Performance Shaping through Cost Cumulants and Neural Networks-based Series Expansion

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Abstract: The performance shaping method is addressed as a statistical optimal control problem. In statistical control, we shape the distribution of the cost function by minimizing *n*-th order cost cumulants. The *n*-th cost cumulant, Hamilton-Jacobi-Bellman (HJB) equation is derived as the necessary condition for the optimality. The proposed method provides an approach to control a higher order cost cumulant for stochastic systems, and generalizes the traditional linear-quadratic-Gaussian and Risk-Sensitive control methods. This allows the cost performance shaping via the cost cumulants. Moreover, the solution of general *n*-th cost cumulant control is provided by numerically solving the HJB equations using neural network method. The results of this paper are demonstrated through a satellite attitude control example.

1 INTRODUCTION

We shape the performance of the system through the statistical properties of the cost function. In linear-quadratic-Gaussian (LQG) control, the performance is optimized by minimizing the mean of the cost function (Fleming and Rishel, 1975). In statistical optimal control, we minimize any cost cumulant to improve the performance of the system. So far the first mean (LQG) and denumerable sum of all the cumulants (risk-sensitive) of the cost function are investigated (Lim and Zhou, 2001). However, there are other statistical parameters that we can vary to shape the performance. This is achieved by minimizing n-th cost cumulants. The study of cost control cumulant was initiated by (Sain, 1966). The authors extended the theory of cost cumulant control to third and fourth cumulants for a *nonlinear* system with *nonquadratic* cost and derived the corresponding HJB equations (Won et al., 2010). HJB equation was derive, but the solution was not determined. In fact, most HJB equations do not have analytical solutions except for the special cases of linear systems with quadratic cost functions. Thus, numerical approximate methods are needed to solve HJB equation. For the first two cumulant case, we solved the HJB equation using neural networks in (Kang and Won, 2010). In this paper, we extend this result to *n*-th cumulants. This is not a simple extension of the results in (Won et al., 2010). There, we developed a procedure to solve higher order cost cumulant problem using the results of the moments. In this paper, we use induction to derive *n*-th cumulant HJB equation, which was not a trivial task. This *n*-th cumulant HJB equation corresponds to the performance shaping idea. Then we solve this HJB equation using a neural network method.

A power series expansion to approximate the value function for an infinite-time horizon deterministic system was given in (Alberkht, 1961). Applying Galerkin approximate method to solve the generalized Hamilton-Jacobi-Bellman (GHJB) was given in (Beard et al., 1997). (Chen et al., 2007) proposed using neural network methods to solve the optimal control problem of a nonlinear finite time system. If we define the weighting and the basis functions as polynomials, the neural network is in fact equivalent to power series expansion in that we use coefficients of the power expansion as the weights in neural networks. However, neural network method has the potential to be more than simple power series expansion by adding additional layers of neurons. In this work, we extend the system dynamics of (Chen et al., 2007) to stochastic systems. Then we solve the HJB equations for the *n*-th cost cumulant control problem using neural network approximation method.

Section 2 states the problem and defines the notations used in this paper. Section 3 develops the HJB equations for the *n*-th cost cumulant control of the

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system. The induction is utilized in the derivation. In order to solve the statistical optimal control problem, we use neural network approximations for HJB equations in Section 4. The simulation results for the satellite attitude control application is presented in Section 5. Finally conclusions are given in the last section.

2 PROBLEM FORMULATION

Consider a stochastic differential equation:

$$dx(t) = f(t,x(t),u(t))dt + \sigma(t,x(t))dw(t), \tag{1}$$

where $t \in [t_0,t_F]$, $x(t_0) = x_0$, $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, and $dw(t)$ is a Gaussian random process of dimension d with zero mean and covariance of $W(t)dt$. The system control is given as

$$u(t) = k(t,x(t)), t \in T. \tag{2}$$

The system cost function is given as:

$$J(t,x(t);k) = \psi(x(t_F)) + \int_t^{t_F} \left[l(s,x(s)) + k'(s,x(s))Rk(s,x(s))\right] ds, \tag{3}$$

where $\psi(x(t_F))$ is the terminal cost, $l(s,x(s))$ is a positive definite function, R is the positive definite matrix. The goal is to find an optimal controller for system (1) which minimizes the n-th cumulant of the cost function (3), such that $V_n(t,x,k(t,x)) = \min_{k \in K_M} \{V_n(t,x,k)\}$.

We now introduce a backward evolution operator $\mathcal{O}(k)$, defined by

$$\mathcal{O}(k) = \frac{\partial}{\partial t} + \sum_{i=0}^{n} f_i(t,x,k(t,x)) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} \left(\sigma(t,x)W(t)\sigma'(t,x)\right)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then the n-th cumulant of cost function, we introduce the n-th moments of cost function which is defined as:

$$M_i(t,x,k(t,x)) = E\left\{J^i(t,x,k(t,x)) \middle| x(t) = x\right\}$$

Next, we introduce some definitions,

Definition 2.1. A function $M_j : Q_0 \to \mathbb{R}^+$ is an admissible j-th moment cost function if there exists an admissible control law k such that $M_j(t,x) = E_{tx}\left\{J^i(t,x,k(t,x))\right\}$ for $t \in T, x \in \mathbb{R}^n$.

Definition 2.2. The admissible moment cost functions $M_1, \ldots, M_j$ defines a class of control laws K_M such that for each $k \in K_M$, $M_1, \ldots, M_j$ satisfy Definition 2.1.

Definition 2.3. A function $V_j : Q_0 \to \mathbb{R}^+$ is an admissible j-th cumulant cost function if there exists an admissible control laws k such that $V_j(t,x) = V_j(t,x,k)$.

Now, we mathematically formulate the statistical optimal control problem as follows. Under the assumption that $M_1, \ldots, M_{n-1}$ exist and are admissible, we find an optimal control $k^* \in K_M$ such that $V_n^*(t,x) = V_n(t,x,k^*)$ satisfies $V_n^*(t,x) \le V_n(t,x,k)$ for all $k(t,x) \in K_M(t,x)$.

The n-th cumulant of the cost function is given by the following recursion formula (Smith, 1995), where we suppress the arguments for the sake of brevity.

$$V_n = M_n - \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} M_{n-1-i} V_{i+1}. \tag{4}$$

It is shown in (4) that the n-th order cumulant can be calculated from the n-th order moment and the lower order moments and cumulants. In the next section, the above definitions and formulas will be used to derive the HJB equations for the n-th order cumulant minimization.

3 *n*-th CUMULANT HJB EQ.

Before we derive the n-th cumulant HJB equation, we introduce the n-th moment HJB equation. (Sain, 1967) derived the n-th moment HJB equation given in recursive form,

$$\mathcal{O}(k)\left[M_n(t,x,k)\right] + nM_{n-1}(t,x,k)L(t,x,k) = 0 \tag{5}$$

Using (5) and the *Definition 2.1*, we have the following theorem which gives the necessary condition for the optimality. Consider on open set $Q_0 \subset (\bar{Q_0})$.

Theorem 3.1. Let $M_j(t,x) \in C_p^{1,2}(Q_0) \cap C(\bar{Q_0})$ be the j-th admissible moment cost function, assume the existence of an optimal controller $k^* \in K_M$ such that

$$M_j^*(t,x) = M_j(t,x,k^*) = \min_{k \in K_M} \left\{M_j(t,x,k)\right\},$$

then k^* and M_j^* satisfy the following HJB equation,

$$\mathcal{O}(k)\left[M_j^*(t,x)\right] + jM_{j-1}(t,x)L(t,x,k^*) = 0, \tag{6}$$

for $t \in T$, $x \in \mathbb{R}^n$ and the terminal condition is given as $M_j^*(t_f,x) = \psi(x(t_f))$.

Proof: See (Won et al., 2010).

The following Lemmas are used to obtain the n-th cumulant HJB equation.

Lemma 3.1. Consider two functions $M_i(t,x)$, $V_j(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$, where i and j are non-negative integers, then

$$\begin{aligned} &\mathcal{O}(k)\left[M_i(t,x)V_j(t,x)\right] \\ &= \mathcal{O}(k)\left[M_i(t,x)\right]V_j(t,x) + M_i(t,x)\mathcal{O}(k)\left[V_j(t,x)\right] \\ &+ \left(\frac{\partial M_i(t,x)}{\partial x}\right)' \sigma(t,x)W\sigma(t,x)' \left(\frac{\partial V_j(t,x)}{\partial x}\right). \end{aligned}$$

Lemma 3.2. Let $V_i(t,x), \ldots, V_{k-1}(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be admissible cost cumulant functions, then

$$\frac{1}{2}\sum_{s=1}^{k-1}\frac{k!}{s!(k-s)!}\left(\frac{\partial V_s}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{k-s}}{\partial x}\right) = \sum_{i=0}^{k-2}\frac{(k-1)!}{i!(k-1-i)!}\left(\frac{\partial V_{k-1-i}}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{i+1}}{\partial x}\right).$$

Proof: Omitted for brevity.

The main result of this paper is given in the following theorem. We use induction to prove the general n-th cumulant optimal control.

Theorem 3.2. *(n-th cumulant HJB equation)* Let $V_1(t,x), V_2(t,x), \ldots, V_{n-1}(t,x) \in C_p^{1,2}(Q_0) \cap C(\bar{Q}_0)$ be an admissible cumulant cost function for the control. Assume the existence of an optimal control law $k^*_{V_k|M} \in K_M$ and an optimal value function $V_n^*(t,x) \in C_p^{1,2}(Q_0) \cap C(\bar{Q}_0)$. Then the minimal n-th cumulant cost function $V_n^*(t,x)$ satisfies the following HJB equation,

$$0 = \min_{k\in K_M}\left\{ \mathcal{O}(k)[V_n^*] + \frac{1}{2}\sum_{s=1}^{n-1}\frac{n!}{s!(n-s)!}\left(\frac{\partial V_s}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{n-s}}{\partial x}\right)\right\}, \tag{7}$$

for $(t,x) \in \bar{Q}_0$, with the terminal condition $V_n^*(t_f,x) = 0$.

Proof: The mathematical induction method is used here to prove the theorem. We proved that when n=2, the second cost cumulant HJB equation satisfies (1). The third and fourth cumulant cases were proved in (Won et al., 2010). Thus, we assume that this theorem holds for the second, third and (n-1)-th cumulant case. We will show that the theorem also holds for the n-th cumulant case. Henceforth, the arguments of M and V are suppressed for brevity.

Let V_n^* be in the class of $C_p^{1,2}(Q) \cap C(\bar{Q})$. Apply the backward evolution operator $\mathcal{O}(k)$ to the recursive formula (4), we have

$$0 = \mathcal{O}(k)[V_n^*] - \mathcal{O}(k)[M_n] + \sum_{i=0}^{n-2}\frac{(n-1)!}{i!(n-1-i)!}\mathcal{O}(k)[M_{n-1-i}V_{i+1}]. \tag{8}$$

From *Theorem 3.1*, we have

$$\mathcal{O}(k)[M_n] + nM_{n-1}L = 0. \tag{9}$$

Then, from *Lemma 3.1*, and letting $i = n-1-i$, $j = i+1$, we obtain

$$\mathcal{O}(k)[M_{n-1-i}V_{i+1}] = \mathcal{O}(k)[M_{n-1-i}]V_{i+1} + M_{n-1-i}\mathcal{O}(k)[V_{i+1}] + \left(\frac{\partial M_{n-1-i}}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{i+1}}{\partial x}\right). \tag{10}$$

Substitute (9), (10) into (8), and we have

$$0 = \mathcal{O}(k)[V_n^*] + nM_{n-1}L + \sum_{i=0}^{n-2}\frac{(n-1)!}{i!(n-1-i)!}\left[\mathcal{O}(k)[M_{n-1-i}]V_{i+1} + M_{n-1-i}\mathcal{O}(k)[V_{i+1}] + \left(\frac{\partial M_{n-1-i}}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{i+1}}{\partial x}\right)\right]. \tag{11}$$

Use (9) again for $\mathcal{O}(k)[M_{n-1-i}]$ in (11) results in

$$0 = \mathcal{O}(k)[V_n^*] + nM_{n-1}L + \sum_{i=0}^{n-2}\frac{(n-1)!}{i!(n-1-i)!}\left[-(n-1-i)M_{n-2-i}V_{i+1}L + M_{n-1-i}\mathcal{O}(k)[V_{i+1}] + \left(\frac{\partial M_{n-1-i}}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{i+1}}{\partial x}\right)\right]. \tag{12}$$

Using the formula in (Stuart and Ord, 1987), $\frac{\partial M_i}{\partial V_j} = \frac{i!}{j!(i-j)!}M_{i-j}$, then

$$\frac{\partial M_{n-1-i}}{\partial x} = \sum_{j=1}^{n-1-i}\frac{(n-1-i)!}{j!(n-1-i-j)!}M_{n-1-i-j}\frac{\partial V_j}{\partial x}.$$

Thus, using the assumption that the theorem holds, from second order to $(n-1)$-th order cumulant case, (12) becomes

$$0 = \mathcal{O}(k)[V_n^*] - \sum_{i=1}^{n-2}\frac{(n-1)!}{i!(n-1-i)!}\left[M_{n-1-i}\sum_{j=0}^{i-1}\frac{i!}{j!(i-j)!}\left(\frac{\partial V_{i-j}}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{j+1}}{\partial x}\right)\right] + \sum_{i=0}^{n-2}\frac{(n-1)!}{i!(n-1-i)!}\left[\sum_{j=1}^{n-1-i}\frac{(n-1-i)!}{j!(n-1-i-j)!}M_{n-1-i-j}\left(\frac{\partial V_j}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{i+1}}{\partial x}\right)\right]. \tag{13}$$

From (13), we notice that both the second and third term on the right hand side contain the moment from $\{M_x\}$, where x = 1, 2, $\ldots$, $n-2$. Therefore, it is feasible to combine two summations with respect to M_x and simplify the equation. For derivation convenience, we use p and q instead of i and j within the bracket of the second term of (13),

$$M_{n-1-i}\sum_{j=0}^{i-1}\frac{i!}{j!(i-j)!}\left(\frac{\partial V_{i-j}}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial V_{j+1}}{\partial x}\right),$$

to distinguish the notations of the second and third term in (13). Note that the notations p, q and i, j are indeed equivalent. Therefore, (13) is rewritten as

$$O(k)[V_n^*] - \sum_{p=1}^{n-2} \frac{(n-1)!}{p!(n-1-p)!} \Bigg[M_{n-1-p} \sum_{q=0}^{p-1} \frac{p!}{q!(p-q)!} \left(\frac{\partial V_{p-q}}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_{q+1}}{\partial x}\right) \Bigg] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \Bigg[\sum_{j=1}^{n-2-i} \frac{(n-1-i)!}{j!(n-1-i-j)!} M_{n-1-i-j} \left(\frac{\partial V_j}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}}{\partial x}\right) \Bigg] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \Bigg[\left(\frac{\partial V_{n-1-i}}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}}{\partial x}\right) \Bigg] = 0. \tag{14}$$

Now, let us focus on the second and third terms of (14). We consider the second and third terms as functions with respect to M_x. We will compare the coefficients of the same M_x on second and third terms, i.e when $M_{n-1-p} = M_{n-1-i-j}$. When $k-1-p = k-1-i-j$, then $p = i+j$, then we will determine the corresponding coefficients associated with $M_{n-1-p} = M_{n-1-i-j}$ via the following procedure. In the second term of (14), the coefficient associated with M_{n-1-p} is

$$-\sum_{q=0}^{p-1} \frac{(n-1)!}{q!(n-1-p)!(p-q)!} \Bigg[\left(\frac{\partial V_{p-q}}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_{q+1}}{\partial x}\right) M_{n-1-p} \Bigg]. \tag{15}$$

For the third term in (14) because $p = i+j$, then for each p in [1, n-2], there are combinations of i and j, such that the summation of which are equal to p such as p=0+p, p=1+(p-1), ... and p=(p-1)+1. Because the range for index i is i = 0, 1, 2, ..., p-1, when we look for the coefficient of $M_{n-1-i-j}$ when $M_{n-1-i-j} = M_{n-1-p}$, we must find summation of the coefficient of $M_{n-1-i-j}$ for all combinations of i and j. Therefore, the corresponding coefficient is $\sum_{i=0, j=p=i}^{i=p-1} C_{M_{n-1-i-j}}$, where $C_{M_{n-1-i-j}}$ is the coefficient for each i and j which lead to the same k-1-i-j. Substitute $p = i + j$, i = 0, 1, 2, ..., p-1, j = p-i back to the third term of (14), then the coefficient associated with $M_{n-1-i-j}$ is

$$\sum_{i=0}^{p-1} \frac{(n-1)! M_{n-1-p}}{i!(n-1-i)!} \Bigg[\frac{(n-1-i)!}{(p-i)!(n-1-p)!} \left(\frac{\partial V_{p-i}}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}}{\partial x}\right) \Bigg]$$

$$= \sum_{i=0}^{p-1} \frac{(n-1)!}{i!(n-1-p)!(p-i)!} \Bigg[\left(\frac{\partial V_{p-i}}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}}{\partial x}\right) M_{n-1-p} \Bigg]. \tag{16}$$

Compare (15) and (16), we notice that since p is equivalent to i and q is equivalent to j, it is obvious that they have exactly format except for the signs. Thus they will cancel with each other when they are summed up. Therefore, the summation of the second and third term on (14) will be zero. Then we apply *Theorem 3.1*, (14) becomes (7). The theorem is proved. □

4 NEURAL NETWORK APPROX.

Neural network method based on series expansion approximate concept for HJB equations is applied to solve the value function of the HJB equations generated in the Section 3.

In our neural network approach, several neural network input function are multiplied by their corresponding weights and then summed up to produce output which is the approximated value function. In this paper, polynomial series expansion $\delta_L(x) = \{\delta_1(x), \delta_2(x), \ldots, \delta_L(x)\}'$ is the neural network input function while the weights of the series expansion is $\overline{w}_L(t) = \{w_1(t), w_2(t), \ldots, w_L(t)\}'$. These weights are time-dependent. The n-th cumulant value function is represented as $V_{nL}^*(x,t) = \overline{w}_L'(t)\delta_L(x) = \sum_{i=1}^{L} w_i'(t)\delta_i(x)$. The subscript L in the value function represents the order of the polynomial series. The higher the order of the series, the closer the approximate value gets to the real value.

In **Theorem 3.2**, the HJB equation for the n-th cumulant case is given. We need to solve this HJB equation to find the optimal controller. The optimal controller k^* for the n-th cumulant case has the form.

$$k^* = -\frac{1}{2} R^{-1} B' \left(\frac{\partial V_1}{\partial x} + \gamma_2 \frac{\partial V_2}{\partial x} \ldots + \gamma_n \frac{\partial V_n^*}{\partial x} \right), \tag{17}$$

with terminal condition $V_n^*(t_f, x) = 0$ where $\gamma_2, \gamma_3, \ldots, \gamma_n$ are the Lagrange multipliers. From (1), we assume that

$$f(t, x(t), k(t, x(t))) = g(t, x) + B(t, x) k(t, x),$$

where the matrices $R(t) > 0$, $B(t,x)$ are continuous real matrices. We substitute k^* back into the HJB equation from the first to n-th cumulant cases, then use neural network series expansion to approximate each HJB equation. Assuming terminal conditions for the value function $V_1, V_2, \ldots, V_n$ are zero.

Then the neural network approximations are used. We approximage V_1 by $V_{1L}(x,t) = \overline{w}'_{1L}(t)\delta_L(x)$, V_2 by $V_{2L}(x,t) = \overline{w}'_{2L}(t)\delta_L(x)$, etc. We obtain differential equations for the weigths. The n-th order one is given as an example.

$$\begin{aligned}
&\dot{\overline{w}}_{nL}(t) = \\
&-\langle\overline{\delta}_{nL}(x),\overline{\delta}_{nL}(x)\rangle_\Omega^{-1}\langle\nabla\overline{\delta}_{nL}(x)g(x),\overline{\delta}_{nL}(x)\rangle_\Omega\overline{w}_{nL}(t)\\
&+\frac{1}{2}\langle\overline{\delta}_{nL}(x),\overline{\delta}_{nL}(x)\rangle_\Omega^{-1}A_L\overline{w}_{nL}(t)\\
&+\sum_{i=2}^{n}\frac{\gamma_i^2}{2}\langle\overline{\delta}_{nL}(x),\overline{\delta}_{nL}(x)\rangle_\Omega^{-1}B_{iL}\overline{w}_{nL}(t)\\
&-\frac{1}{2}\sum_{i=1}^{n-1}\frac{n!}{i!(n-i)!}\langle\overline{\delta}_{nL}(x),\overline{\delta}_{nL}(x)\rangle_\Omega^{-1}F_{iL}\overline{w}_{(n-i)L}\\
&-\frac{1}{2}\langle\overline{\delta}_{nL}(x),\overline{\delta}_{nL}(x)\rangle_\Omega^{-1}G_L\overline{w}_{nL}.
\end{aligned} \tag{18}$$

The terminal conditions $\overline{w}_{1L}(t_f),\ldots,\ \overline{w}_{nL}(t_f)$ are assumed zero and the quantities A_L, B_{iL}, C_L, F_{iL} and G_L are defined as follows,

$$\begin{aligned}
A_L &= \sum_{s=1}^{L} w_{1s}(t)\langle\nabla\overline{\delta}_{1L}(x)BR^{-1}B'\nabla\overline{\delta}_{1s}(x)\overline{\delta}_{1L}(x)\rangle_\Omega,\\
B_{iL} &= \sum_{s=1}^{L} w_{is}(t)\langle\nabla\overline{\delta}_{iL}(x)BR^{-1}B'\nabla\overline{\delta}_{is}(x)\overline{\delta}_{1L}(x)\rangle_\Omega,\\
C_L &= \langle tr\left(\sigma W\sigma'\nabla\left(\nabla\overline{\delta}'_{1L}(x)\overline{w}_{1L}(t)\right)\right),\overline{\delta}_{1L}(x)\rangle_\Omega,\\
F_{iL} &= \sum_{s=1}^{L} w_{is}(t)\langle\nabla\overline{\delta}_{(n-i)L}(x)\sigma W\sigma'\nabla\overline{\delta}_{is}(x)\overline{\delta}_{nL}(x)\rangle_\Omega,\\
G_L &= \langle tr\left(\sigma W\sigma'\nabla\left(\nabla\overline{\delta}'_{nL}(x)\overline{w}_{nL}(t)\right)\right),\overline{\delta}_{nL}(x)\rangle_\Omega.
\end{aligned}$$

To solve the n-th cumulant neural network equation, we need to simultaneously solve all the n-th cumulant equation by converting the PDEs to neural network ODEs. Then, we solve the approximated ODEs with the corresponding Lagrange multipliers γ_2, γ_3, γ_4 to γ_n.

5 SIMULATION

In this section, we will use the neural network to calculate the first, second third and fourth cumulant optimal control for a satellite attitude control application studied in (Won, 1999). This is a linear system with quadratic cost function. We define the state variables as follows: $\overline{x} = [x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}] = [\phi,\theta,\psi,\omega_x,\omega_y,\omega_z,\Omega_1,\Omega_2,\Omega_3,\Omega_4]$. ϕ, ψ and θ are the *roll*, *yaw* and *pitch* Euler angles of the satellite. ω_x, ω_y and ω_z are the angular velocities of the satellite. Ω_1, Ω_2, Ω_3 and Ω_4 are the reaction wheel velocities. We consider the fourth cumulant control and analyze the system state space form and derive the HJB equations for the fourth cumulant control. Let $V_{1L}(x,t) = \overline{w}'_{1L}(t)\delta_L(x)$ to approximate $V_1(x,t)$, $V_{2L}(x,t) = \overline{w}'_{2L}(t)\delta_L(x)$ to approximate $V_2(x,t)$, etc. The basis function $\delta_L(x)$ is chosen from the expansion of a polynomial generating function $(x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_a)^2$ which contains 55 elements. Thus, the weights to be determined are defined as $\overline{w}_{1L}(t) = \{w_1,w_2,\ldots,w_{55}\}$, $\overline{w}_{2L}(t) = \{w_{56},w_{57},\ldots,w_{110}\}$, $\overline{w}_{3L}(t) = \{w_{111},w_{112},\ldots,w_{165}\}$, $\overline{w}_{4L}(t) = \{w_{166},w_{167},\ldots,w_{220}\}$. Using neural network method, we solve for the approximated $V_{1L}(x,t)$, $V_{2L}(x,t)$, $V_{3L}(x,t)$ and $V_{4L}(x,t)$.

Here we simulate the first four cost cumulant control, i.e n = 4. Because we use Lagrange multiplier method to derive the second, third and fourth cost cumulant HJB equations, we solve the HJB equations by assigning different values for the Lagrange multipliers γ_2, γ_3 and γ_4. In all simulation, we fix γ_2 = γ_3 = 0.001, while γ_4 vary from 0 to 0.0001. In Fig. 1(a), we observe that first cumulant V_1 increases with increases in γ_4 and the smallest V_1 is obtained when γ_4 is 0. For brevity we did not show second and third cumulant value functions as a function of *gamma*$_4$. The second cumulant V_2 decreases with with increases in γ_4 and reaches a minimum when γ_4 = 0.0001. This is related to the variance of the cost function. It is observed that V_3 decreases with increases in γ_4. Fig. 1(b) shows the fourth cumulant V_4 as γ_4 varies in value. It is observed V_4 decreases with increases in γ_4. Fig. 1(c) shows that the neural network weights converge to constants when integrated backward in time.

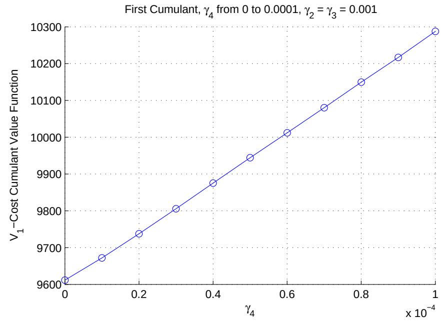
Depending on the desired statistical properties of the cost function, the engineer will choose the appropriate Lagrange multiplier values. In the satellite attitude control case, the mean and the variance are the important factors. Thus γ_4 equal to 0.0001 is a good choice. From the value functions, the corresponding optimal controller k^* is determined by substituting V_1, V_2, V_3 and V_4 back to the following equation,

$$k^* = -\frac{1}{2}R^{-1}B'\left(\frac{\partial V_1}{\partial x}+\gamma_2\frac{\partial V_2}{\partial x}+\gamma_3\frac{\partial V_3}{\partial x}+\gamma_4\frac{\partial V_4^*}{\partial x}\right). \tag{19}$$

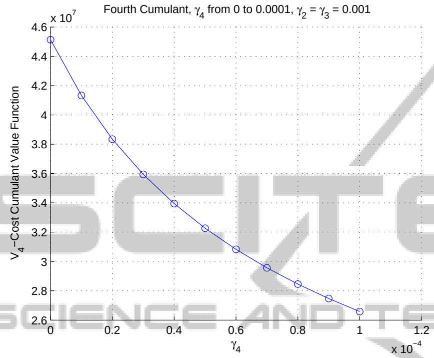
The optimal controller k^*(t,x) is the minimal fourth cumulant controller. Similarly, the minimal first cumulant controller (LQG control) is found by letting γ_2 = γ_3 = γ_4 = 0. Following similar approach, we can determine higher order cumulant optimal controller.

6 CONCLUSIONS

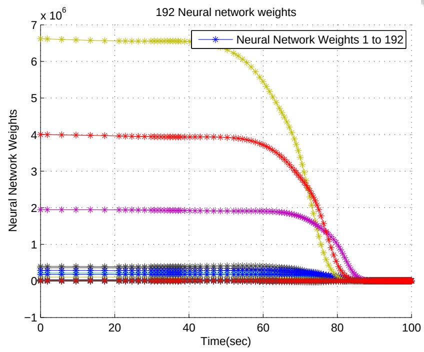
In this paper, we studied the statistical optimal control



(a) V1



(b) V4



(c) Weights

Figure 1: Value Functions and Neural Network Weights.

problem. Statistical control shapes the cost cumulants to improve the performance of the controller. We derived the necessary condition for minimizing the n-th cost cumulant for a given system. By shaping the density function, we improve the system performance. The HJB equation for the n-th cumulant minimization is derived. The optimal controller for the n-th cumulant minimization problems is found by solving the corresponding HJB equations. The n-th cumulant HJB equation is numerically solved using neural network method. In the satellite attitude control problem, the performance is shaped using the Lagrange multiplier, which decreased the fourth cumulant, while the first cumulant increased.

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REFERENCES

Alberkht, E. G. (1961). On the Optimal Stabilization of Nonlinear Systems. *PMM - Journal of Applied Mathematics and Mechanics*, 25:1254–1266.

Beard, R., Saridis, G., and Wen, J. (1997). Sufficient Conditions for the Convergence of Galerkin Approximations to the Hamilton-Jacobi Equation. *Automatica*, 33(12):2159–2177.

Chen, T., Lewis, F. L., and Abu-Khalaf, M. (2007). A Neural Network Solution for Fixed-Final Time Optimal Control of Nonlinear Systems. *Automatica*, 43:482–490.

Fleming, W. H. and Rishel, R. W. (1975). *Deterministic and Stochastic Optimal Control*. Springer-Verlag, New York.

Kang, B. and Won, C. (2010). Nonlinear Second Cost Cumulant Control using Hamilton-Jacobi-Bellman Equation and Neural Network Approximation. In *Proc. of the American Control Conference*, Baltimore, MD.

Lim, A. E. and Zhou, X. Y. (2001). Risk-Sensitive Control with HARA Utility. *IEEE Transactions on Automatic Control*, 46(4):563–578.

Sain, M. K. (1966). Control of Linear Systems According to the Minimal Variance Criterion—A New Approach to the Disturbance Problem. *IEEE Transactions on Automatic Control*, AC-11(1):118–122.

Sain, M. K. (1967). Performance Moment Recursions, with Application to Equalizer Control Laws. In *Proc. of Annual Allerton Conference on Circuit and System Theory*, Monticello, IL.

Smith, P. J. (1995). A Recursive Formulation of the Old Problem of Obtaining Moments from Cumulants and Vice Versa. *The American Statistician*, 49(2):217–218.

Stuart, A. and Ord, J. K. (1987). *Kendall's Advanced Theory of Statistics:Distribution Theory*. Oxford University Press, New York, 5th edition.

Won, C., Diersing, R. W., and Kang, B. (2010). Statistical Control of Control-Affine Nonlinear Systems with Nonquadratic Cost Function:HJB and Verification Theorems. *Automatica*, 46(10):1636–1645.

Won, C.-H. (1999). Comparative Study of Various Control Methods for Attitude Control of a LEO Satellite. *Aerospace Science and Technology*, 3(5):323–333.