{3}T_0$, and $\frac{9}{3}T_0$ seconds from the first sample. However, the time durations of the consecutive bunches are $\frac{11}{3}T_0,\ \frac{12}{3}T_0$, and $\frac{13}{3}T_0$ seconds, respectively, and hence the gaps between consecutive bunches are $\frac{2}{3}T_0, \frac{3}{3}T_0$, and $\frac{4}{3}T_0$, respectively. It is evident from Fig. 1 that the number of samples in the frame satisfies the Nyquist criterion with $\frac{1}{T_0}$ as the Nyquist rate. Generally, a frame of $M_1M_2T_0$ $[s]$ contains M_2 bunches each containing M_1 samples. Within a bunch, the consecutive samples are separated by $\tau_1T_0, \tau_2T_0, \cdots, \tau_{M_1-1}T_0$ from the first sample and we consider $\tau_0 = 0$. The time durations of the bunches are denoted by $\delta_1T_0,\ \delta_2T_0,\ \cdots,\ \delta_{M_2}T_0$. To analyze the bunched nonuniform sampling scheme, we first consider the generation of bunched nonuniform samples of a bandlimited signal $x(t)$. Without of loss of generality, we assume the maximum frequency of $x(t)$ as $\frac{1}{2T_0}$ Hz. Figure 2 depicts a discrete time model for the generation of bunched nonuniform samples from uniform samples. In Fig. 2, the variable θ denotes the digital frequency in radians. By assuming $M = M_1M_2$, we obtain an alternative discrete time model (Sommen and Janse, 2008), which relates uniform samples and bunched nonuniform samples as shown in Fig. 3. In Fig. 3, the pairs of modulation and demodulation terms are used to avoid phase jumps of aliased signals after down sampling operation within the fundamental interval $\theta \in [-\pi, \pi)$ (Sommen and Janse, 2008). The delay elements in Figs. 2 and 3 are represented in the frequency domain with respect to their corresponding sampling rates. In Fig. 3, the frequency responses of the delays for the p-th branch are defined as,

$$\Delta_{s,p}(e^{j\theta}) = e^{-j\tau_p\left(\theta - \frac{M-1}{M}\pi\right)},\ \ p = 0, 1, \cdots, M_1 - 1, \tag{1}$$

$$\Delta^k_{s,p}(e^{j\theta}) = e^{-j\left(\theta - \frac{M-1}{M}\cdot M_1\pi\right)\frac{\delta_k}{M_1}},\ \ k = 1, 2, \cdots, M_2 - 1. \tag{2}$$

3 FREQUENCY DOMAIN RELATION BETWEEN UNIFORM AND BUNCHED NONUNIFORM SAMPLES

In this section, we provide a relation between uniform and bunched nonuniform samples in the frequency domain. With the help of alternative discrete time model as well as Eqs. (1) and (2), we write,

$$\underline{Y}_{sM\times 1}(e^{j\theta}) = \frac{1}{M}\cdot \boldsymbol{\Delta}(e^{j\theta})_{M\times M}\cdot \boldsymbol{W}_{M\times M}\cdot \boldsymbol{X}_{M\times 1}(e^{j\frac{\theta}{M}}) \tag{3}$$

where,

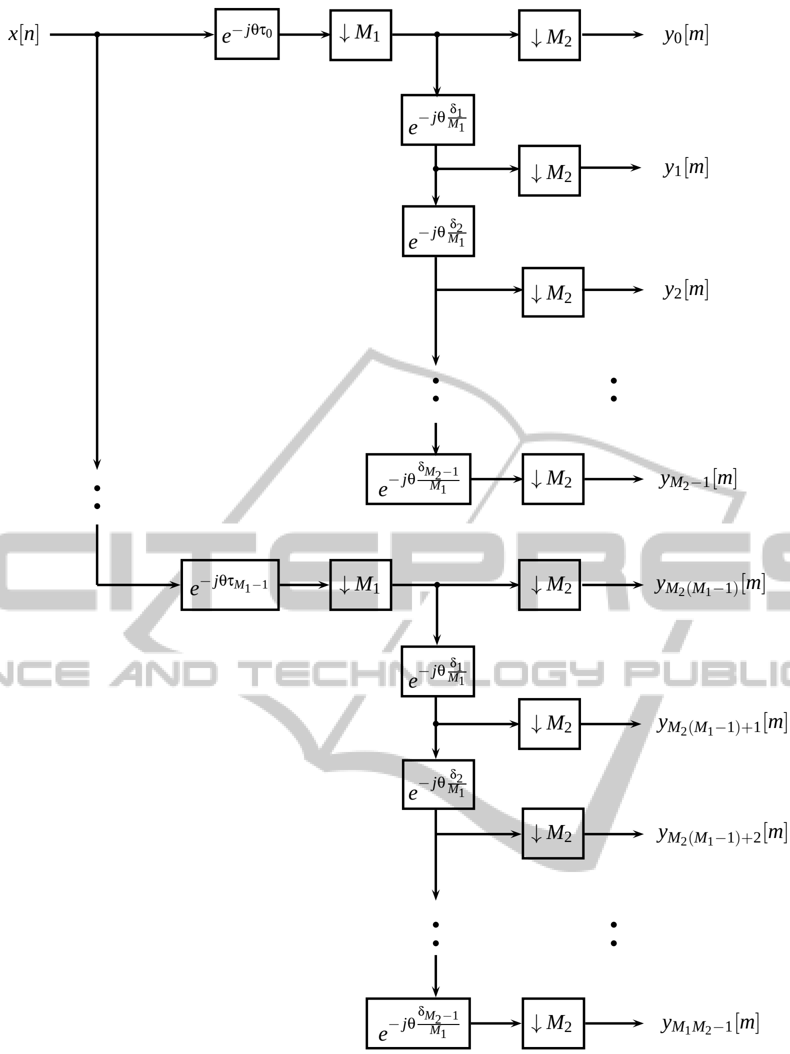

Figure 2: Discrete time model of bunched nonuniform sampling scheme.

$$\underline{Y}_{sM\times 1}(e^{j\theta}) = \left[Y_{s,0}(e^{j\theta}),\cdots,Y_{s,M-1}(e^{j\theta})\right]^T \qquad (4)$$

$$\begin{aligned}\mathbf{\Delta}_{M\times M}(e^{j\frac{\theta}{M}}) = \mathrm{diag}\Big\{ & e^{-j\theta\frac{\tau_0}{M}}, e^{-j\theta\left(\frac{\tau_0+\delta_1}{M}\right)}, \\ & e^{-j\theta\left(\frac{\tau_0+\delta_1+\delta_2}{M}\right)},\cdots, e^{-j\theta\left(\frac{\tau_0+\delta}{M}\right)}, \\ & \cdots, e^{-j\theta\frac{\tau_{M_1-1}}{M}},\cdots, e^{-j\theta\left(\frac{\tau_{M_1-1}+\delta}{M}\right)}\Big\}\end{aligned} \qquad (5)$$

$$\delta = \sum_{p=1}^{M_2-1}\delta_p \qquad (6)$$

$$\begin{aligned}\boldsymbol{W}_{M\times M} = \Big[& \underline{W}_M^{\tau_0}, \underline{W}_M^{\tau_0+\delta_1}, \underline{W}_M^{\tau_0+\delta_1+\delta_2}, \cdots, \underline{W}_M^{\tau_0+\delta}, \cdots, \\ & \cdots, \underline{W}_M^{\tau_{M_1-1}}, \cdots, \underline{W}_M^{\tau_{M_1-1}+\delta}\Big]^T\end{aligned} \qquad (7)$$

$$\underline{W}_M^{\tau_p} = \left[W_M^{-\left(\frac{M-1}{2}\right)\tau_p},\cdots,W_M^{\left(\frac{M-1}{2}\right)\tau_p}\right]^T \qquad (8)$$

$$\begin{aligned}\underline{X}_{M\times 1}(e^{j\frac{\theta}{M}}) = \Big[& X\left(e^{j\frac{\theta}{M}}\cdot W_M^{\frac{(M-1)}{2}}\right),\cdots, \\ & \cdots, X\left(e^{j\frac{\theta}{M}}\cdot W_M^{\frac{-(M-1)}{2}}\right)\Big]^T\end{aligned} \qquad (9)$$

$$\underline{Y}_{M\times 1}(e^{j\theta}) = \left[Y_0(e^{j\theta}),\cdots,Y_{M-1}(e^{j\theta})\right]^T \qquad (10)$$

$$\begin{aligned} &= \Big[Y_{s,0}(e^{j\theta}\cdot e^{j(M-1)\pi}),\cdots, \\ & \cdots Y_{s,M-1}(e^{j\theta}\cdot e^{j(M-1)\pi})\Big]^T.\end{aligned} \qquad (11)$$

In the above definitions, $W_M = e^{-j\frac{2\pi}{M}}$ represents the twiddle factor. In Eq. (3), the vector $\underline{Y}_{sM\times 1}(e^{j\theta})$ represents unmodulated bunched nonuniform samples in the frequency domain. The matrices $\mathbf{\Delta}_{M\times M}$ and $\boldsymbol{W}_{M\times M}$ are full rank matrices. The vector $\underline{X}(e^{j\frac{\theta}{M}})$ in Eq. (3) consists of M number of uniformly distributed, downsampled frequency bands

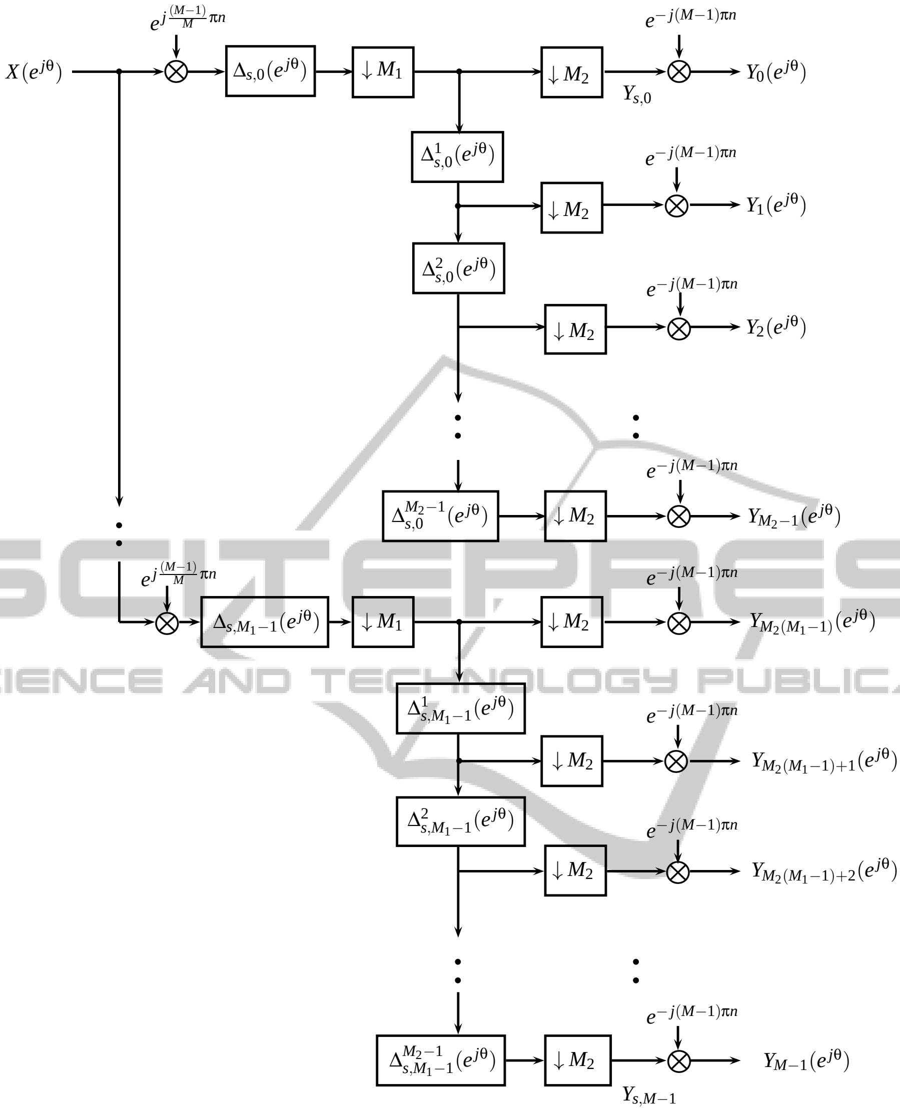

Figure 3: Alternative discrete time model of bunched nonuniform sampling scheme.

of the input signal $\boldsymbol{X}(e^{j\theta})$ when applied to the analysis part of the uniform discrete Fourier transform (DFT) modulated filter bank (Sommen and Janse, 2008; Vaidyanathan, 1993). The representation of $\boldsymbol{X}(e^{j\theta})$ in terms of $\underline{X}(e^{j\frac{\theta}{M}})$ enhances the efficiency of processing by optimizing the number of computations. Using Eqn. (3), we draw a structure to relate uniform and bunched nonuniform samples as shown in Fig. 4. This structure can be considered as an extended version of the analysis part of uniform DFT modulated filter bank presented in (Sommen and Janse, 2008). In Fig. 4, filters with frequency responses $H_M^p(e^{j\theta}),\ p = 0, 1, \cdots, M-1$, are the polyphase components of the prototype filter used in the uniform DFT filter bank (Sommen and Janse, 2008). The prototype filter is a lowpass filter with cutoff frequency $\frac{\pi}{M}$. We note that these polyphase filters are allpass filters, with the frequency responses as that of fractional delay filters (Sommen and Janse, 2008), i.e., for $p = 0, 1, \cdots, M-1$,

$$H_M^p(e^{j\theta}) = e^{j\frac{\theta}{M}p}, \quad -\pi \le \theta < \pi. \tag{12}$$

The block $\boldsymbol{F_s}^{-1}$ in Fig. 4 represents a shifted M -point inverse DFT matrix (Sommen and Janse, 2008), which is defined as

$$\boldsymbol{F_s} = \left[\underline{\mathrm{W}}_M^0, \cdots, \underline{\mathrm{W}}_M^p, \cdots, \underline{\mathrm{W}}_M^{M-1}\right]^T \tag{13}$$

$$\underline{W}_M^p = \left[W_M^{-\frac{(M-1)}{2}p}, \cdots, W_M^{\frac{(M-1)}{2}p}\right]^T. \tag{14}$$

The reconstruction of uniform samples is simple and perfect by inverting linear Eqn. (3). From Eqn. (3),

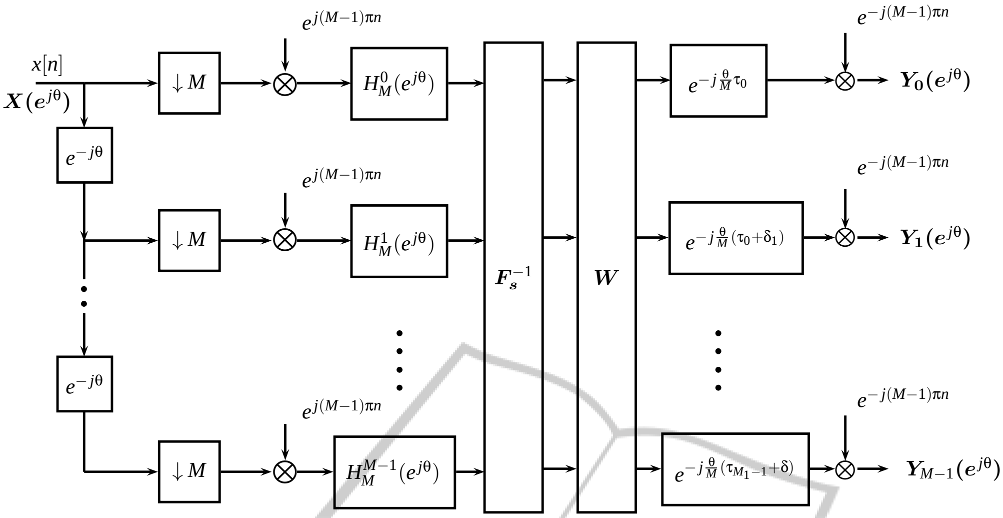

Figure 4: Generation of bunched nonuniform samples.

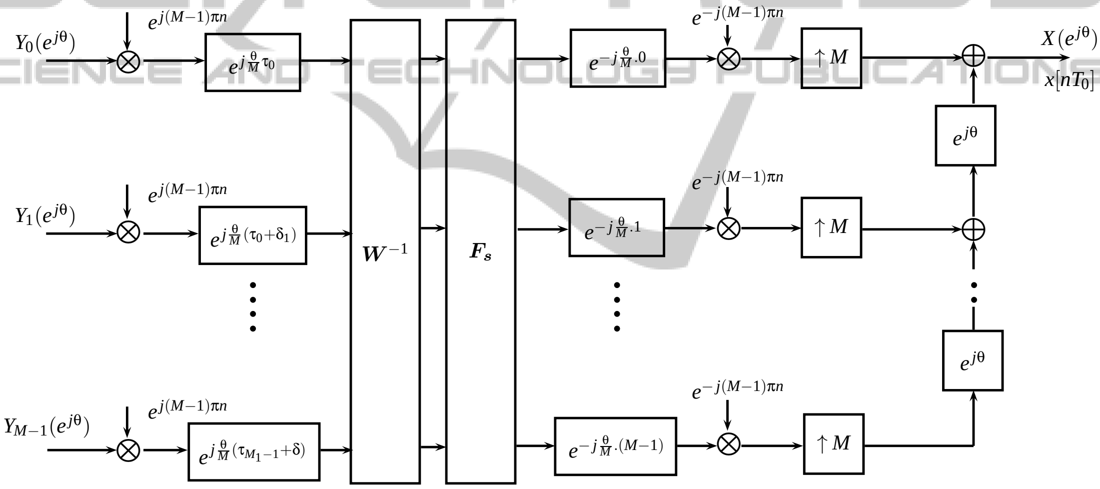

Figure 5: Reconstruction of uniform samples from bunched nonuniform samples.

we obtain

$$\frac{1}{M}\underline{X}_{M\times 1}(e^{j\frac{\theta}{M}}) = \boldsymbol{W}_{M\times M}^{-1}\cdot\boldsymbol{\Delta}_{M\times M}^{-1}\cdot\underline{Y}_{sM\times 1}(e^{j\theta}). \quad (15)$$

We use the synthesis part of the uniform DFT modulated filter bank in order to obtain $\boldsymbol{X}(e^{j\theta})$ from $\underline{X}(e^{j\frac{\theta}{M}})$. The reconstruction of uniform samples $x[nT_0]$ from bunched nonuniform samples is depicted in Fig. 5, which is an extended version of the synthesis part of the uniform DFT modulated filter bank presented in (Sommen and Janse, 2008).

4 SIMULATION RESULTS AND DISCUSSIONS

We have carried out simulation studies to verify the performance of the proposed structures shown in Figs. 4 and 5. Table 1 provides the set of parameters considered for simulations.We have considered 6,144 samples of two different signals $x_1[n] = \sin[0.1\pi n] + 2\sin[0.6\pi n]$ and $x_2[n] = 10^3\sin[0.6\pi n]/(\pi n) + 10^2\sin(0.7\pi n)/(\pi n)$ as inputs for the simulations. Causal versions of the structures presented in Figs. 4 and 5 are implemented with fractional delay filters (Laakso et al., 1996) of order 184. We have calculated the signal-to-error ratio (SER) as defined in (Prendergast et al., 2004) and the absolute

Table 1: Parameters considered for simulation.

T_0 [sec]	M_1	M_2	τ_0	τ_1	τ_2	τ_3	δ_1	δ_2
1	4	3	0	2/3	4/3	6/3	11/3	12/3

Table 2: Performances of proposed structures.

Input	*AME* [*dB*]	*SER* [*dB*]
$x_1[n]$	−161	163
$x_2[n]$	−179	154

mean error (AME) defined in (Sommen and Janse, 2008) by considering 512 number of samples of both the inputs and the respective reconstructed uniform samples $\tilde{x}[n]$. We compare $x[n]$ and $\tilde{x}[n]$ by calculating the absolute error function $e[n]$ for 512 samples, where

$$e[n] = |x[n] - \tilde{x}[n]|. \quad (16)$$

The SER and AME for both the test signals are shown in Table. 2 and these are calculated by considering all possible shifts caused by the implementation of causal fractional delay filters. Figure 6 shows the magnitude and phase spectra for 512 samples of $x_1[n]$ and its reconstructed uniform samples. It can be seen from Fig. 6 that the results reflect the theory proposed for the perfect reconstruction of uniform samples from bunched nonuniform samples. Figures 7 and 8 depict zoomed versions of $e[n]$ for both the test signals $x_1[n]$, $x_2[n]$, respectively. It is evident from Figs. 7 and 8 that the reconstruction errors are very small for both the input signals which are of the order of 10^{-8}, 10^{-9}, respectively. Even though it is proved that perfect reconstruction is possible theoretically, simulations show small reconstruction errors which are due to finite length implementation of causal fractional delay filters.

5 CONCLUSIONS

We have discussed the problem of reconstruction of uniform samples from bunched nonuniform samples using the synthesis part of a uniform discrete Fourier transform (DFT) modulated filter bank. We considered a general case of unequal spacing between bunches of nonuniform samples. The scheme proposed for the reconstruction of uniform samples from bunched nonuniform samples and the simulation results obtained show the efficiency of the signal processing approach followed for the reconstruction of uniform samples from bunched nonuniform samples.

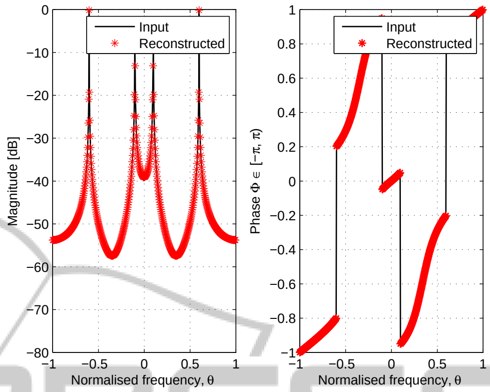

Figure 6: Magnitude and phase spectra of $x_1[n]$ and its reconstructed uniform samples.

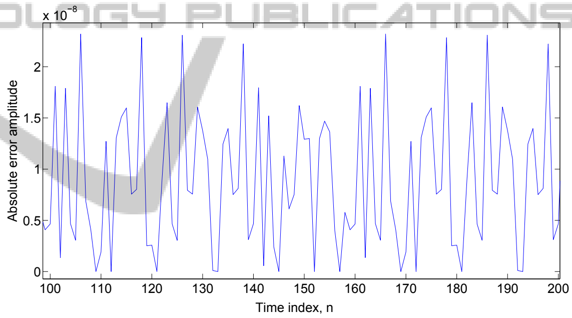

Figure 7: Absolute error function of $x_1[n]$.

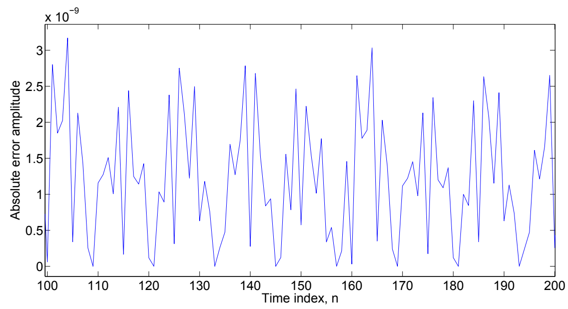

Figure 8: Absolute error function of $x_2[n]$.

ACKNOWLEDGEMENTS

The authors wish to thank the three anonymous reviewers for their constructive criticisms and suggestions which have greatly enhanced the presentation and quality of the paper.

REFERENCES

Black, W. and Hodges, D. (1980). Time interleaved converter arrays. *Solid-State Circuits, IEEE Journal of*, 15(6):1022 – 1029.

Jerri, A. J. (1977). The shannon sampling theorem—its various extensions and applications: a tutorial review. *Proceedings of the IEEE*, 65(11):1565 – 1596.

Johansson, H. and Löwenborg, P. (2006). Reconstruction of nonuniformly sampled bandlimited signals by means of time-varying discrete-time fir filters. *EURASIP J. Appl. Signal Process.*, pages 105–105.

Laakso, T., Valimaki, V., Karjalainen, M., and Laine, U. (1996). Splitting the unit delay fir/all pass filters design. *IEEE Signal Processing Magazine*, 13(1):30 – 60.

Marvasti, F. A. (2001). *Nonuniform sampling: Theory and Practice*. Plenum Pub Corp.

Oppenheim, A., Schafer, R., and Buck, J. (1999). *Discrete-time signal processing*. Prentice-Hall, Inc., Englewood Cliffs, NJ, USA, 2nd edition.

Ouderaa, E. V. D. and Renneboog, J. (1988). Some formulas and applications of nonuniform sampling of bandwidth-limited signals. *IEEE Transactions on Instrumentation and Measurement*, IM-37:353–357.

Papoulis, A. (1977). Generalized sampling expansion. *IEEE Transactions on Circuits and Systems*, CAS-24:652–654.

Prendergast, R., Levy, B., and Hurst, P. (2004). Reconstruction of bandlimited periodic nonuniformly sampled signals through multirate filter banks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 51(8):1612 – 1622.

Singh, V. and Rajpal, N. (2007). Datacompression using non - uniform sampling. In *International Conference on Signal Processing, Communications and Networking*, pages 603 –607.

Sommen, P. and Janse, K. (2008). On the relationship between uniform and recurrent nonuniform discrete-time sampling schemes. *IEEE Transactions on Signal Processing*, 56(10-2):5147–5156.

Vaidyanathan, P. P. (1993). *Multirate systems and filter banks*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA.

Wang, A., Wang, S., and Chen, M. (2004). A novel nonuniform sampling method and processing system to overcome the nyquist rate limit and suppress the noise. *Proc. of the World congress on Intelligent Control and Automation*, pages 3842–3845.