Discrete Asymptotic Reachability via Expansions in Non-integer Bases

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Abstract: Aim of this paper is to show the connection between the theory of expansions in non-integer bases and discrete control systems. This idea is supported by an example of application, in the framework of robotics. We show how a model of multi-phalanx self-similar robot hand may be studied by means of results and techniques coming from non-standard numeration systems and related tools, like Iterated Function Systems (IFS) and, more generally, fractal geometry.

1 INTRODUCTION

Discrete control systems are discrete in time control systems with a finite control set. They are employed for local approximations of continuous systems, but also to model intrinsically discrete phenomena. The study of the dynamics of a polyhedron rolling on a plane or on a spheric surface, whose interest is motivated by robotics, or systems with a bounded transmission channel for controls are examples of applications of discrete control systems to practical problems.

On the other hand the theory of expansions in non-integer bases knew an increasing interest from researchers in mathematics and theoretical computer science, due its applications to, among others, computer arithmetics, data compression and cryptography. Expansions in non-integer bases were introduced in (Rényi, 1957) and they generalize the classical positional number systems, e.g., binary and decimal numeration system, to the choice of a non-integer base and of an arbitrary digit set.

Aim of this paper is to show the connection between the theory of expansions in non-integer bases and discrete control systems. This idea is supported by an example of application, in the framework of robotics. We show how a model of multi-phalanx self-similar robot hand may be studied by means of results and techniques coming from non-standard numeration systems and related tools, like Iterated Function Systems (IFS) and, more generally, fractal geometry.

The paper is organized as follows. Section 2 contains an overview on expansions in non-integer bases. To the best of our knowledge, the first application of theoretical results in non-integer base expansions in control theory are due to Chitour and Piccoli (Chitour and Piccoli, 2001): in Section 3 we recall some of their results on this topic. Section 4 and Section 5 are devoted to a model of robot hand, whose properties are investigated by means of arguments and results coming from the theory of expansions in non-integer bases.

2 EXPANSIONS IN NON-INTEGER BASES

A *positional number system* is a couple (λ, A), s.t. the *base* λ is greater than 1 in modulus and the *alphabet* A is a subset of $\mathbb{C}$. A value $x \in \mathbb{C}$ is *representable* if there exists an infinite sequence with digits in A, named *expansion* of x in base λ,

$$x = \sum_{j=1}^{\infty} \frac{c_j}{\lambda^j}$$

The set $\mathbf{R}(\lambda, A) = \left\{ \sum_{j=1}^{\infty} \frac{c_j}{\lambda^j} \mid c_j \in A \right\}$ is called *representable set* or, sometimes, fundamental domain.

Example 1. *In the case of binary numeration system we have* $\mathbf{R}(2, \{0,1\}) = [0,1]$. *More generally if* $\lambda < 2$ *then* $\mathbf{R}(\lambda, \{0,1\}) = [0, 1/(\lambda-1)]$ *(Rényi, 1957).*

Example 2. *If* $\lambda = 3$ *and* $A = \{0,2\}$ *then* $\mathbf{R}(\lambda, A)$ *is the Middle Third Cantor set.*

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Example 2 points out the intimate relation between non-integer base expansions and self-similar fractal sets. Self-similar sets, namely sets that are similar to a part of themselves, are classically generated by iterated function systems (Falconer, 1990). An iterated function system (IFS) is a set of contractive functions $f_j : \mathbb{C} \to \mathbb{C}$. We recall that a function in a metric space (X, d) is a contraction, if for every $x, y \in X$

$$d(f(x), f(y)) < c \cdot d(x, y)$$

for some $c < 1$. In (Hutchinson, 1981) Hutchinson showed that every finite IFS, namely every IFS with finitely many contractions, admits a unique non-empty compact fixed point R w.r.t the Hutchinson operator

$$\mathcal{F} : S \mapsto \bigcup_{j=1}^{J} f_j(S)$$

Moreover for every non-empty compact set $S \subseteq \mathbb{C}$

$$\lim_{k \to \infty} \mathcal{F}^k(S) = R$$

The *attractor* R is a self-similar set. This result was lately generalized to the case of infinite IFS (Mihail and Miculescu, 2009).

Remark 1. *If conv*(R) *is the convex hull of* R*, then* $\mathcal{F}^k conv(R) \downarrow R$ *for* $k \to \infty$.

Example 3. *The Middle Third Cantor set* $C := R(3, \{0, 2\})$ *is the attractor of* $\mathcal{F} = \{f_1, f_2\}$*, where*

$$f_1 : x \mapsto \frac{x}{3} \quad f_2 : x \mapsto \frac{1}{3}(2 + x).$$

that is

$$C = f_1(C) \cup f_2(C) = \mathcal{F}(C).$$

Remark that $[0, 1]$ *is the convex hull of* C*. We have*

$$\mathcal{F}^n([0, 1]) \downarrow C.$$

In general any representable set $\mathbf{R}(\lambda, A)$ is the attractor of the IFS of the IFS $\mathcal{F}_{\lambda,A} := \{f_i : x \mapsto \frac{1}{\lambda}x - a_i \mid a_i \in A\}$ and this yields an operative way to approximate $\mathbf{R}(\lambda, A)$ from above given its convex hull – see Remark 1.

Example 4. *If* $\lambda \in \mathbb{R}$ *and if* A *is a finite subset of* $\mathbb{R}$ *then conv*$(\mathbf{R}(\lambda, A)) = [\min A/(\lambda - 1), \max A/(\lambda - 1)]$.

Beyond the representability issue, the theory of expansion in non-integer bases includes several unexpected results laying on, among others, ergodic theory, automata theory, algebraic number theory. For instance it is well known that representations in decimal numeration system are univoque, i.e., there exists only one decimal representation for (almost) every number. The exceptional cases are only represented by the ambiguity $0.999\cdots = 1$. When we consider non-integer bases and real digit set the scenario is deeply different: indeed if the base is sufficiently small, then there exists for almost every number, *a continuum of different expansions* (Sidorov, 2003). Moreover the Golden Ratio G plays a special role in the case $A = \{0, 1\}$: indeed if $\lambda \leq G$ then every number in $\mathbf{R}(\lambda, A)$ can be represented in at least two different ways. The redundancy of these numeration systems leads to the study of particular expansions and related symbolic dynamical systems. For instance the *greedy expansions* privilege the choice of great digits (Rényi, 1957) and they minimize the truncation error in the case of positive real bases. Particular expansions in negative and complex base are also discussed in (Komornik and Loreti, 2007) and (Komornik and Loreti, 2010).

We conclude this section with a result on representability in complex base, that we will apply in Section 5.

Theorem 1. *(Lai, 2011) The set of representable numbers in base* $\lambda = \rho e^{i2\pi/n}$ *and arbitrary finite alphabet* $A = \{a_1 < \cdots < a_m\} \subset \mathbb{R}$ *is a convex polygon (containing the origin and with 2n edges if n is even and with n edges if otherwise) if and only if*

$$\max_{i=1,\ldots,m-1} a_{i+1} - a_i \leq \frac{\max A - \min A}{p^n - 1}. \qquad (1)$$

Remark 2. *If* $A = \{0, 1\}$ *then (1) is equivalent to* $\rho \leq 2^{1/n}$.

3 DISCRETE CONTROL SYSTEMS AND EXPANSIONS IN NON-INTEGER BASES

In (Chitour and Piccoli, 2001), the controllability of linear discrete control systems is investigated. Among other results, the paper contains a deep investigation of the unidimensional case, i.e., the study of the system

$$x_{k+1} = \lambda x_k + a_k \quad a_k \in A$$

with $|\lambda| > 1$ and $A \subset \mathbb{R}$. In this case the reachable set is $R(\lambda, A) = \{\sum_{j=0}^{n} a_j \lambda^j \mid a_j \in A, n \in \mathbb{N}\}$. To explain the relation between $R(\lambda, A)$ and the expansions in non-integer base, we introduce the notion of *integer part* in base λ. Let $x \in \lambda^N \mathbf{R}(\lambda, A)$ for some N, then

$$x = c_{-N}\lambda^{N-1} + \cdots + c_{-1}\lambda + c_0 + c_1\lambda^{-1} + \cdots$$

for some (c_j) with digits in A. The numerical value $c_{-N}\lambda^{N-1} + \cdots + c_{-1}\lambda + c_0$ is called *integer part* of x in base λ.

Remark 3. *Due of the redundancy of expansions in non-integer bases (Sidorov, 2003), a real number may have distinct (or none) integer parts in base* λ.

In other words, $R(\lambda, A)$ *is the set of all the integer parts of any real number in base* λ *and with alphabet* A.

Also by means of this relation with the theory of expansions in non-integer bases, Chitour and Piccoli pointed out a connection between the algebraic properties of λ and the topology of $R(\lambda, A)$. Before stating some of their results, we recall the following definitions.

Definition 1. *An* algebraic integer λ *is the real solution of a monic polynomial with integer coefficients, namely*

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_0 = 0$$

for some $a_0, \ldots, a_n \in \mathbb{Z}$ *. If the polynomial* $P(X) := X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n$ *is irreducible, namely it cannot be factorized as product of polynomials with integer coefficients, and if* $P(\lambda) = 0$*, then* $P(X)$ *called the* minimal polynomial *of* λ.

A real or complex number $\bar{\lambda}$ *is a* algebraic conjugate *of* λ *if it is a root of the minimal polynomial of* λ*, namely*

$$P(\bar{\lambda}) = 0.$$

A Pisot number *is an algebraic integer* $\lambda > 1$ *whose algebraic conjugates are less than* 1 *in modulus.*

Denote $\lfloor\lambda\rfloor$ *the integer part of* λ*. A* regular number *is a real number* λ *such that for every* $P(X)$ *with coefficients in* $A_\lambda = \{0, \pm 1, \ldots, \pm\lfloor\lambda\rfloor\}$

$$P(\lambda) \neq 0.$$

Example 5. *The Golden Mean, i.e., the greatest root of the polynomial* $P(X) = X^2 - X - 1$ *is a Pisot number.*
All transcendental numbers are regular.

Theorem 2. *(Chitour and Piccoli, 2001) For every* $\lambda > 1$ *it is possible to design an appropriate control set A such that* $R(\lambda, A_\lambda)$ *is dense in* $\mathbb{R}$*. In particular if* λ *is regular and setting* $A_\lambda := \{0, \pm 1, \ldots, \pm\lfloor\lambda\rfloor\}$ *we have that* $R(\lambda, A_\lambda)$ *is dense in* $\mathbb{R}$.

Algebraic properties of λ are relevant also in the following result.

Theorem 3. *(Chitour and Piccoli, 2001) Let* λ *be a* Pisot number*. If* $A \subset \mathbb{Z}$ *then* $R(\lambda, A)$ *is a discrete.*

4 A MODEL FOR A ROBOT HAND BASED ON EXPANSIONS IN NON-INTEGER BASES

We introduce a model for a robot hand, which main features are the following: an arbitrary number of fingers, moving on parallel planes excepting the index

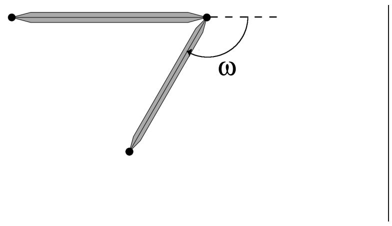


Figure 1: $\rho = 2^{1/3}$ and $\omega = 2\pi/3$. We have a full-extension configuration, i.e., $e_1 = e_2 = 1$, and the rotation controls $r_1 = 0$ and $r_2 = 1$. In particular $r_2 = 1$ implies that the angle between the first phalanx and the second phalanx is $\pi - \omega = \pi/3$.

finger and the opposable thumb. Every finger is characterized by an arbitrary number of phalanxes and a constant ratio between phalanxes. The motion of every phalanx is ruled by a couple of controls belonging to the unit interval. In particular every phalanx can extend and/or rotate or simply do nothing, the latter case corresponding to a null couple of controls. The physical parameters describing a particular finger are

- $\omega \in (0, 2\pi) \setminus \{\pi\}$ representing the greatest rotation (so that $\pi - \omega$ is the greatest angle between phalanxes);
- $\rho > 1$ representing the scaling factor of phalanxes;

while the control features are the following:

- x_k is the position of the k-th junction on the complex plane;
- $u_k \in E = \{e_0 = 0 < e_1 < \cdots < e_m = 1\}$ is the extension control for the k-th phalanx;
- $v_k \in R = \{r_0 = 0 < r_1 < \cdots < r_n = 1\}$ is the rotation control for the k-th phalanx;

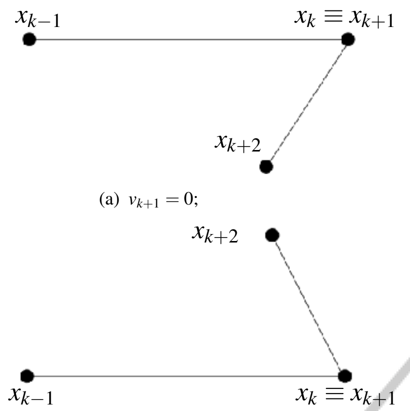
We set the dynamics of a finger on the complex plane, so that rotations are modeled with complex exponentiation, and we assume with no loss of generality x_0, the initial junction of the finger, to be set in the origin. The resulting discrete control system is

$$\begin{cases} x_k = x_{k-1} + \dfrac{u_k}{\rho^k} e^{-i\sum_{n=1}^{k} v_n\, \omega} \\ x_0 = 0. \end{cases} \tag{2}$$

We briefly remark some features of the finger derived from (2). The complex value $x_k - x_{k-1} = \frac{u_k}{\rho^k} e^{-i\sum_{n=1}^{k} v_n\, \omega}$ represents the k-phalanx. In particular its lenght depends on the extension control u_k:

$$|\, x_k - x_{k-1} \,| = \frac{u_k}{\rho^k}.$$

On the other hand, this phalanx is oriented on the complex plane according to the argument of $x_k - x_{k-1}$, that is $\sum_{n=1}^{k} v_n\ \omega$. This implies that the system keeps memory of all rotations and, in



(a) $v_{k+1} = 0$;

(b) $v_{k+1} = 1$.

Figure 2: In both cases $u_{k+1} = 0$, $u_{k+2} = 1$ and $v_{k+2} = 1$.

particular, unextended rotations affect subsequent motions (see Figure 2). We approach the study of the reachable set of System (2) by means of the theory of expansions in non-integer bases. We have the following correspondence

base	$\leftrightarrow$	physical properties of the finger
alphabet	$\leftrightarrow$	control set
representability	$\leftrightarrow$	reachability

Indeed, by setting

$$\lambda := \rho e^{i\,\omega} \tag{3}$$

we have that $R_k(\omega,\rho)$, the *reachable set* in time k, satisfies

$$R_k(\omega,\rho) = \left\{ x_k = \sum_{j=1}^{k} \frac{c_j}{\lambda^j} \;\middle|\; c_j = u_j e^{i\omega(j - \sum_{n=1}^{j} v_n)}; u_j \in E; v_n \in R \right\}.$$

Equation above implies that reachable points are representable numbers in base λ and with alphabet

$$A := \{u_j e^{i\omega(j-\sum_{n=1}^{j} v_n)};\ u_j \in E,\ v_n \in R\}. \tag{4}$$

Consider now the asymptotic reachable set

$$R_\infty(\omega,\rho) := \overline{\bigcup_{k=1}^{\infty} R_k(\omega,\rho)} = \left\{ x_k = \sum_{j=1}^{\infty} \frac{c_j}{\lambda^j} \;\middle|\; c_j = u_j e^{i\omega(j-\sum_{n=1}^{j} v_n)}; u_j \in E; v_n \in R \right\}$$

From the theory of expansions in non-integer bases we have the following result.

Proposition 1. *Let $\rho \in (1,\infty)$, $\omega \in [0,2\pi]$, $E,R \subset [0,1]$ and let λ and A be respectively like in (3) and (4). Then $R_\infty(\omega,\rho)$ is the attractor of the IFS*

$$\mathcal{F}_{\rho,\omega,E,R} := \{f_i : 1/\lambda(x + a_i) \mid a_i \in A\}.$$

In particular, for $k \to \infty$

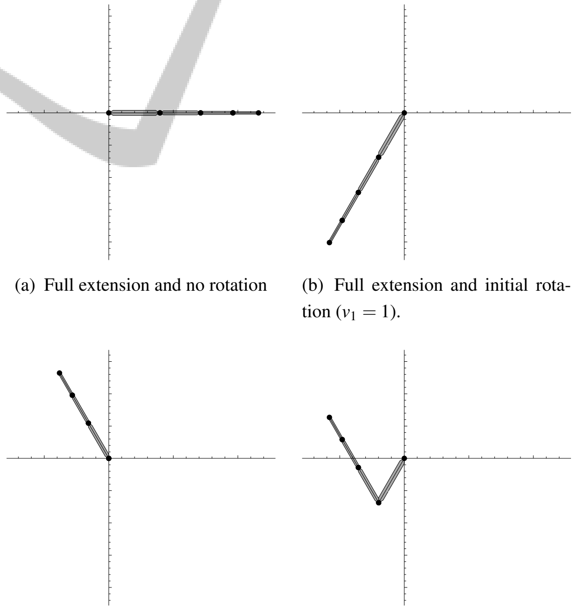
$$\mathcal{F}^k_{\rho,\omega,E,R}(conv(R_\infty(\rho,\omega,E,R)) \downarrow R_\infty(\rho,\omega,E,R).$$

5 THE BINARY CASE

Through this Section we focus on the case $E = R = \{0,1\}$, namely the case of binary controls. Since E and R are fixed, we shall omit them in the notations.

5.1 Convex Hull of the Reachable Set: A Particular Case

Through a combinatorial approach it is possible to determine the convex hull of the reachable set $R_k(\rho,\omega)$, by characterizing the extremal configurations (see Figure 3).



(a) Full extension and no rotation

(b) Full extension and initial rotation ($v_1 = 1$).

(c) $u_1 = 0, v_1 = v_2 = 1$ and full extension and no-rotation of latter phalanxes.

(d) full extension, $v_1 = v_2 = 1$ and no-rotation of latter phalanxes.

Figure 3: Extremal configurations for $R_4(2^{1/3}, 2\pi/3)$.

Theorem 4. *For every $k \in \mathbb{N}$ and for every $\rho > 1$ the convex hull of $R_k(\rho, 2\pi/3)$ is a polygon whose extremal points are reachable points obtained by the following control sequences*

- *full extension, no rotation:*

$$\mathbf{u} = (1,1,\ldots,1) \quad \mathbf{v} = (0,0,0,\ldots,0)$$

- *full extension, one initial rotation:*

$$\mathbf{u} = (1,1,\ldots,1) \quad \mathbf{v} = (1,0,0,\ldots,0)$$

- *full extension, two initial rotations:*

$$\mathbf{u} = (1,1,\ldots,1) \quad \mathbf{v} = (1,1,0\ldots,0)$$

- *no initial extension, two initial rotations:*

$$\mathbf{u} = (0,1,\ldots,1) \quad \mathbf{v} = (1,1,0\ldots,0)$$

5.2 Reachability around the Origin

We are interested on the topological properties of $R_\infty(\rho,\omega)$ and we get started by asking whether it contains an open ball centered in the origin, that is to say, if there exists a region of the plane that can be reached by the finger with arbitrary precision in finite time.

To this end we consider a particular class of configurations called *full-extension configurations*. They represent all the configurations where the rotation control is constantly equal to one, i.e., $v_j = 1$ for every $j \in \mathbb{N}$. The corresponding alphabet A^{fe} is $\{0,1\}$ and the corresponding reachable set is

$$R^{fe}_\infty(\rho,\lambda) := \{\sum_{j=1}^{\infty} \frac{a_j}{\lambda^j} \mid a_j \in \{0,1\}\}$$

By applying Theorem 1, we have the following result.

Theorem 5. *The representable set in base $\lambda = \rho e^{i\omega}$ and with alphabet A is contained in $R_\infty(\rho,\omega)$. Moreover if $\omega = 2\pi/n$, with $n \geq 3$ and if $\rho \leq 2^{1/n}$ then $R^{fe}_\infty(\rho,\omega)$ is a polygon with $2n$ edges if n is odd and with n edges if n is even.*

5.3 Asymptotic Reachable Set and IFS

In view of Proposition 1 the asymptotic reachable set

$$R_\infty(\rho,\omega) = \left\{ \sum_{j=1}^{\infty} \frac{u_k}{\rho^k} e^{-i\omega\sum_{n=1}^{j} v_n} \mid u_j, v_n \in \{0,1\} \right\}$$

is the fixed point of the IFS

$$f_1 : x \mapsto \frac{x}{\rho} \qquad f_2 : x \mapsto \frac{e^{-i\omega}}{\rho}x,$$
$$f_3 : x \mapsto \frac{1}{\rho}(1+x) \qquad f_4 : x \mapsto \frac{e^{-i\omega}}{\rho}(1+x).$$

We now explain the role of the above IFS and its relation with the controls. Let $x \in R_\infty(\rho,\omega)$. Then

$$x = \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{-i\omega\sum_{n=1}^{j} v_n}$$

and for every $h = 1,\ldots,4$

$$f_h(x) = \frac{e^{-i\omega v_0}}{\rho}(u_0 + x) = \sum_{j=0}^{\infty} \frac{u_j}{\rho^j} e^{-i\omega\sum_{n=1}^{j} v_n}$$

for some $u_0, v_0 \in \{0,1\}$ In particular to every function in the IFS corresponds a couple of control sequences

$$f_1 \leftrightarrow u = 0 \text{ and } v = 0 \quad f_2 \leftrightarrow u = 0 \text{ and } v = 1$$
$$f_3 \leftrightarrow u = 1 \text{ and } v = 0 \quad f_4 \leftrightarrow u = 1 \text{ and } v = 1.$$

To apply f_h to a point $x \in R_\infty(\rho,\omega)$ is equivalent to prefix the corresponding control couple (u_0, v_0) to the control sequence of x.

Example 6. *Let $x \in R_\infty(2^{1/3},\pi/6)$ with control sequences*

$$\mathbf{u} = (1,1,1,1,0,0,\ldots) \qquad \mathbf{v} = (0,1,0,1,0,0,\ldots)$$

then $f_4(x) \in R_\infty(2^{1/3},\pi/6)$ and its control sequences are

$$\mathbf{u} = (1,1,1,1,1,0,0,\ldots) \qquad \mathbf{v} = (1,0,1,0,1,0,0,\ldots)$$

See also Figure 4.

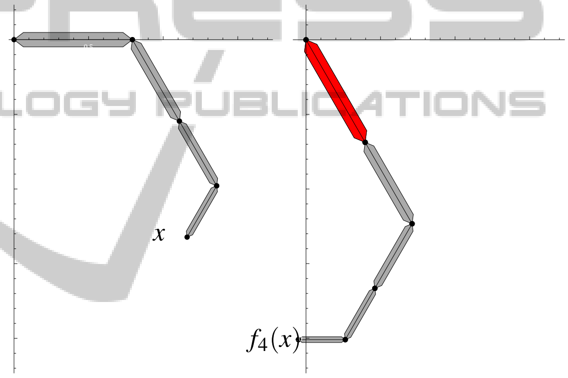


Figure 4: To apply f_4 corresponds to prepend a phalanx with controls $(1,1)$ to the finger associated to x.

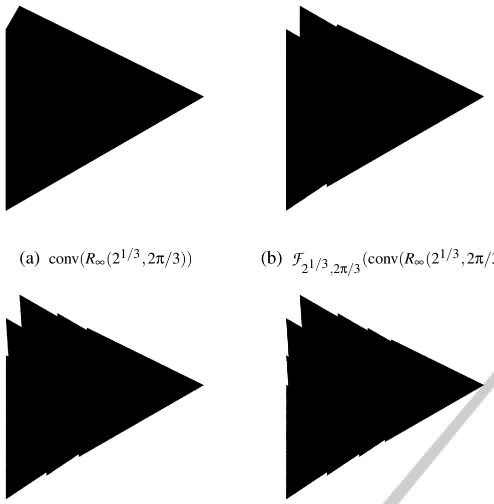
We now turn our attention to the approximation from above of $R_\infty(\rho,\omega)$. By Proposition 1 we have that

$$\mathcal{F}^k_{\rho,\omega}(\mathrm{conv}(R_\infty(\rho,\omega,E,R)) \downarrow R_\infty(\rho,\omega)$$

where $\mathcal{F}_{\{\rho,\omega\}} = \{f_{u,v} : x \mapsto e^{-i\omega v}/\rho(u+x) \mid u,v \in \{0,1\}\}$. In the case $\omega = 2\pi/3$, $\mathrm{conv}(R_\infty(\rho,\omega))$ is explicitly characterized (Theorem 3), thus we have an operative method to approximate $R_\infty(\rho,\omega)$ by simply iterating $\mathcal{F}_{\rho,\omega}$ over $\mathrm{conv}(R_\infty(\rho,\omega))$ – see Figure 5.

6 CONCLUSIONS AND OPEN PROBLEMS

Theory of expansions in non-integer bases turned out to be a promising tool for the investigation of discrete control systems, both in the analysis of the (asymptotic) reachable set and in the implementation of control algorithms. We applied some techniques coming



(a) $\mathrm{conv}(R_\infty(2^{1/3}, 2\pi/3))$ (b) $\mathcal{F}_{2^{1/3},2\pi/3}(\mathrm{conv}(R_\infty(2^{1/3}, 2\pi/3)))$

(c) $\mathcal{F}^2_{2^{1/3},2\pi/3}(\mathrm{conv}(R_\infty(2^{1/3}, 2\pi/3)))$ (d) $\mathcal{F}^3_{2^{1/3},2\pi/3}(\mathrm{conv}(R_\infty(2^{1/3}, 2\pi/3)))$

Figure 5: An approximation of $R_\infty(2^{1/3}, 2\pi/3)$ through IFS.

from the theory of expansions in non-integer base to the study a robot hand model, whose main feature is an arbitrary number of self-similar phalanxes. In particular, density conditions of the reachable set around the origin are established and the asymptotic reachable set is characterized as the attractor of a particular Iterated Function System. The latter result yields a technique for the approximation from above of the asymptotic reachable set.

We conclude the present paper with some open problems and perspectives related to the robot hand model. The topology of the asymptotic reachable set should be further investigated, by investigating the fractal properties of its boundary and possibly by extending density conditions of Theorem 5. These theoretical results could be then exploited to get an ad-hoc calibration of parameters in order to avoid forbidden areas for bio-medical applications. A class of control algorithms for this system was introduced in (Lai and Loreti, 2012) and it is strongly related to representation techniques in non-integer numeration system. It is left to further investigations to provide optimal control algorithms.

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