Applying Hyperbolic Wavelets in Frequency Domain Identification

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Keywords: System Identification, Discrete–time Systems, Frequency–domain Representations, Wavelets, Hyperbolic Geometry.

Abstract: The paper elaborates a hyperbolic wavelet construction for representing signals in the Hardy space H^2 on the unit disc. An efficient computing scheme based on the matrix form of the representation is worked out. The wavelet coefficients can be computed on the basis of discrete time–domain measurements. This wavelet is used to reconstruct poles of functions in H^2 as the basis of nonparametric frequency–domain identification of discrete–time signals and systems.

1 INTRODUCTION

Representations of discrete-time signals and systems in the frequency domain are used in many fields of science and technology, e.g. in detection and changes in systems, system identification, and control design. The stable representations of signals and systems of finite energy result in complex analytic functions defined on the unit disc of the Hardy space H^2. The identification of H^2 signals is usually based on physical measurements in the time–domain. Convenient methods for system identification can be obtained in the case when an orthogonal basis of the space H^2 is used. A well-known orthonormal basis in H^2 is the trigonometric system that forms the basis of classical Fourier–transform representations and associated identification methods. Orthogonal bases can also be generated by rational functions and this concept leads to rational orthogonal bases (ROBs) that have gained great significance besides H^2 also in H^∞ system identification (Heuberger et al., 2005). Application of ROBs requires *a priori* information on the locations of system poles. This paper elaborates a method to obtain representations of H^2 functions that does not use strict *a priori* assumptions. A promising opportunity to realize this arises from some wavelet-type construction that utilize the hyperbolic geometry generated by the so-called Blaschke functions. The goal is to apply hyperbolic wavelet methods to identify poles of functions in H^2.

2 RATIONAL ORTHOGONAL BASES

The Blaschke function in $H^2(\mathbb{D})$ is defined as

$$B_b(z) := \frac{z-b}{1-\overline{b}z} \quad (z \in \mathbb{C}, b \in \mathbb{D}),$$

where b is called the parameter of the Blaschke-function. The parameter b is identical to the zero and $b^* = 1/\overline{b}$ is the pole of B_b.

The most important feature of the Blaschke function is that $B_b : \mathbb{T} \to \mathbb{T}$ and $B_b \mathbb{D} \to \mathbb{D}$ are bijections, as a consequence the Blaschke functions to be inner functions in the space $H^2(\mathbb{D})$.

The discrete Laguerre-system is complete orthonormed system in $H^2(\mathbb{D})$ defined by

$$\phi_n(z) = \frac{\sqrt{1-|b|^2}}{1-\overline{b}z} B_b^n(z), \quad (n = 0, 1, \ldots).$$

If the pole locations of the system are exactly known one obtains finite rational representations (Soumelidis et al., 2002b). Rational orthogonal bases have intensively been discussed in the context of H^2 and H^∞ identification of systems (Heuberger et al., 2005), and efficient methods have been elaborated that solved the identification problem in the case when — at least approximately — the pole locations are known. Special attention paid on the problems of pole selection and validation (Bokor et al., 1999; e Silva, 2005) as well as methods have been found to refine the pole locations starting from an approximate placement (Soumelidis et al., 2002a), however the general problem identifying poles has not been solved so far.

Soumelidis A., Bokor J. and Schipp F..
Applying Hyperbolic Wavelets in Frequency Domain Identification.
DOI: 10.5220/0004043705320535
In *Proceedings of the 9th International Conference on Informatics in Control, Automation and Robotics* (ICINCO-2012), pages 532-535
ISBN: 978-989-8565-21-1

3 HYPERBOLIC WAVELETS

It is known that the Blaschke functions form a group with respect to the function composition, i.e. $(B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2})(z) := B_{\mathfrak{b}_1}(B_{\mathfrak{b}_2}(z))$. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2} = B_{\mathfrak{b}_1 \circ \mathfrak{b}_2}$. The $(\mathbb{B}, \circ)$ results in a group isomorphic with the group group of the Blaschke functions. The neutral element of the group $(\mathbb{B}, \circ)$ is $\mathfrak{e} := (0,1) \in \mathbb{B}$ and the inverse element of $\mathfrak{b} = (b, \varepsilon) \in \mathbb{B}$ is $\mathfrak{b}^{-1} = (-b\varepsilon, \overline{\varepsilon})$.

This group can be associated with the congruence transforms of the Poincaré model of the hyperbolic geometry (see e.g. (Ahlfors, 1973)). It can be proved that the map

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - \overline{z}_1 z_2|} = |B_{z_1}(z_2)|$$
$$(B_{z_1} := B_{(z_1,1)}, z_1, z_2 \in \mathbb{D})$$

is a metric on $\mathbb{D}$, called pseudohyperbolic metric (Ahlfors, 1973). Moreover the Blaschke functions B_b $(b \in \mathbb{D})$ are isometries with respect to this metric, i.e.

$$\rho(B_b(z_1), B_b(z_2)) = \rho(z_1, z_2)$$
$$(b \in \mathbb{D}, z_1, z_2 \in \mathbb{D}).$$

It is also well–known that the Hardy space $H := \mathrm{H}^2(\mathbb{D})$ is Hilbert space with respect to the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \overline{g}(e^{it})\, dt \quad (f, g \in H),$$

and the power functions $h_n(z) := z^n$ $(z \in \mathbb{C}, n \in \mathbb{N})$ form an orthonormal basis in the space. By defining the multiplier function

$$R_{\mathfrak{b}}(z) := \frac{\sqrt{\varepsilon(1 - |b|^2)}}{1 - \overline{b}z}$$
$$(z \in \overline{\mathbb{D}}, \mathfrak{b} := (b, \varepsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}),$$

introduce the mapping

$$U_{\mathfrak{b}} f := R_{\mathfrak{b}^{-1}} f \circ B_{\mathfrak{b}^{-1}} \quad (\mathfrak{b} \in \mathbb{B}, f \in H). \qquad (1)$$

$(U_{\mathfrak{b}}, \mathfrak{b} \in \mathbb{B})$ can be considered as a unitary representation of the group $(\mathbb{B}, \circ)$ on the Hilbert space H with properties

(i) $U_{\mathfrak{b}_1}(U_{\mathfrak{b}_2} f) = U_{\mathfrak{b}_1 \circ \mathfrak{b}_2} f \quad (\mathfrak{b}_1, \mathfrak{b}_2 \in \mathbb{B}, f \in H)$,

(ii) $\|U_{\mathfrak{b}} f\| = \|f\| \quad (f \in H, \mathfrak{b} \in \mathbb{B})$,

(iii) $\mathfrak{b} \to U_{\mathfrak{b}} f \in H$ $(f \in H, \mathfrak{b} \in \mathbb{B})$ is continuous.

See for proofs in (Pap and Schipp, 2006), and an introduction to the unitary group representations in (Wawrzyńczyk, 1984). From the properties (i) to (iii) follows that $U_{\mathfrak{b}}$ maps any complete orthogonal system in H into complete orthogonal system in the same space. Particularly the system

$$L_n^b := U_{\mathfrak{b}^{-1}} h_n \quad (n \in \mathbb{N}, \mathfrak{b} := (b, 1) \in \mathbb{B})$$

form an orthogonal basis in H that is called *discrete Laguerre system*.

The unitary group representations allow us to introduce the concept of the *wavelets* in the Hilbert space H (Goupillaud et al., 1984), (Meyer, 1990), and (Daubechies, 1992). The continuous wavelet transform on a function $f \in \mathrm{L}^2(\mathbb{R})$ is formed by taking translation and dilation of a function ψ named the *mother wavelet*; the integral operator with the kernel

$$\psi^{pq}(x) := \frac{\psi((x-q)/p)}{\sqrt{p}}, \; x \in \mathbb{R},$$

$p \in (0, \infty), q \in \mathbb{R}$ is called *wavelet transform*:

$$(\mathcal{W}_\psi f)(p, q) := \frac{1}{\sqrt{p}} \int_{\mathbb{R}} f(x) \overline{\psi} \left(\frac{x-q}{p} \right) dx =$$
$$= \langle f, \psi^{pq} \rangle \quad (f \in \mathrm{L}^2(\mathbb{R})),$$

where $\langle \cdot, \cdot \rangle$ means the inner product of the Hilbert-space $\mathrm{L}^2(\mathbb{R})$. Using the unitary representation $U_{\mathfrak{b}}$ $(\mathfrak{b} \in \mathbb{B})$ defined by (1) one obtains

$$(\mathcal{W}_\varphi f)(\mathfrak{b}) = \langle f, U_{\mathfrak{b}} \varphi \rangle \quad (f, \varphi \in \mathrm{H}^2(\mathbb{D}), \mathfrak{b} \in \mathbb{B}),$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the Hardy space $\mathrm{H}^2(\mathbb{D})$. This construction can be referred as *Blaschke* or *hyperbolic wavelet*.

Particularly the hyperbolic wavelets generated by the power–functions

$$\varepsilon_n(t) := e^{\mathrm{i}nt} \quad (n \in \mathbb{Z}, t \in \mathbb{R}),$$

can be interpreted as the Laguerre–Fourier coefficients, i.e. the Laguerre representation of any function $f \in H$ can be considered as a hyperbolic wavelet transform.

Any function $f \in H$ can be expressed in the trigonometrical system in the form

$$f = \sum_{n=0}^{\infty} \widehat{f}(n) \varepsilon_n(t),$$

where

$$\widehat{f}(n) := \langle f, \varepsilon_n \rangle \quad (n \in \mathbb{N})$$

is the n-th trigonometric Fourier–coefficients of the function f. Consequently the Fourier–coefficients of

$$F_{\mathfrak{b}} := U_{\mathfrak{b}} f = \sum_{n=0}^{\infty} \widehat{f}(n) U_{\mathfrak{b}} \varepsilon_n$$

can be obtained as

$$\widehat{F}_{\mathfrak{b}}(m) := \langle U_{\mathfrak{b}} f, \varepsilon_m \rangle =$$
$$= \sum_{n=0}^{\infty} \langle U_{\mathfrak{b}} \varepsilon_n, \varepsilon_m \rangle \widehat{f}(n) = \sum_{n=0}^{\infty} u_{mn}(\mathfrak{b}) \widehat{f}(n),$$

where $u_{mn}(\mathfrak{b}) = \langle U_{\mathfrak{b}}\varepsilon_n, \varepsilon_m\rangle$ $((m,n) \in \mathbb{N}^2)$ form the matrix of the representation $U_{\mathfrak{b}}$ in the trigonometrical basis. By introducing the matrix

$$\mathfrak{U}_{\mathfrak{b}} = [u_{mn}(\mathfrak{b})]_{(m,n)\in\mathbb{N}^2},$$

the mapping $f \to F_{\mathfrak{b}}$ can be expressed in the space of the Fourier–coefficients with the matrix–transform

$$\widehat{F}_{\mathfrak{b}} = \mathfrak{U}_{\mathfrak{b}}\widehat{f} \quad (f \in H, \mathfrak{U}_{\mathfrak{b}} = \{u_{mn}(\mathfrak{b})\}). \qquad (2)$$

Since the transform is unitary, i.e. $U_{\mathfrak{b}^{-1}} = U_{\mathfrak{b}}^{-1} = U_{\mathfrak{b}}^{*}$, the elements

$$u_{mn}(\mathfrak{b}) = \langle U_{\mathfrak{b}}\varepsilon_n, \varepsilon_m\rangle = \overline{u_{mn}(\mathfrak{b})} \quad (m,n \in \mathbb{N}).$$

can be expressed by the Jacobi–polynomials, i.e. $u_{mn}(\mathfrak{b}) =$

$$= (-1)^m\sqrt{1-r^2}e^{\mathrm{i}(n-m)\alpha}r^{|n-m|}P_{\min\{m,n\}}^{(0,|n-m|)}(2r^2-1).$$

The Fourier–coefficients in $\widehat{f}$ correspond to discrete–time signal data points that can be interpreted as the uniformly sampled form of the continuous–time physical signals. Computation of the elements in $\mathfrak{U}_{\mathfrak{b}}$ can be performed by using recursions for any parameter selection $\mathfrak{b} \in \mathbb{B}$, and in advance to taking the measurements.

4 IDENTIFYING POLES

Suppose that the system under consideration contains only a single pole of multiplicity 1, in this case the conjugated Laguerre–Fourier coefficients are given as $\widehat{F}_{\mathfrak{b}}(m) = L_m^b(a)$, and the quotients

$$q_m(b) = \frac{\widehat{F}_{\mathfrak{b}}(m+1)}{\widehat{F}_{\mathfrak{b}}(m)} = B_b(a) \quad (m \in \mathbb{N}),$$

form a constant sequence and its elements equal to a Blaschke function applied to a. This fact can be used to identify the position of inverse pole a,

$$a = B_{b^{-1}}(q_m(b)),$$

where $B_{b^{-1}}$ is the inverse of B_b, i.e. a is given by applying a hyperbolic transform corresponding to the inverse group element belonging to b.

This concept can be extended to multiple poles, it will be shown that in the case of multiple poles there exist a region $D_i \in \mathbb{D}$ where the sequence of the quotients generated by the conjugated Laguerre–Fourier coefficients converge. A theorem can be set up as follows:

Theorem 1. *For any rational function f in any point b of D the limit*

$$(\mathcal{Q}f)(b) := \lim_{n\to\infty} q_m(b) \quad (f \in \mathfrak{R})$$

exists, and

$$(\mathcal{Q}f)(b) = B_b(a_i), \quad b \in D_i \quad (i = 1,2,\ldots,P).$$

In the case of poles of multiplicity 1 *for the speed of convergence the estimation*

$$|q_m(b) - B_b(a_i)| = O(q_i^n) \ (n \in \mathbb{N}, b \in D_i, q_i < 1)$$

can be given.

The proof can be found in (Schipp and Soumelidis, 2011).

The result of Theorem 1 can be used to reconstruct the poles of function f by

$$B_b^{-1}((\mathcal{Q}f)(b)) = a_i \quad (b \in D_i, i = 1,2,\cdots,P). \quad (3)$$

By this way all the poles can be reconstructed that possess nonempty region, i.e $D_i \neq \emptyset$. The procedure goes like this:

1. Estimation of the Laguerre–Fourier coefficients belonging to parameter b based on measurements.
2. Reconstruction the poles as a limit of quotients of consecutive Laguerre–Fourier coefficients.

The estimation of the Laguerre–Fourier coefficients of function f with parameter b can efficiently be computed according to the form (2) applied on the time–domain signal measurements. Finding multiple poles can be done by selecting a sequence of parameters b arranged randomly or in arbitrary order.

5 A NUMERICAL EXAMPLE

The identification of the poles of a simulated function is presented. The set of (inverse) poles belonging to the function is $\{a_1 = 0.8, a_{2,3} = 0.8 * e^{\pm\mathrm{i}\frac{\pi}{4}}\}$ with the associated residues $\{\lambda_1 = 1.5, \lambda_{2,3} = 1\}$.

Figure 1 presents a visualization of the iteration processes for finding specific poles. The sequence given by (3) is drawn in the complex plane by white points. The sequence converges towards pole designated by a_2. The convergence can be checked on the lower two diagrams in Figure 2where the absolute value and the phase of the sequences against the indices can be seen. The upper diagram in these figures depicts the absolute value of the Laguerre-Fourier coefficients belonging to the specific selection of b. The reconstruction error – defined as a root-mean-square difference – is in the magnitude $10^{-5} \ldots 10^{-7}$.

Figure 1 also presents the regions D_i that belongs to the poles a_i. Poles a_1 and a_3 can be identified by selecting parameter b within the other two regions.

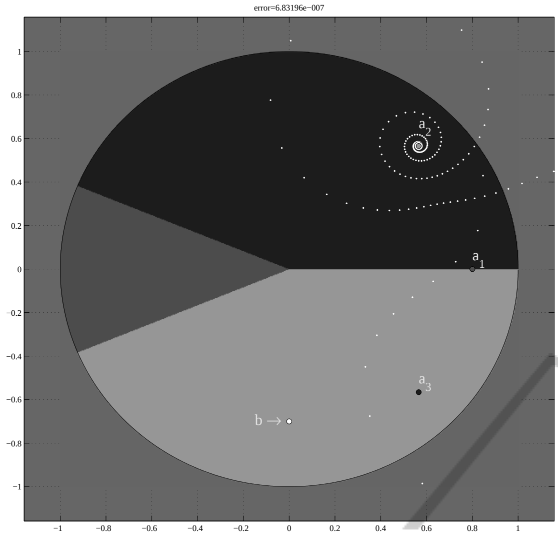

Figure 1: Finding pole a_2.

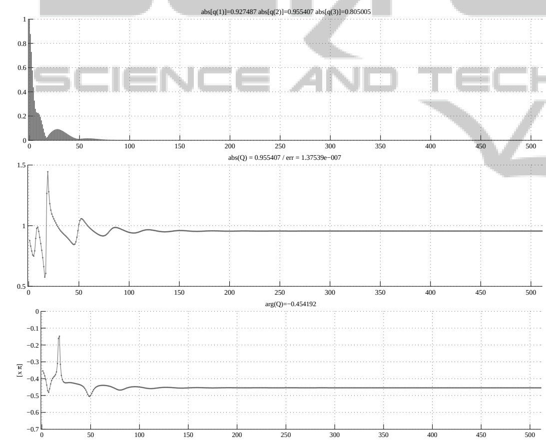

Figure 2: Modulus of the L-F coefficients, modulus and phase of sequence q_n.

6 CONCLUSIONS

A hyperbolic wavelet concept for representing signals belonging to the space of functions H^2 on the unit disc has been constructed, and an efficient computing scheme based upon the matrix form of the representation has been elaborated. The wavelet coefficients can be computed on the basis of discrete time–domain measurements. The wavelet construct can be used in reconstructing poles belonging to functions in $\mathrm{H}^2(\mathbb{D})$, which forms the basis of nonparametric frequency–domain identification of discrete–time signals and systems.

ACKNOWLEDGEMENTS

This work is supported by the Control Engineering Research Group of HAS at Budapest University of Technology and Economics. The European Union and the European Social Fund have provided financial support to the project under the grant agreement no. TÁMOP-4.2.1./B-09/1/KMR-2010-0003.

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