THE BRAIDED RECONSTRUCTION THEOREMS

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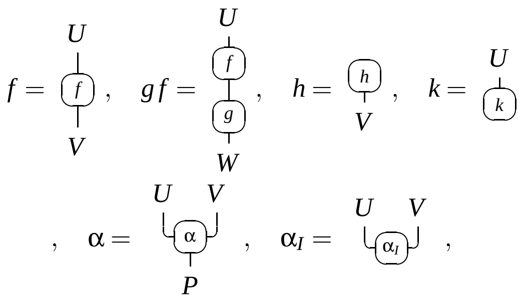
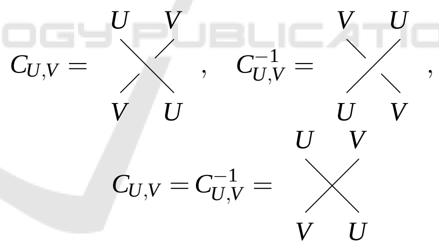
Abstract: In this paper, we introduce the method (transmutation) turning an ordinary (co)quasitriangular Hopf algebra into a braided Hopf algebra, and give the other one which is dual to it.

1 INTRODUCTION

Braided tensor categories become more and more important. They have been applied in conformal field, vertex operator algebras, isotopy invariants of links (see(Huang, 2005; Huang and Kong, 2004; Bakalov and Kirillov, 2001; Hennings, 1991; Kauffman, 1997; Radford, 1994)). So studying in braided tensor categories is interesting, some jobs have been done(see (Zhang, 2003; Shouchuan and Yange, 2008; Xu and Zhang)). In this paper, we will turn an ordinary (co)quasitriangular Hopf algebra into a braided Hopf algebra, which is due to S. Majid (Majid, 1995). Of course, our results only hold in symmetric braided tensor categories, others need be studied furthermore. Since every braided tensor category is always equivalent to a strict braided tensor category, we can view every braided tensor as a strict braided tensor and use braiding diagrams freely.

Some Notations. Let $(\mathcal{D}, \otimes, I, C)$ be a braided tensor category, where I is the identity object and C is the braiding, with the inverse C^{-1}. For $f : U \to V$, $g : V \to W$, $h : I \to V$, $k : U \to I$, $\alpha : U \otimes V \to P$, $\alpha_I : U \otimes V \to I$ are morphisms in $\mathcal{D}$, we denote them by:

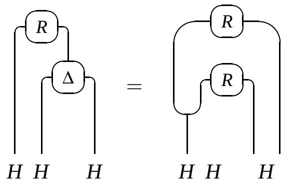




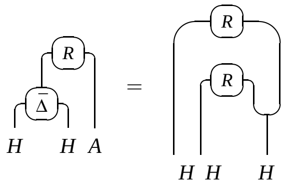
where U, V, W are in $\mathcal{D}$.

Let $\mathcal{C}$ be a tensor category, the braided bialgebra $(H, \underline{R}, \Delta)$ in $\mathcal{C}$ is called quasitrangular bialgebra, if (H, Δ, ε) is a co-algebra and satisfy:

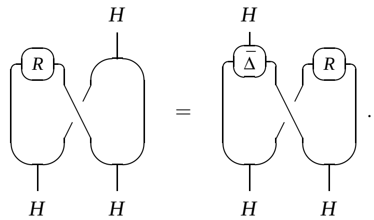
(QT1):



(QT2):



(QT3):



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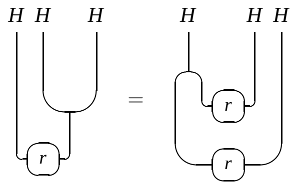
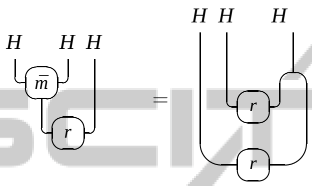
$(H,R,\overline{\Delta})$ is also named that braided quasitrangular bialgebra in $\mathcal{C}$, with $R \in Hom_{\mathcal{C}}(I, H\otimes H)$ has convolution-invertible.

Dually, braided bialgebra $(H,r,\overline{m})$ is called braided co-quasitrangular bialgebra in $\mathcal{C}$, if $(H,\overline{m},\varepsilon)$ is an algebra and satisfy:

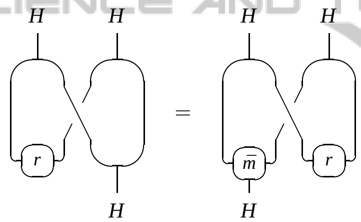
(CQT1):



(CQT2):



(CQT3):



and $r \in Hom_{\mathcal{C}}(H\otimes H, I)$ has convolution-invertible too.

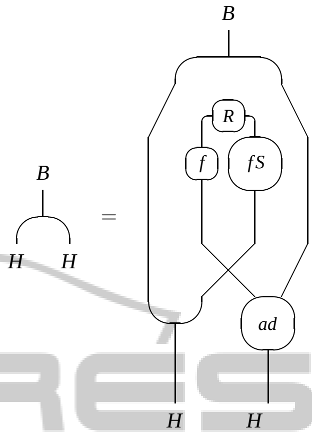
2 THE BRAIDED RECONSTRUCTION THEOREM I

Let $\mathcal{C}$ be a tensor category, $\mathcal{D}$ a braided tensor category and (F,μ_0,μ) a tensor functor from $\mathcal{C}$ to $\mathcal{D}$ with $\mu_0 = id_I$. Let $Nat(G,T)$ denote all the natural transformations from functor G to functor T. Assume that there is an object B of $\mathcal{D}$ and a natural transformation α in $Nat(B\otimes F, F)$. Here $(B\otimes F)(X) = B\otimes F(X)$ for any object X in $\mathcal{D}$.

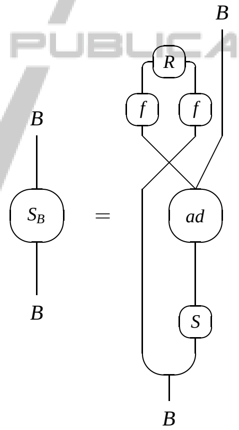
Lemma 2.1. *((Zhang)) H be a bialgebra in symmetric braided tensor category, then (i) (H,R) is quasitriangular Hopf algebra iff $({}_H\mathcal{M}, C^R)$ is braided tensor category. (ii) (H,r) is co-quasitriangular Hopf algebra iff $({}^H\mathcal{M}, C^r)$ is braided tensor category.*

Theorem 2.2. *Let $\mathcal{X}$ be a symmetric braided tensor category, H be a Hopf algebra and (H_1,R) be a quasitriangular Hopf algebra in $\mathcal{X}$. Let f be a bialgebra homomorphism from H_1 to H. Then*

(i) There exists a bialgebra B(braided Hopf algebra if H has left dual), written as $B(H_1,f,H)$, living in $({}_{H_1}\mathcal{M}, C^R)$. Here $B(H_1,f,H) = H$ as algebra, its counit is ε_H, and its comultiplication and antipode are:



and



respectively.

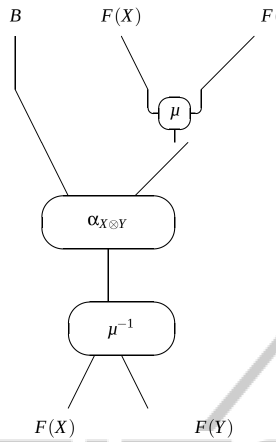
(ii) If H is a braided quasitriangular bialgebra, then B is a braided quasitriangular bialgebra. In particular, when $H = H_1$ and $f = id_H$, $B(H_1,f,H)$ is a braided group, called the braided group analogue of H and written as $\underline{H}$.

Proof (i)Set $\mathcal{C} = {}_H\mathcal{M}$, $\mathcal{D} = ({}_{H_1}\mathcal{M}, C^R)$. Let F be the functor by pull-back along f. That is, for any $(X,\alpha_X) \in {}_H\mathcal{M}$, we obtain an H_1-module (X,α'_X) with $\alpha'_X = \alpha_X(f\otimes id_X)$, written as $(X,\alpha'_X) = F(X)$. For any morphism $g \in Hom_{\mathcal{C}}(U,V)$, define $F(g) = g$. B is a left B-module by adjoint action. Let B_L denote the left regular B-module. Obviously, α is a natural transformation from $B\otimes F$ to F. Now, we show that θ_V is injective for any $V \in {}_{H_1}\mathcal{M}$. If $\theta_V(g) = \theta_V(h)$.It is straightforward since g and h are H_1-module homomorphisoms from V to B. Similary, we can show that $\theta_V^{(2)}$ and $\theta_V^{(3)}$ are injective.

Obviously, B is a braided bialgebra living in $({}_{H_1}\mathcal{M}, C^R)$ determined by braided reconstruction.

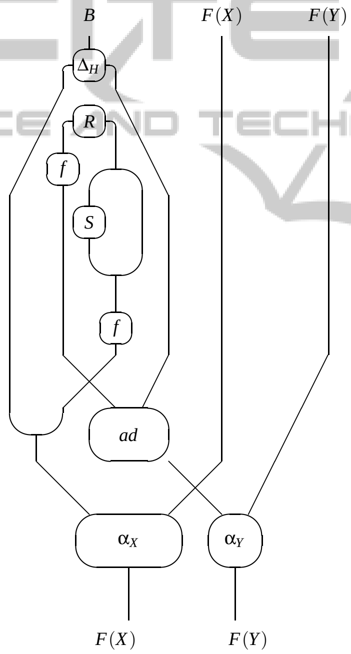
Now we prove that, the comultiplication of B is the same as stated. That is, we need to show that:

$\theta_B^{(2)}(\Delta_B)_{X\otimes Y}$ =



(1)

The left side of (1) =



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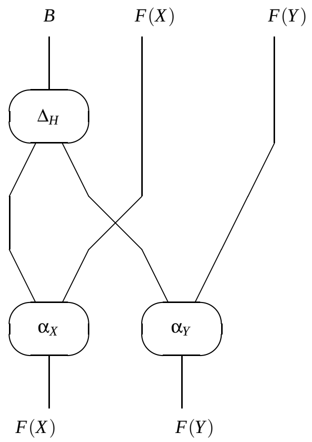
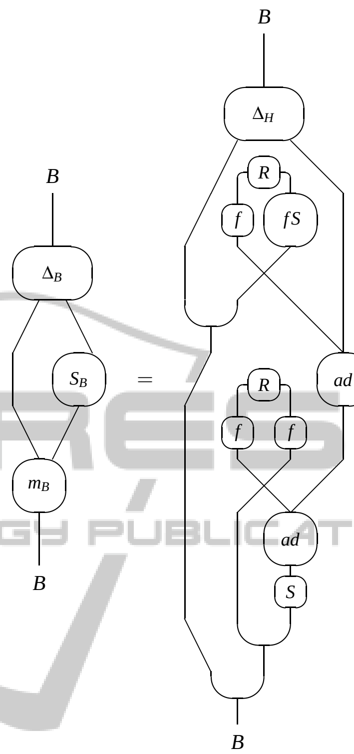
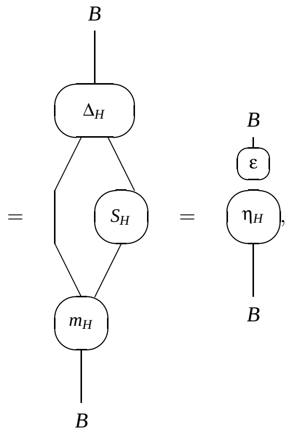


= the right side of (1).

If H has left dual, set $\mathcal{C} = \{M \in {}_H\mathcal{M} \mid M$has left dual$\}$, it is clear that $\mathcal{C}$ is a rigid tensor category(every object has a left dual). Thus B is a braided Hopf algebra.

And





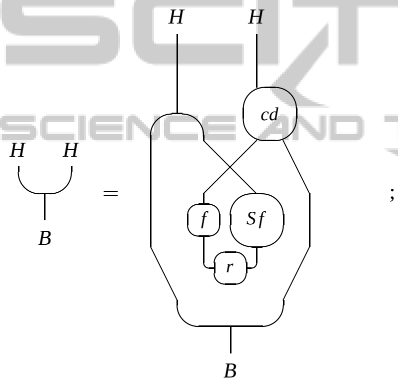
which show that the definition of the antipode of B is reasonable.

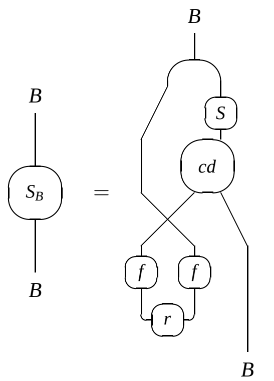
If H is a quasitriangular Hopf algebra, ${}_H\mathcal{M}$ is braided tensor category by lemma 2.1, then by proposition 1.1 (iii), B is a quasitriangular Hopf algebra. If $H_1 = H$, F is identical functor in ${}_H\mathcal{M}$, and let $\Delta = \Delta_B = \underline{\Delta}$, $R_B = \eta \otimes \eta$, then $\underline{H}$ is a braided group. □

3 THE BRAIDED RECONSTRUCTION THEOREM II

The notation like $\mathcal{C}$, $\mathcal{D}$, (F,μ_0,μ), $Nat(G,T)$ are same as stated in section 1. Assume that B is a object of $\mathcal{D}$, and ϕ is natural transformation in $Nat(F, B\otimes F)$. Here $(B\otimes F)(X) = B\otimes F(X)$ for any $X \in \mathcal{D}$.

Theorem 3.1. *Let $\mathcal{X}$ be a symmetric braided tensor category, H be a Hopf algebra and (H_1, r) be a coquasitriangular Hopf algebra in $\mathcal{X}$. Let f be a bialgebra homomorphism from H_1 to H. Then*
(i) There exists a bialgebra B(braided Hopf algebra if H has right dual), written as $B(H_1, f, H)$, living in $({}^{H_1}\mathcal{M}, C^r)$. Here $B(H_1, f, H) = H$ as coalgebra, its unit is η_H, and its multiplication and antipode are:





respectively.

(ii) If H is a braided coquasitriangular bialgebra, then B is a braided coquasitriangular bialgebra. In particular, when $H = H_1$ and $f = id_H$, $B(H_1, f, H)$ is a braided group, called the braided group analogue of H and written as $\underline{H}$.

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