Wave Vibration Analysis of Classical Multi-story Planar Frames

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Keywords: Wave Vibration, Multi-story Frame, Bending Vibration, Longitudinal Vibration.

Abstract: This paper concerns free vibration analysis of in-plane vibrations in classical multi-story planar frame structures. An exact analytical solution is obtained using wave vibration approach. The coupling effects between bending and longitudinal vibrations in frames are taken into account. Classical beam theories are applied in modeling the flexural and longitudinal vibrations. Reflection matrices at “sliding” and “rolling” boundaries, as well as reflection and transmission matrices at the “L” and “T” joints are obtained. Numerical examples are presented along with comparisons to results available in literature.

1 INTRODUCTION

Due to their complexity, vibrations in multi-story planar frame structures are often analyzed either based on approximated discrete models such as lumped mass/elasticity models, or using numerical approach such as the Finite Element Analysis (FEA) approach. There are very limited analytical studies based on distributed models found in the literature.

Lumped mass/elasticity may be suitable for finding the fundamental frequency of a multi-story frame, but they are prone to large errors and are therefore not suitable for identifying higher modes of vibration. The FEA approach is usually applied in modeling multi-story frames (Vertes, 1985, and Meirovitch, 2001), its accuracy is dependent upon the number of meshes or nodes per structural element. In general, higher modes demand more nodes, consequently the FEA approach is only suitable for relatively low frequencies. A branch mode method was developed for studying in-plane vibrations in multi-story frames (Gladwell, 1964) with longitudinal vibrations neglected in the analysis.

In this paper, vibrations in multi-story planar frame structures are obtained analytically from wave vibration standpoint (Graff, 1975; Cremer et. al., 1987, and Doyle, 1989). Reflection matrices at “sliding” and “rolling” boundaries, as well as reflection and transmission matrices at the “L” and “T” joints are discussed. The coupling effects of flexural and longitudinal motions at joints are taken into account. This study is based on the classical vibration theories, as a result, it is suitable to relatively low frequencies.

2 EQUATION OF MOTION AND WAVE PROPAGATION

Consider the forces and moments acting on a uniform element of a beam lying along the *x*-axis. When applying classical beam/rod related theories, the equations of motion for bending and longitudinal vibrations are (Inman, 1994; and Ginsberg, 2001)

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\rho A\frac{\partial^2 y(x,t)}{\partial t^2}=q(x,t), \quad (1a)$$

$$\rho A\frac{\partial^2 u(x,t)}{\partial t^2}-EA\frac{\partial^2 u(x,t)}{\partial x^2}=p(x,t), \quad (1b)$$

where x is the position along the beam axis, t is time, $y(x,t)$ and $u(x,t)$ are the transverse and longitudinal deflections of the centerline of the beam; $q(x,t)$ and $p(x,t)$ are the externally applied transverse and longitudinal forces; E and ρ are the Young’s modulus and mass density; respectively. I is the area moment of inertia of cross section, A is the cross-sectional area.

The shear force $V(x,t)$, bending moment $M(x,t)$, and longitudinal force $F(x,t)$ at any section of the beam are related to the transverse

Mei C..
Wave Vibration Analysis of Classical Multi-story Planar Frames.
DOI: 10.5220/0004035204880493
In *Proceedings of the 9th International Conference on Informatics in Control, Automation and Robotics* (ICINCO-2012), pages 488-493
ISBN: 978-989-8565-21-1

deflection $y(x,t)$, bending slope $\psi(x,t)$, and longitudinal deflection $u(x,t)$ by

$$V=-EI\frac{\partial^3 y(x,t)}{\partial x^3},\ M=EI\frac{\partial \psi(x,t)}{\partial x},\ F=EA\frac{\partial u(x,t)}{\partial x}, \qquad (2)$$

where $\psi=\frac{\partial y(x,t)}{\partial x}$ according to the classical Euler-Bernoulli beam theory.

2.1 Free Wave Propagation

First, consider the free bending vibration problem when no external force is applied to the beam. Assuming time harmonic motion and using separation of variables, the solution to Eq. (1a) can be written in the form $y(x,t)=y_0 e^{-ikx}e^{i\omega t}$, where ω is frequency and k is the wavenumber. A set of bending wavenumbers is found as

$$k^2=\pm\sqrt{\rho A\omega^2/EI}\,. \qquad (3)$$

Now consider the free longitudinal vibration problem when no external force is applied to the beam. Again assuming time harmonic motion and using separation of variables, the solution to Eq. (1b) can be written in the form $u(x,t)=u_0 e^{-ikx}e^{i\omega t}$.

The longitudinal wavenumber is found as

$$k=\pm\sqrt{\rho/E}\,\omega\,. \qquad (4)$$

2.2 Propagation Matrix

Consider two points A and B on a uniform beam a distance *x* apart. Waves propagate from one point to the other, with the propagation being determined by the appropriate wavenumber. Denoting the positive and negative going wave vectors at points A and B as $\mathbf{a}^+$ and $\mathbf{a}^-$ and $\mathbf{b}^+$ and $\mathbf{b}^-$, respectively, they are related by

$$\mathbf{a}^-=\mathbf{f}(x)\mathbf{b}^-;\ \mathbf{b}^+=\mathbf{f}(x)\mathbf{a}^+ \qquad (5)$$

where $\mathbf{f}(x)$ is the propagation matrix for a distance *x*.

3 REFLECTION AND TRANSMISSION OF COUPLED BENDING AND LONGITUDINAL WAVES

Waves incident upon discontinuities (such as boundaries and joints) are reflected and transmitted. In this section, reflection matrices at "sliding" and "rolling" boundaries, and reflection and transmission matrices at "L" and "T" joints are studied.

3.1 Wave Reflection at Boundaries

An incident wave is reflected at a boundary, as shown in Figure 1. The incident wave $\mathbf{a}^+$ and the reflected wave $\mathbf{a}^-$ are related through the reflection matrix **r** by

$$\mathbf{a}^-=\mathbf{r}\mathbf{a}^+, \qquad (6)$$

where **r** can be determined by considering equilibrium at the boundary.

For "sliding" boundary, the equilibrium conditions at the boundary are

$$\psi=0,\ V(x,t)=0,\ u(x,t)=0. \qquad (7)$$

The equilibrium conditions at a "rolling" boundary are

$$y(x,t)=0,\ M(x,t)=0,\ F(x,t)=0. \qquad (8)$$

The reflection matrices at classical boundaries such as clamped and free boundaries are derived in (Mei, 2010).

3.2 Wave Reflection and Transmission at an "L" Joint

Wave transmission and reflection at an angle joint in general introduce wave mode conversion. At an "L" joint, for example, an incident bending wave induces reflected and transmitted bending and axial waves in the members attached to the joint. This is evident from the coupled equilibrium and continuity relations below.

Figure 2 shows the free body diagram of an "L" joint in planar motion. The equilibrium conditions are

$$\begin{aligned} F_2-V_1&=m\ddot{y}_J \\ -V_2-F_1&=m\ddot{u}_J \\ M_2-M_1+V_1\frac{h_2}{2}+V_2\frac{h_1}{2}&=J\ddot{\psi}_J \end{aligned} \qquad (9)$$

where *F* is the axial force in the beam and *h* the beam thickness. Subscripts 1 and 2 refer to beam 1 and beam 2, u_J, y_J, and ψ_J are the displacements and rotation of the joint as indicated in the figure. The first two of these equations include the mass of the joint, while the third includes the moment of inertia of the joint and the moments induced by the off-set shear forces.

Figure 1: Sliding (a) and rolling (b) boundaries.

The continuity equations at the joint are

$$u_1 = u_J\,, u_2 = y_J\,, y_1 = y_J - \frac{h_2}{2}\psi_J\,,$$
$$y_2 = -u_J + \frac{h_1}{2}\psi_J\,, \psi_1 = \psi_J\,, \psi_2 = \psi_J \tag{10}$$

A set of positive going waves $\mathbf{a}^+$ incident upon the L-joint from one beam gives rise to transmitted and reflected waves $\mathbf{b}^+$ and $\mathbf{b}^-$, which are related to the incident waves through the transmission and reflection matrices **t** and **r** by

$$\mathbf{b}^+ = \mathbf{T}\mathbf{a}^+\,,\ \mathbf{a}^- = \mathbf{R}\mathbf{a}^+\,. \tag{11}$$

The transmission and reflection matrices $\mathbf{T}_{12}$ and $\mathbf{R}_{11}$ corresponding to an incident wave from beam 1 and the transmission and reflection matrices $\mathbf{T}_{21}$ and $\mathbf{R}_{22}$ corresponding to an incident wave from Beam 2 can be obtained from solving Eqs. (9) to (11).

3.2 Wave Reflection and Transmission at a “T” Joint

Similarly, wave transmission and reflection at a “T” joint also introduce wave mode conversion. The transmission and reflection matrices are obtained from considering the continuity and equilibrium conditions at the joint. The free body diagram of a “T” joint in planar motion is shown in Figure 3. The continuity equations at the joint are

$$u_1 = u_J,\ u_2 = y_J,\ u_3 = u_J,\ y_1 = y_J - \psi_J h_2/2,$$
$$y_2 = -u_J + \psi_J h_1/2,\ y_3 = y_J + \psi_J h_3/2, \tag{12}$$
$$\psi_1 = \psi_J,\ \psi_2 = \psi_J,\ \psi_3 = \psi_J.$$

The equilibrium conditions are

$$V_3 + F_2 - V_1 = m\ddot{y}_J,\ F_3 - V_2 - F_1 = m\ddot{u}_J,$$
$$M_3 + M_2 - M_1 + V_3\frac{h_2}{2} + V_1\frac{h_2}{2} + V_2\frac{h_1}{2} = J\ddot{\psi}_J\,. \tag{13}$$

There exist three sets of reflection and trans- mission relations, corresponding to incident waves from each of the three beam elements respectively. The reflection and transmission relations can be found from Eqs. (12) and (13). More details can be found in (Mei, 2010).

4 WAVE VIBRATION ANALYSIS OF A MULTISTORY FRAME

For a multi-story frame that is symmetrical about a vertical line through the centers of the spans, the vibration modes are either symmetrical or anti-symmetrical. It has been shown that vibrating in symmetrical modes, the mid-points of the cross-members behave as sliding ends; while vibrating in anti-symmetrical modes, the mid-points of the cross-members behave as rolling ends (pinned vertically but allowing translational motion in the horizontal direction). As a result, when the frame is vibrating in a symmetrical mode, each half of it will have the same modal form as the isolated half frame shown in Figure 4(a); and when it is vibrating in one of its anti-symmetrical modes, each half of it can be treated as if it is an isolated half-frame having the form shown in Figure 4(b) (Gladwell, 1964, and Bishop and Johnsona, 1960).

4.1 Free Wave Vibration Analysis

From wave vibration standpoint, vibrations propagate along a uniform waveguide (or structural element), and are reflected and transmitted at discontinuities (such as joints and boundaries). Assembling these propagation, reflection, and transmission matrices offers a concise and systematic approach for analyzing coupled bending and longitudinal vibrations in a multi-story frame structure.

Figure 4(a) illustrates an *n*-story frame vibrating at its symmetrical modes. The half frame model consists of *n* horizontal and *n* vertical beam elements. And the discontinuities in the half frame model include one “L” joint, (*n-1*) “T” joints, one classical boundary, and *n* sliding boundaries.

- The *n* pairs of propagation relations along the uniform vertical beam elements are

$$\mathbf{a}_i^+ = \mathbf{f}(L)\mathbf{A}_{i-1}^+\,,\ \mathbf{A}_{i-1}^- = \mathbf{f}(L)\mathbf{a}_i^-\,, \tag{14a}$$

where i = 1, 2, ..., n.

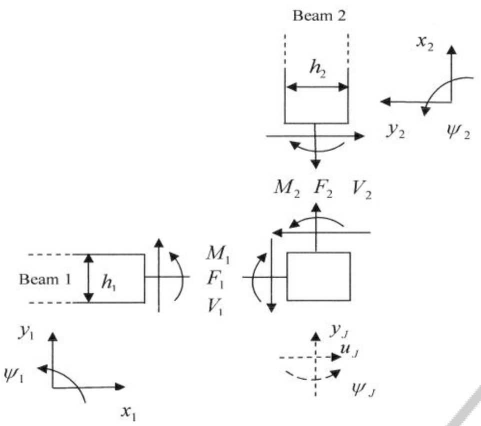

Figure 2: Free body diagram of an "L" joint.

- The n pairs of propagation relations along the uniform horizontal beam elements are

$$\mathbf{c}_{Ri}^{+}=\mathbf{f}(L_H/2)\mathbf{c}_{Li}^{+},\ \mathbf{c}_{Li}^{-}=\mathbf{f}(L_H/2)\mathbf{c}_{Ri}^{-},\qquad (14b)$$

where i = 1, 2, ..., n.

- The reflection and transmission relations of the waves at "T" joints are

$$\begin{aligned}\mathbf{a}_i^{-}&=\mathbf{r}_{11}\mathbf{a}_i^{+}+\mathbf{t}_{31}\mathbf{A}_i^{-}+\mathbf{t}_{21}\mathbf{c}_{Li}^{-},\\ \mathbf{A}_i^{+}&=\mathbf{r}_{33}\mathbf{A}_i^{-}+\mathbf{t}_{13}\mathbf{a}_i^{+}+\mathbf{t}_{23}\mathbf{c}_{Li}^{-},\\ \mathbf{c}_{Li}^{+}&=\mathbf{r}_{22}\mathbf{c}_{Li}^{-}+\mathbf{t}_{12}\mathbf{a}_2^{-}+\mathbf{t}_{32}\mathbf{A}_1^{-},\end{aligned}\qquad (15c)$$

where i = 1, 2, ..., n-1.

- The reflection and transmission relations of the waves at "L" joint are

$$\begin{aligned}\mathbf{c}_{Ln}^{+}&=\mathbf{R}_{22}\mathbf{c}_{Ln}^{-}+\mathbf{T}_{12}\mathbf{a}_n^{+},\\ \mathbf{a}_n^{-}&=\mathbf{R}_{11}\mathbf{a}_n^{+}+\mathbf{T}_{21}\mathbf{c}_{Ln}^{-}.\end{aligned}\qquad (15d)$$

- The reflection relations at the sliding boundaries are

$$\mathbf{c}_{Ri}^{-}=\mathbf{r}_{sliding}\mathbf{c}_{Ri}^{+},\qquad (15e)$$

where i = 1, 2, ..., n.

- The reflection at the classical boundary is

$$\mathbf{A}_0^{+}=\mathbf{r}_0\mathbf{A}_0^{-}.\qquad (15f)$$

Writing Eqs. (13) in matrix form gives

$$\mathbf{Az}=\mathbf{0},\qquad (16)$$

where **A** is a (24n) by (24n) coefficient matrix and **z** is a 24n wave component vector. Setting the determinant of the coefficient matrix **A** to zero gives the natural frequencies of the multi-story frame.

Figure 4(b) illustrates an n-story frame vibrating at its anti-symmetrical modes. The analysis follows

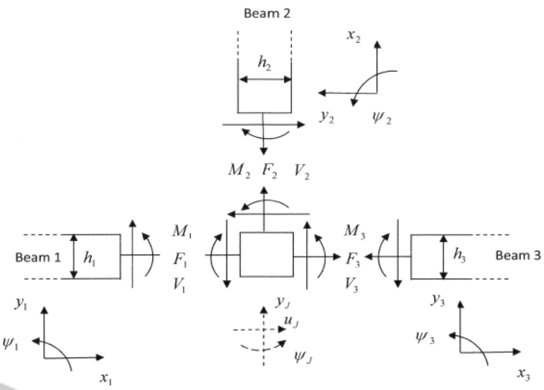

Figure 3: Free body diagram of a "T" joint.

a similar procedure, the only difference is that the n sliding boundaries are replaced by n rolling boundaries.

4.2 Numerical Examples

Two example multi-story frame structures are studied, one being a three-story frame and the other a two-story frame. For comparison purpose, the physical properties of the three-story frame are chosen to be the same as those in (Vertes, 1985) and they are as follows: Lengths of vertical and horizontal beams are 6.0m and 8.0m respectively, cross sectional area $A=0.3m^2$, area moment of inertia $I=0.01m^4$, Young's modulus $E=210GN/m^2$, and mass density $\rho=25kN/m^3$. The physical properties of the two-story frame are chosen to be the same as those in (Petyt, 1990) and they are as follows: Lengths of vertical and horizontal beams are 22.86cm and 45.72cm respectively, the cross section of the beam elements is 0.3175cm by 1.27cm, Young's modulus E is $206.84GN/m^2$, and mass density ρ is $7830kg/m^3$. The boundary conditions are clamped-clamped.

The natural frequencies of the two example frames are listed in Tables 1 and 2, with comparisons to the related references respectively. The examples show good agreement with the results presented in the available literature.

5 CONCLUSIONS

In this paper, in-plane vibrations in multi-story planar frames are analyzed using the wave approach. The vibrations are modeled using classical vibration theories. The coupling effect between bending and

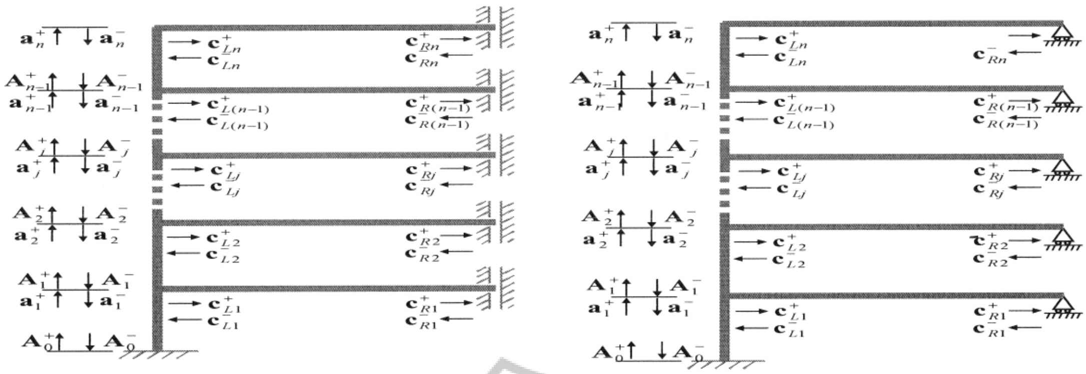

Figure 4: Half frames for symmetrical (a), and anti-symmetrical (b) mode analysis.

Table 1: Natural frequencies of the 3-story frame. (superscript a denotes anti-symmetrical modes).

Modes	Natural frequencies (rad/s)	
	Present	(Vertes,1985)
1	11.7[a]	11.7
2	39.4[a]	39.4
3	73.1[a]	73.2
4	104.0	N/A
5	121.8	121.0

Table 2: Natural frequencies of the 2-story frame.

Modes	Natural frequencies (Hz)		$\Omega = \left(\sqrt[4]{\frac{\rho A}{EI}\omega^2} L \right)^2$	
	Present	(Petyt, 1990)	Present	(Gladwell, 1964)
Anti-symmetrical	15.1421	15.14	1.0554	1.0554
	53.3183	53.32	3.7164	3.7165
	155.3018	155.48	10.8248	10.8262
	186.1038	186.51	12.9717	12.9819
	270.0581	270.85	18.8235	18.8256
	345.8450		24.1060	24.1133
	590.5131		41.1597	41.2266
	652.7439	N/A	45.4973	45.5200
	794.7385		55.3946	55.4848
	905.8716		63.1407	N/A
Symmetrical	56.1226		3.9118	3.9124
	67.2203		4.6854	4.6866
	212.5325		14.8139	14.8186
	291.5674		20.3227	20.3441
	381.6300	N/A	26.6002	26.6516
	410.3624		28.6029	28.6143
	699.3279		48.7443	48.7989
	834.6623		58.1773	58.3556
	986.1119		68.7336	68.9144

longitudinal vibrations is taken into account. Reflection matrices at "sliding" and "rolling" boundaries, as well as reflection and transmission matrices at the "L" and "T" joints are discussed. With the availability of the propagation, reflection, and transmission matrices, vibration analysis of multi-story planar frames becomes systematic and concise: it involves a simple assembly of the involved matrices. The procedures are illustrated using two numerical examples, both show good agreement with the results presented in the available literature.

ACKNOWLEDGEMENTS

The author gratefully acknowledges the support on this project from the CMMI Division of the NSF through Grant #0825761.

REFERENCES

Bishop R. E. D. and Johnson D. C., 1960, The Mechanics of Vibration, *Cambridge University Press*, Great Britain.

Cremer L., Heckl M. and Ungar E. E., 1987, Structure-Borne Sound, Berlin: Springer-Verlag.

Doyle J. F., 1989, Wave Propagation in Structures, New York: Spring – Verlag.

Ginsberg J. H., 2001, Mechanical and Structural Vibrations, New York: John Wiley and Sons, Inc.

Gladwell G. M. L., 1964, "The Vibration of Frames," *Journal of Sound and Vibration*, 4, pp. 402-465.

Graff K. F., 1975, Wave Motion in Elastic Solids, Ohio *State University Press*.

Inman D. J., 1994, Engineering Vibrations, New Jersey: Prentice-Hall, Inc.

Mei C., 2010, "In-plane vibrations of classical planar frame structures – an exact wave-based analytical solution," *Journal of Vibration and Control*, 16(9), pp. 1265-1285.

Meirovitch L., 2001, Fundamentals of Vibrations, McGraw Hill, New York.

Petyt M., 1990, Introduction to Finite Element Vibration Analysis, *Cambridge University Press*, Great Britain.

Vertes G., 1985, Structural Dynamics, *Elsevier Science Publishers B.V.*, The Netherlands.