Robust Stability Analysis of a Class of Delayed Neural Networks

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Abstract: This paper studies the global robust stability of delayed neural networks. A new sufficient condition that ensures the existence, uniqueness and global robust asymptotic stability of the equilibrium point is presented. The obtained condition is derived by using the Lyapunov stability and Homomorphic mapping theorems and by employing the Lipschitz activation functions. The result presented establishes a relationship between the network parameters of the neural system independently of time delays. We show that our results is new and improves some of the previous global robust stability results expressed for delayed neural networks.

1 INTRODUCTION

In recent years, neural networks proved to be a useful system which has been successfully applied to various practical engineering problems such as optimization, image and signal processing, and associative memory design. In the design of neural networks for solving practical problems, the key factor associated with the dynamical behavior of neural networks is the characterization of the equilibrium point in terms of the network parameters and activation functions. In some special applications of neural networks such as designing neural networks for solving optimization problems, the equilibrium point of the designed neural network must be unique and globally asymptotically stable. On the other hand, when an electronically implemented neural network is used in real time applications, we might get faced with two undesired physical event that may affect the dynamics of neural networks. The first event is the time delays time delays that occur during the signal transmission between the neurons, the other one is the deviations due the to the tolerances of the electronic components used in the implementation of neural networks. The readers can find a detailed robust stability analysis of delayed neural networks under various assumptions on the activation functions and present various robust stability conditions for different classes of neural networks in (Arik and Tavsanoglu, 2000); (Cao and Wang, 2003); (Cao and Wang, 2005); (Ensari and Arik, 2010); (Forti and Tesi, 1995); (Li et al., 2003); (Liao and Wang, 2000);(Liao et al., 2002); (Liao and Yu,1998); (Liao et al., 2001); (Mohamad, 2001); (Ozcan and Arik, 2006); (Singh, 2007); (Sun and Feng, 2003); (Wang and Michel, 1996); (Yi and Tan, 2002).

The neural network model we consider in this paper is described by the following equations:

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + u_i,\ i = 1,2,...,n \quad (1)$$

where n is the number of the neurons, $x_i(t)$ denotes the state of the neuron i at time t, $f_i(\cdot)$ denote activation functions, a_{ij} and b_{ij} are the weight coefficients, τ_j are the delay parameters, u_i is the constant input to the neuron i, c_i is the charging rate for the neuron i.

Neural network model (1) can be written in the vector-matrix form as follows

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + u \quad (2)$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in R^n$, $C = diag(c_i > 0)_{n\times n}$ is a positive diagonal matrix, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $u = (u_1, u_2, ..., u_n)^T$ and $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t)))^T$ and $f(x(t-\tau)) = (f_1(x_1(t-\tau_1)), f_2(x_2(t-\tau_2)), ..., f_n(x_n(t-\tau_n)))^T$.

It will be assumed that the matrices C, A and B in (2) are uncertain but their elements have the lower and upper bounds. That is to say, C, A and B are assumed to have the parameter ranges defined as

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follows :

$$C_I := \{0 < \underline{C} \leq C \leq \overline{C}, i.e., 0 < \underline{c}_i \leq c_i \leq \overline{c}_i\}$$
$$A_I := \{A = (a_{ij}) : \underline{A} \leq A \leq \overline{A}, i.e., \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}\} \quad (3)$$
$$B_I := \{B = (b_{ij}) : \underline{B} \leq B \leq \overline{B}, i.e., \underline{b}_{ij} \leq b_{ij} \leq \overline{b}_{ij}\}$$

We also assume that $f_i(\cdot)$ are Lipschitz continuous, i.e., there exist some positive constants $\ell_i > 0$ such that

$$|f_i(x) - f_i(y)| \leq \ell_i |x - y|, \ i = 1, 2, ..., n, \ \forall x, y \in R, x \neq y$$

The class of Lipschitz activation functions is denoted by $f \in \mathcal{L}$.

The following two lemmas will play an important role in determining the sufficient conditions for the global robust exponential stability of the equilibria of neural networks (1) and (2) :

Lemma 1 (Cao and Wang, 2005). Let the matrices A and B in (3) be defined in the intervals $A \in [\underline{A}, \overline{A}]$ and $B \in [\underline{B}, \overline{B}]$. Then, the following inequalities hold :

$$||A||_2 \leq ||A^*||_2 + ||A_*||_2$$
$$||B||_2 \leq ||B^*||_2 + ||B_*||_2$$

where $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, $B^* = \frac{1}{2}(\overline{B} + \underline{B})$ and $B_* = \frac{1}{2}(\overline{B} - \underline{B})$.

Lemma 2 (Ensari and Arik, 2010). Let the matrices A and B in (3) be defined in the intervals $A \in [\underline{A}, \overline{A}]$ and $B \in [\underline{B}, \overline{B}]$. Then, the following inequalities hold :

$$||A||_2 \leq \sqrt{||A^*||_2^2 + ||A_*||_2^2 + 2||A_*^T |A^*| ||_2}$$
$$||B||_2 \leq \sqrt{||B^*||_2^2 + ||B_*||_2^2 + 2||B_*^T |B^*| ||_2}$$

where $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, $B^* = \frac{1}{2}(\overline{B} + \underline{B})$ and $B_* = \frac{1}{2}(\overline{B} - \underline{B})$.

Lemma 3 (Singh, 2007). Let the matrices A and B in (3) be defined in the intervals $A \in [\underline{A}, \overline{A}]$ and $B \in [\underline{B}, \overline{B}]$. Then, the following inequalities hold :

$$||A||_2 \leq ||\hat{A}||_2$$
$$||B||_2 \leq ||\hat{B}||_2$$

where $\hat{A} = (\hat{a}_{ij})_{n \times n}$ with $\hat{a}_{ij} = max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}$ and $\hat{B} = (\hat{b}_{ij})_{n \times n}$ with $\hat{b}_{ij} = max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$.

Lemma 4 (Forti and Tesi, 1995). If $H(x) \in C^0$ satisfies the following conditions

(i) $H(x) \neq H(y)$ for all $x \neq y$,

(ii) $||H(x)|| \rightarrow \infty$ as $||x|| \rightarrow \infty$,

then, $H(x)$ is homeomorphism of R^n.

We also make use of the following vector norm and matrix norm in the proof of our main result. Let $v = (v_1, v_2, ..., v_n)^T \in R^n$ and $W = (w_{ij})_{n \times n}$. Then, we have

$$||v||_2 = \left\{ \sum_{i=1}^{n} |v_i|^2 \right\}^{1/2}, ||Q||_2 = [\lambda_{\max}(Q^T Q)]^{1/2}$$

Throughout this paper, for $v = (v_1, v_2, ..., v_n)^T \in R^n$, $|v|$ will denote $|v| = (|v_1|, |v_2|, ..., |v_n|)^T$. For any matrix $W = (w_{ij})_{n \times n}$, $|W| = (|w_{ij}|)_{n \times n}$. If W is positive definite, then, $\lambda_m(W)$ and $\lambda_M(W)$ will denote the minimum and maximum eigenvalues of W, respectively.

2 GLOBAL ASYMPTOTIC ROBUST STABILITY ANALYSIS

In this section, we present new sufficient conditions for the existence, uniqueness and global robust stability of the equilibrium point for the neural systems (1). We proceed with following result:

Theorem 1: Let $f \in \mathcal{L}$. Then, the neural network model (2) is globally asymptotically robust stable, if the following condition holds

$$\Omega^* = r - ||P||_2 - ||Q||_2 > 0$$

where $r = \frac{c_m}{\mu_M}$ with $c_m = min(\underline{c}_i)$ and $\mu_M = max(\mu_i)$, and

$$||P||_2 = min\{||A^*||_2 + ||A_*||_2, \sqrt{||A^*||_2^2 + ||A_*||_2^2 + 2||A_*^T |A^*| ||_2}, ||\hat{A}||_2\}$$

$$||Q||_2 = min\{||B^*||_2 + ||B_*||_2, \sqrt{||B^*||_2^2 + ||B_*||_2^2 + 2||B_*^T |B^*| ||_2}, ||\hat{B}||_2\}$$

Proof: For the map

$$H(x) = -Cx + Af(x) + Bf(x) + u$$

we have

$$H(x) - H(y) = -C(x - y) + A(f(x) - f(y)) + B(f(x) - f(y))$$

If we multiply both sides of (20) by $(x - y)^T$, then we get :

$$\begin{aligned}
&(x-y)^T (H(x) - H(y)) \\
= &-(x-y)^T C(x-y) \\
&+(x-y)^T A(f(x) - f(y)) \\
&+(x-y)^T B(f(x) - f(y)) \\
\leq &-c_m ||x-y||_2^2 \\
&+(||A||_2 + ||B||_2)||x-y||_2 ||f(x) - f(y)||_2
\end{aligned}$$

The fact that $||f(x)-f(y)||_2 \leq \mu_M||x-y||_2$ implies

$$\begin{aligned} & (x-y)^T(H(x)-H(y)) \\ \leq \quad & -c_m||x-y||_2^2+\mu_M(||A||_2+||B||_2)||x-y||_2^2 \end{aligned}$$

Since $||A||_2 \leq ||P||_2$, $||B||_2 \leq ||Q||_2$, we obtain

$$\begin{aligned} & (x-y)^T(H(x)-H(y)) \\ \leq \quad & -c_m||x-y||_2^2+\mu_M(||P||_2+||Q||_2)||x-y||_2^2 \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \frac{1}{\mu_M}(x-y)^T(H(x)-H(y)) \\ \leq \quad & -(r-(||P||_2+||Q||_2))||x-y||_2^2 \\ = \quad & -\Omega^*||x-y||_2^2 \end{aligned} \tag{4}$$

implying that

$$(x-y)^T(H(x)-H(y)) < 0, \forall x \neq y$$

from which it can be directly concluded that $H(x) \neq H(y)$ when $x \neq y$.

In order to show that $||H(x)|| \rightarrow \infty$ as $||x|| \rightarrow \infty$, we let $y=0$ in (21), in which case, we can write

$$\frac{1}{\mu_M}x^T(H(x)-H(0)) \quad \leq \quad -\Omega^*||x||_2^2$$

from which one can derive that

$$||x||_\infty||H(x)-H(0)||_1 \quad \geq \quad \Omega^*\mu_M||x||_2^2$$

Using $||x||_\infty \leq ||x||_2$ and $||H(x)-H(0)||_1 \geq ||H(x)||_1-||H(0)||_1$, it follows that $||H(x)||_1 \geq \Omega^*\mu_M||x||_2+||H(0)||_1$. Since $||H(0)||_1$ is finite, we conclude that $||H(x)|| \rightarrow \infty$ as $||x|| \rightarrow \infty$. Hence, under the condition of Theorem 1, neural network (1) has a unique equilibrium point.

We will now simplify system (1) as follows : we let $z_i(\cdot)=x_i(\cdot)-x_i^*,\ i=1,2,...,n$ and note that the $z_i(\cdot)$ are governed by :

$$\begin{aligned} \dot{z}_i(t) = & -c_iz_i(t)+\sum_{j=1}^{n}a_{ij}g_j(z_j(t)) \\ & +\sum_{j=1}^{n}b_{ij}g_j(z_j(t-\tau_{ij})),\ i=1,2,...,n \end{aligned} \tag{5}$$

where $g_i(z_i(\cdot))=f_i(z_i(\cdot)+x_i^*)-f_i(x_i^*),\ i=1,2,...,n$. It can easily be verified that the functions g_i satisfy the assumptions on f_i i.e., $f \in \mathcal{L}$ implies that $g \in \mathcal{L}$. We also note that $g_i(0)=0, i=1,2,...,n$. It is thus sufficient to prove the stability of the origin of the transformed system (4) instead of considering the stability of x^* of system (1).

For $\tau_{ij}=\tau_j$, (4) can be expressed in the matrix-vector form as follows :

$$\dot{z}(t)=-Cz(t)+Ag(z(t))+Bg(z(t-\tau)) \tag{6}$$

where $z(t)=(z_1(t),z_2(t),...,z_n(t))^T \in R^n$ is state vector of transformed neural system, $g(z(t))=(g_1(z_1(t)),g_2(z_2(t)),...,g_n(z_n(t)))^T$ and $g(z(t-\tau))=(g_1(z_1(t-\tau_1)),g_2(z_2(t-\tau_2)),...,g_n(z_n(t-\tau_n)))^T$.

Now construct the following positive definite Lyapunov functional

$$V(z(t)) \quad = \quad z^T(t)z(t)+k\sum_{i=1}^{n}\int_{t-\tau_i}^{t}z_i^2(\zeta)d\zeta$$

where the k is a positive constant to be determined later. The time derivative of the functional along the trajectories of system (5) is obtained as follows

$$\begin{aligned} \dot{V}(z(t)) = \quad & -2z^T(t)Cz(t)+2z^T(t)Ag(z(t)) \\ & +2z^T(t)Bg(z(t-\tau))+k||z(t)||_2^2 \\ & -k||z(t-\tau)||_2^2 \\ \leq \quad & -2c_m||z(t)||_2^2+2||A||_2||z(t)||_2||g(z(t))||_2 \\ & +2||B||_2||z(t)||_2||g(z(t-\tau))||_2 \\ & +k||z(t)||_2^2-k||z(t-\tau)||_2^2 \\ \leq \quad & -2c_m||z(t)||_2^2+2\mu_M||A||_2||z(t)||_2^2 \\ & +2\mu_M||B||_2||z(t)||_2||z(t-\tau)||_2 \\ & +k||z(t)||_2^2-k||z(t-\tau)||_2^2 \\ \leq \quad & -2c_m||z(t)||_2^2+2\mu_M||A||_2||z(t)||_2^2 \\ & +\mu_M||B||_2||z(t)||_2^2+\mu_M||B||_2||z(t-\tau)||_2^2 \\ & +k||z(t)||_2^2-k||z(t-\tau)||_2^2 \\ \leq \quad & -2c_m||z(t)||_2^2+2\mu_M||P||_2||z(t)||_2^2 \\ & +\mu_M||Q||_2||z(t)||_2^2 \\ & +\mu_M||Q||_2||z(t-\tau)||_2^2+k||z(t)||_2^2 \\ & -k||z(t-\tau)||_2^2 \end{aligned}$$

Letting $k=\mu_M||Q||_2$ results in

$$\begin{aligned} \dot{V}(z(t)) \quad \leq \quad & -2(c_m-\mu_M||P||_2-\mu_M||Q||_2)||z(t)||_2^2 \\ = \quad & -2\mu_M(r-||P||_2-||Q||_2)||z(t)||_2^2 \\ = \quad & -2\mu_M\Omega^*||z(t)||_2^2 \end{aligned}$$

It is easy to see that $\dot{V}(z(t))<0$ for all $z(t) \neq 0$, and $\dot{V}(z(t))=0$ if and only if $z(t) \neq 0$. In addition, $V(z(t))$ is radially unbounded since $V(z(t)) \rightarrow \infty$ as $||z(t)|| \rightarrow \infty$. Thus, it follows that the origin system (5), or equivalently the equilibrium point of system (2) is globally asymptotically stable.

We will now compare our result obtained in Theorem 1 with a previously reported corresponding stability result which is given in the following:

Theorem 2 (Ozcan and Arik, 2006). Let $f \in \mathcal{L}$. Then, the neural network model (2) is globally asymptotically robust stable, if

$$\sigma = r - (||A^*||_2 + ||A_*||_2 + ||B^*||_2 + ||B_*||_2) > 0$$

where $r = \frac{c_m}{\mu_M}$ with $c_m = min(\underline{c}_i)$and $\mu_M = max(\mu_i)$.

Since $||P||_2 \leq ||A^*||_2 + ||A_*||_2$, $||Q||_2 \leq ||B^*||_2 + ||B_*||_2$, Theorem 1 directly implies the result of Theorem 2.The result of Theorem 2 can be considered a special case of the result of Theorem 1.

3 CONCLUSIONS

By using a proper Lyapunov functional, we have obtained a easily verifiable delay independent sufficient condition for the global robust stability of the equilibrium point. We have also compared our result with the previous corresponding robust stability results published in the previous literature, proving that our condition is new and generalizes previously reported results.

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