DESIGNING PERSIAN FLORAL PATTERNS USING CIRCLE PACKING

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Abstract: In this paper, we present a novel approach toward generating floral patterns. We extract the essence of a pattern aside from its appearance and geometry into combinatorial elements. As a result, existing patterns can be reshaped while preserving their essence. Furthermore, we can create new patterns that adhere to high level concepts such as imperfect symmetry and visual balance. By decomposing floral patterns into a configuration of circles and angles, we can reconstruct this patterns on different surfaces given a conformal mapping.

1 INTRODUCTION

Persian floral patterns have been used to decorate books, carpets and buildings for many centuries. Examples of these patterns are illustrated in fig.1. These patterns were not only used on the plane but also on spherical geometry to decorate domes. Like other instances of Islamic art, these patterns were designed by skilled mathematicians, and they exhibit fascinating geometric and mathematical properties. This art was taught by traditional practitioners through apprenticeship. Therefore, few resources are remained about the design process of these patterns.

Figure 1: Example of Persian floral patterns (Takestani, 2002).

Designing floral patterns can be divided into two aspects. First, stems are used to compose and structure the space. Second, flowers and other ornamental elements are added to locations specified by the stem structure. Although a pattern is never complete without the second step, it is the first step that carries most of the beauty of the design. The composition should comply with aesthetic concepts such as balance and repetition. With an aesthetic composition in hand, one can use pre-designed sets of flowers to decorate a pattern. The main challenge of designing floral patterns lies in finding the underlying composition or the structure of the stems.

By analyzing existing floral patterns and studying traditional literature, we realized that the underlying structure and design of these patterns are based on circles. Spiral forms, flowers and even the shape of the leaves can be expressed using circle arcs. As we explain in section 3, we can capture the essence of a pattern into a configuration of circles. Therefore, The properties and appearance of a design can be controlled by changing its underlying circular configuration. This led us to relate our problem to a well-established field of mathematics known as circle packing.

Circle packing is a relatively recent field in mathematics that studies the configuration of circles showing specified patterns of tangencies. Circle packing is based on rich mathematical foundations such as conformal structures and discrete analytic functions. Capturing the essence of a floral pattern in a packing of circles provides a powerful framework to generate and manipulate these beautiful patterns.

By taking advantage of circle packing concepts and tools, we can analyse, generate and edit floral patterns and also formulate specific properties in their structure. Furthermore, we can construct patterns on non-Euclidean geometries and project designs on

different surfaces without distortion using conformal mappings.

In what follows, we first discuss related work. Next, in section 3 we analyze floral patterns and their circular structure. In section 4 we present concepts and definitions from circle packing and explain how we apply them to control the properties of floral patterns. In section 5 we provide a method for designing a Persian floral pattern algorithmically. And finally, section 6 concludes our discussion.

2 RELATED WORK

Ornamental design has been investigated in the context of computer graphics for many years. One of the earliest works in this area is an algorithmic approach to generate 17 symmetry groups on the open plane (Alexander, 1975). Later, Frieze patterns from group theory were used to generate ornaments in bounded areas (Glassner, 1996).

Separation of composition and ornaments was first applied in (Smith, 1984). In this work the branching structures of plants are generated using parallel rewriting grammars called graphtals; then, in a post-processing step the result is visually enhanced by adding leaves and flowers.

Although L-systems (Prusinkiewicz and Lindenmayer, 1996) is widely used to generate plants, it is not a suitable approach toward generating floral ornaments (Wong et al., 1998).

Generating spiral curves is investigated in (Xu and Mould, 2009). These curves are generated based on the trajectory of charged particles in a magnetic field and are called magnetic curves.

Islamic and Persian patterns are studied in many works in computer graphics, because of their geometric and mathematical nature. Star patterns along with a number of algorithms to generate them are investigated in (Kaplan and Salesin, 2004; Kaplan, 2008). Animation of Persian floral patterns is studied in (Etemad et al., 2008). In (Djibril and Thami, 2008), classifying symmetric patterns in decorative Islamic art is discussed. A discussion on computer aided art can be found in (Geng and Geng, 2010).

Our work is closely related to the computer generated floral patterns of (Wong et al., 1998). This work presents a grammar based approach to generate floral patterns by defining a set of elements and their growth rules. Starting with user-specified seed points, the main role of the system is to decide which elements to grow. In order to make this decision, the system finds the largest empty circle inside the boundary and then invokes the elements near that circle.

There are a number of draw backs to this method: although aesthetic principles like visual balance are discussed in this work, the presented algorithm does not provide a mechanism to support these principles. To address this issue, an interactive approach is presented, which still relies on artistic input for an aesthetic partitioning (Obispo and Anderson, 2007; Anderson and Wood, 2008). Moreover, the provided algorithm is costly because finding the largest circle in every iteration is computationally expensive.

The approach we present here is not grammar based and is able to generate patterns not only on the plane but also on non euclidean geometries. Moreover, we are able to control high level properties of the result such as visual balance and partial symmetry.

3 ANALYSIS OF PERSIAN FLORAL PATTERNS

As discussed in section 1, ornaments have two different aspects: composition and appearance. The Composition involves partitioning the space inside the boundaries of a design in a visually appealing way. There are a number of guide lines to compose an aesthetic design (Wong et al., 1998):

Repetition, either in the form of simple translation or complex symmetries, implies a presence of order and design in the pattern. Repetition appears in Persian floral patterns in the form of bilateral symmetry and also imperfect congruency created by the spirals.

Balance is another rule of aesthetic design, which is related to visual weight of the components of the design. In the case of floral patterns, visual weight is mostly influenced by the size, number of turns and density of elements of the spirals.

Adhering to the Boundary is another important rule in ornament design. A pattern that exhibits this property evokes a sense of intelligence in the design elements, which form according to their boundary.

Conventionalization and Growth are also two common property of ornaments (Wong et al., 1998). Conventionalization involves representing abstract forms of nature in the ornament. Floral patterns are an abstraction of vines, climbing and winding to form the composition of the design. Growth is the means of connecting different parts of the design. In floral patterns, growth is evident in spiral forms that represent the stems. In floral patterns, and more specifically Persian floral patterns, spirals are the main element of composition (Etemad et al., 2008).

Besides the composition, appearance is another aspect of floral patterns. Ornamental elements such as flowers and leaves are added to the pattern based on the structure to decorate the result. These elements can be reused in different designs. Hundreds of these elements are available in several books, including: (Jones, 1987; Aghamiri, 2004; Takestani, 2002; Behzad, 1998; Honarvar, 2005).

Figure 2: An example of Persian floral pattern decomposed into underlying configuration of circles.

Fig.2 shows a Persian floral pattern from Jame mosque in Isfahan, Iran. This pattern consists of four repeated parts using bilateral symmetry. To simplify this pattern, we have removed the leaves and changed the background in (fig.2:top, right). This pattern consists of seven spirals. Representing the spirals with a circle, the pattern is simplified to a configuration of circles (fig.2:bottom, right).

These circles and their tangencies form a graph in which every node is a center of a circle, and every edge represents either tangency or overlap of two circles. We call this graph the combinatorics of the design. The stem represents a tree (an abstraction of a plant in nature). Likewise, it represents a tree in the combinatorics graph. The edges of this tree represent transitions of the stem from one circle to another.

We have simplified the design given in fig.2 into a circle packing. This enables us to not only distinguish between the appearance and the geometry but also between the geometry and the combinatorics of the pattern. In other words, such a decomposition enables us to capture the essence of a design separate from the geometry and shape, which is the most important contribution in this paper.

An important mathematical tool to manipulate circles is the conformal transformation. This type of transformation preserves the angle between lines. Therefore, they map circles into circles. By captur-

ing a design in a configuration of circles, we can use conformal mapping to transfer a design to another geometries or manifold without distortion.

In order to produce a design, we need information about spirals, their connections and also leaves and flowers. However, we should keep in mind that we can only use circles and angles because these are the only things that stay unchanged through conformal mappings.

For the sake of simplicity throughout this paper, we denote circle S centered at c and of radius r with $S(c,r)$. Since we are interested in define our ornaments only based on circles and angles, we define point P on the circle $S(c,r)$ with $P(S,\alpha)$ where α is the angle of the arc between P and the x-axis, as shown in fig. 3. Therefore, an arc on the circle between two points can be expressed by its starting angle and the incremental angle to the end point. For example, the arc A on circle S starting from $P(S,\alpha)$ to $Q(S,\theta)$ is denoted by $A(c,r,\alpha,\theta-\alpha)$.

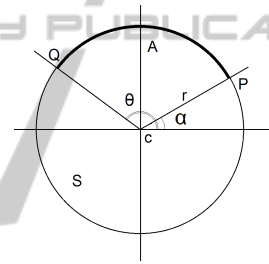Figure 3: Circle S centered at c and radius r is denoted by $S(c,r)$. Arc A from P to Q is denoted by $A(c,r,\alpha,\theta-\alpha)$.

Spirals consist of circular arcs with different radii and centers. Fig.4 shows a spiral and the circle arcs that form it. This spiral consists of three circle arcs: $A_1(c_1,r_1,\pi,-\pi)$, $A_2(c_2,r_2,0,-\frac{2\pi}{3})$ and $A_3(c_3,r_3,-\frac{2\pi}{3},-\frac{5\pi}{3})$, $r_1 > r_2 > r_3$.
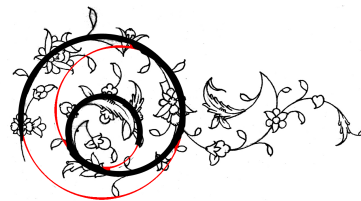

Figure 4: Spirals are constructed based on circles.

There are different forms of spirals depending on the arc angles of each arc. It is common in Persian floral patterns to reduce the radius of the inner circle by a constant value (i.e $r_1 = r_2 + c = r_3 + 2c$) (Takestani, 2002).

The center c_2 of the circle $S_2(c_2, r_2)$ which is inner tangent to circle $S_1(c_1, r_1)$ at the point $p(S_2, \theta)$ can be found using following equations:

$$c_{2x} = c_{1x} + (r_1 - r_2)\cos(\theta)$$
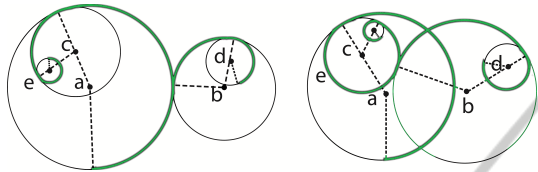$$c_{2y} = c_{1y} + (r_1 - r_2)\sin(\theta) \qquad (1)$$

Figure 5: Connections between spirals. right: Spirals are tangent. left: Spirals overlap.

The connection between two spirals depends on their outer circles. The outer circles of the two spirals should be either tangent or overlapped in order to make a connection. Two types of connection are shown in fig. 5. An important rule in floral pattern design is that two connected spirals should have different rotation directions (Honarvar, 2005).
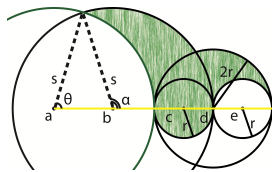
Figure 6: Design of a leaf based on circle arcs.

Although our main task here is to express the structure of the ornament using circles and angles, we realized that even the leaves and flowers in Persian floral pattern may be expressed in such terms. Fig. 6 shows how a traditional-style leaf is designed with five circles arcs. Of course, not all the possible ornamental elements can be expressed efficiently using only arcs.

4 CIRCLE PACKING

So far, we described how we represent a floral pattern with circle arcs and their tangencies. We can use this analysis to generate these types of patterns automatically. In order to create a new design, the underlying configuration of circles must first be designed. Therefore, we need a method to fill a bounded area with circles of different size. Moreover, we want to be able to control high level properties of the resulting circles.

There are different versions of the circle packing problem depending on their constraints. For example, some methods work with uniform circles only; or some methods find the maximum radii for a number of circles inside a bounded area. A collection of some of these different circle packing problems can be found in (Castillo et al., 2008). Among these works, the circle packing problem introduced by (Thurston, 1985) is the best match for our problem. This approach studies the configuration of circles that exhibit a specific pattern of tangency. Here, we present a brief introduction to terminology and methods of this type of circle packing. A thorough discussion on this circle packing (which we simply call it circle packing from this point) can be found in (Stephenson, 2005).

A packing of circles has two aspects: combinatorics and geometry. The geometry of a packing is defined by the location of the circles and their radii, whereas the combinatorics are defined by the tangency relations among circles. The tangencies studied here are external; meaning that tangent circles have disjoint interiors. Moreover, we limit our discussion to tangencies among triples of circles; the tangency relations between an n-tuble of circles can be converted into triples by adding extra circles (Stephenson, 2005).

Combinatorics are encoded in abstract simplicial 2-complexes K which triangulate oriented topological surfaces. Geometric realization of K is referred to as the carrier of the packing which comes from connecting the center of the circles with geodesics.
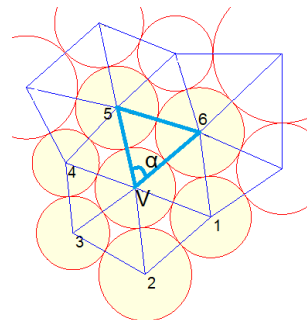
Figure 7: A circle packing and its carrier.

Vertices of K can be interior or boundary. A vertex V along with its neighbors $U_1, U_2, .., U_n$ in K is called a flower and neighbours $U_1, U_2, .., U_n$ are called petals. We use lower case letters to refer to the radius of a circle (e.g. v is the radius of circle represented by vertex V). There is a face (2-complex) associated with each tangent triple on K and there is an angle α associated with vertex V in each face that contains V (see fig.7). Angle $\alpha(v; u_i, u_j)$ in face $< V, U_i, U_j >$

can be computed using law of cosines (Collins and Stephenson, 2003):

$$\alpha(v; u_i, u_j) = 2 \arcsin\left(\sqrt{\frac{u_i}{u_i + v} \cdot \frac{u_j}{u_j + v}}\right). \quad (2)$$

The summation $\theta(v) = \sum \alpha(v; u_i, u_j)$ for all faces adjacent to V is called the angle sum of V. To have a non-overlapping circle packing on the plane, the angle sum of every interior vertex must be 2π. This is can be seen clearly by looking at the carrier as a geometric realization of the combinatorics. Similarly, the geometric interpretation of a vertex having angle sum higher than 2π on the plane is an overlapped flower. Moreover, If the angle sum is less than 2π the flower can form on the sphere, and if it is greater than 2π the flower can form on the Poincare disc (Dubejko and Stephenson, 1995).

An abstract simplicial 2-complex K can have a feasible geometric circle packing if and only if it represents a triangulation of a space. If K has this necessary and sufficient condition, then it can result in different geometric configurations of circles. Therefore, we can impose more restrictions to obtain a unique packing. These restrictions can be on the size or the angle sum of the boundary circles. We use these restrictions to control the boundary shape of our floral patterns. Finding a packing that satisfies given boundary conditions is called the boundary value problem (the Dirichlet problem) and there exists a unique (up to automorphism) solution for this problem (Collins and Stephenson, 2003).

The algorithm to find this packing is based on the angle sums. This algorithm has two steps. The first step is to find the correct radii for all the circles, and the second step is to lay out the circles. This algorithm is efficient and guaranteed to converge to the desired packing (Collins and Stephenson, 2003).

As stated earlier, to form a non-univalent planner packing (i.e. a packing on the plane which does not overlap), the angle sum of every inner circle must be 2π. Starting from an arbitrary value for the radii, first we initialize boundary radii with defined values. Then, we iteratively refine the radius of every circle without a radius restriction to a new value until the angle sum condition is met (by a threshold). Assume vertex V has k neighbors and its current angle sum is θ. There is a radius r_e so that if all the neighbors of V had this radius, the angle sum would still be θ. Then, there is a radius r_{new} so that if V has r_{new} as its radius and all the neighbors are of radius r_e the angle sum would be A_v. Replacing radii of circles with r_{new} iteratively will adjust the angle sum of the vertices and eventually result in a set of radii so that the circles

with that radii can fit beside each other. The process above is summarized in this equation:

$$r_{new} = \frac{1 - \sin(\frac{A_v}{2k})}{\sin(\frac{A_v}{2k})} \cdot \frac{\sin(\frac{\theta}{2k})}{1 - \sin(\frac{\theta}{2k})} \quad (3)$$

In this formula k is the degree of vertex V and A_v is the target angle sum for this vertex, which is 2π for inner vertices in the case of planer packings. This algorithm carries the angle sum from circles with a higher angle sum to circles with a lower angle sum until the error is less than a threshold.

The next step is to lay out the circles. Having the combinatorics K and the radii computed above, we traverse the combinatorics and place the circles, which will now fit together to form the packing. Starting from an edge of a face on K, two circles associated with that edge are located arbitrarily and the third circle of that face is placed based on the position of the first two. We then lay out faces that have a common edge with this face, and similarly, we can place all the circles based on the previous circles. The choice of the first edge and the order of traversing the faces does not change the final packing (Stephenson, 2005).

As explained in the previous section, the underlying circles of a floral pattern can overlap. We can create overlapped packings by adding constant values to circles' radii after the packing or to u_i and u_j in equation 2 while calculating the radii. Overlapped packings can be converted to floral patterns as depicted in fig. 5.

We use the algorithm and concepts explained above to create new designs with specified high level properties such as imperfect symmetry and visual balance. Repetition and symmetry can be formulated in the combinatorics of a pattern. Symmetric combinatorics result in a partial or perfect symmetric floral pattern.

Visual balance of the design is a factor of the size of the spirals and the number of turns they make. There is a relation between the size of a circle (relative to its petals) and the degree of its corresponding vertex in the combinatorics. This fact is supported by the Ring Lemma (Rodin and Sullivan, 1987) : Let $S(r_0, c_0)$ be an interior circle in a packing, and $S_i(r_i, c_i)$ for $i = 1..k$ ($k \geq 3$) be its k petals. Then, $r_0/r_i \leq a(k)$ in which, $a(k)$ is a increasing function of k. This implies that by increasing the degree of a vertex, we can increase the radius of its corresponding circle in comparison to its neighbors. Therefore, a visually balanced result can be achieved using a balanced combinatorics in which degrees of the vertices do not deviate much. As an exaggerated example, if all the vertices are of degree 6 (with no boundary condition) then all of the resulting circles have the same
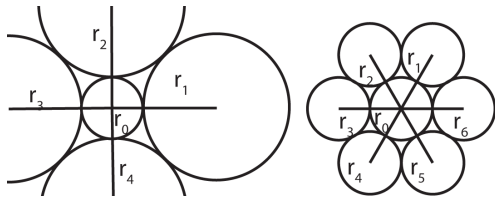
size.

Figure 8: Relation between the degree of a vertex and the size of its corresponding circle compared to the petals.

5 GENERATING FLORAL PATTERNS

With the analysis presented in section 3 and the concepts provided in section 4, we propose a method to generate floral patterns automatically.

Given boundary B, the problem is to generate a floral pattern that fills this boundary and adheres to high level properties such as balance and symmetry. First, we find M, the basic motif of B. Then, we define the combinatorics K and set boundary conditions. Next, we generate the circle packing. Having the circle packing we find the desired tree T in K. Then we traverse T and set the spiral types for each circle. Finally, we apply the symmetry and repetition rules defined in first step to fill B. We will describe each step in more details.

Symmetry is an important characteristic of the floral ornaments. As the first step, we should decide what type of symmetry should be used for the given boundary. There are a number of methods to detect the symmetry in a shape automatically (Masuda et al., 1993; Lee et al., 2008). Using these techniques, we can detect the symmetry and specify the main motif in the boundary. We can focus only on the motif and then repeat the generated pattern. Alternatively, we can formulate the symmetry in the combinatorics of the underlying circle packing and the tree. This method results in an imperfect symmetry.

The combinatorics can come from an existing pattern, or it can be defined so that it exhibits some specific properties, described in section 4. The same combinatorics can result in different floral patterns depending on the boundary.

After specifying the combinatorics, we solve the boundary value problem. To specify the boundary values, we need a correspondence between the points on the given boundary (B) and the boundary of K. Therefore, B is converted into a polygon having an equal number of vertices to the boundary vertices of

K. For every boundary circle, we can either fix the radius or the angle sum. As shown in fig.9 right, the decision is based on the importance of the polygon's angle at that vertex. If the center of the circle is collinear with its neighbors by a threshold, we fix the radius. Otherwise, we fix the angle sum of that circle.

Using the algorithm discussed in section 4 we generate the circle packing. At this point, some circles may exceed the boundary. To solve this issue, we propose three approaches. First, we can push the circles to fit within the boundary, which will result in a non-univalent packing (i.e. overlapping circles). The second approach is to scale the packing to fit the boundaries which will result in empty spaces left in some areas of the pattern (usually corners). The third approach is to "cookie cut" the packing, by which we mean removing the circles which fall outside of the boundary. This approaches can be used in combination with each other.

The next step, is to specify a tree inside the combinatorics' graph. This tree plays an important role in the final appearance of the floral pattern, and of course, the choice is not unique. Based on our experiments, we present guidelines that can be used to create more appealing results. Having the boundary spirals as the leaves is more desired since it complies more with conventionalization. Furthermore, it helps to create a balanced composition by decreasing the visual weight on the boundaries. Setting boundary spirals as parts of the stem may result in larger arcs with higher numbers of turns. Moreover, a combinatorial symmetric tree results in a symmetric or imperfect symmetric design depending on the size of the circles.

Similar to the discussion in section 4, the vertices of the tree should have a balanced degree distribution to achieve a visually balanced result. Similar to the combinatorics, the tree can be extracted from an existing pattern designed by an artist. Many floral patterns can be generated from a single combinatorics and a single tree.
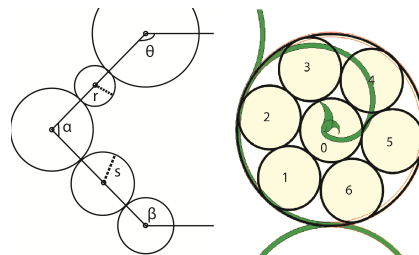

Figure 9: Right: Arranging boundary circles. Left: Partitioning inner section of a spiral using circle packing.

After defining the tree, we traverse it and specify

the properties of the spirals (i.e. type, thickness and number of turns). Stem's thickness is usually uniform in Persian floral patterns. It can also be defined based on the depth of the corresponding vertex in the combinatorial tree. To achieve a balanced composition, number of the turns of a spiral should be proportional to the size of its outer circle.

Placement of ornaments is also done in this step. We use circle packing within the circle (Wang et al., 2002) to partition the space inside a spiral in order to allocate space for the ornaments. As an example, fig. 9 shows a spiral partitioned based on the circles inside its bounding circle. Circles 1, 2, 3 and 5 are available to place an ornament, but circle 4 and 0 are occupied by the stem and circle 6 is not attached to the stem.

Figure 10: An example on a pattern constructed on sphere.

We can construct these patterns in different geometries. As an example, in fig.10, We have created a pattern on a sphere. For patterns on the sphere, the combinatorics should be spherical; meaning that there should be no boundary vertices. To create spherical patterns, we remove one of the vertices from the combinatorics graph and use its petals as the combinatorics' boundary. Next, we create a planar packing on the unit disc using the combinatorics and Poincare disc metrics. Then we use stereographic projection to map patterns on the unit disc to one of the hemispheres.

In order to map a unit disc on the hemisphere, we use a modified version of stereographic projection. We place the center of the stereographic sphere on the $(0,0,0)$ instead of $(0,0,1)$. Assuming $P(u,v)$ is a point on the plane the projected point $Q(x,y,z)$ on the sphere is given by following equations:

$$z = \frac{1-(u^2+v^2)}{1+(u^2+v^2)}, \quad x = u(1+z), \quad y = v(1+z) \quad (4)$$

Then the equator is considered as the circle removed from the combinatorics since it is tangent to all the boundary circles. The circle represented by equator is

larger than other circles. We can use a Mobius transformation (automorphisms on sphere) to normalize the size of the results. This method provides seamless packing on the sphere (Stephenson, 2005). Since we only map the arcs and this projection is conformal, there is no distortion in the final results (i.e. circles remain circles).

We can cover arbitrary surfaces with floral patterns given a conformal parametrization. Methods such as (Bobenko et al., 2003) provide us with a circle packings covering more general surfaces. We can use these packing to construct floral patterns that cover an object.
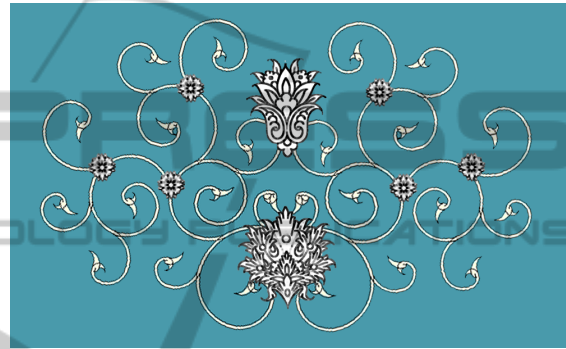

Figure 11: Final results by placing ornaments on the structure.

6 CONCLUSIONS

In this work, we presented an analysis of Persian floral patterns based on circle packing. This approach helps us to separate the semantics and combinatorics of the design from its geometry and capture the essence of a design independent of its appearance (i.e. the detailed ornamentation and the rendering style). Furthermore, the elegant mathematics supporting this method enables us to generate new designs that exhibit high level characteristics such as visual balance and imperfect symmetry. The presented methodology may be applicable to other styles of floral patterns.

We have implemented the proposed method to generate Persian floral patterns. Most of the pictures in this paper are generated using this implementation. It is important to notice that the domain and possibilities of the presented approach extend beyond this implementation. Fig.11 shows an example of the final results with added ornament elements.

As future work, we would like to explore the possibility of interactive and sketch-based pattern design,

allowing the creation of patterns that adhere to structures provided by the user. Furthermore, animating the generated patterns procedurally is another possible direction for future works.

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