Fuzzy Singleton Congestion Games

Litsa Alexandra and Sbabou Samir
Center of Research in Economics and Management, University of Caen, Caen, France

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Abstract: In traditional game theory, the players attempt to maximize their utility functions. However, in real world there are many situations where payoffs have uncertainty and are fuzzy in nature. In this paper, we analyze fuzzy singleton congestion games with fuzzy payoff functions using tools from fuzzy set theory. We model this kind of games and we employ the 'technique for order performance by similarity to ideal solution' with fuzzy data in order to rank fuzzy payoffs, and consequently alternatives, with respect to multiple criteria. Based on this ranking we find possible solutions of these games which correspond to fuzzy Nash equilibria. Our method is adequate to model real-life situations, where players make more subtle decisions.

1 INTRODUCTION

Congestion games are a special class of non-cooperative games introduced by Rosenthal in 1973. This kind of games provides a natural framework for a wide range of economics and computer science applications such as resource allocation, routing and network design problems. In Rosenthal's model (Rosenthal, 1973), a set of players competes for a set of resources and the payoff of each resource depends only on the number of players using it. The utility a player derives from a combination of resources is the sum of the payoffs associated with each resource included in his choice. A slightly different formulation of congestion games was introduced by Milchtaich in 1996 under the name of congestion games with player-specific payoff functions. As a matter of fact, all players are restricted to the selection of a single resource and either they all share the same utility function - symmetric case - or each of them has his own payoff function - nonsymmetric case. In these games, the specific payoff functions are decreasing to the number of players. A key game-theoretic property of these games is that they always admit at least one Nash equilibrium. In this paper, we focus on congestion games in the sense of Milchtaich, also called singleton congestion games.

This kind of games can be seen as decision making problems. In routing problems for example, each individual has to choose a specific road to reach his destination. Thus, he has to take a decision with respect to different kinds of constraint like the time he has to reach the destination or the cost of his trip. The latter aspect calls for multiple criteria. A process of decision making, consisting in deriving the best option from a feasible set, is present in just about every conceivable human task. However, the basic model of a decision in the classical decision theory has very little in common with the real decision making as all information is supposed to be precisely known - crisp data. The only component in which uncertainty is permitted is the occurrence of the different states of nature, for which probabilistic descriptions are allowed. Nevertheless, when the uncertainty is of qualitative nature, the use of other techniques is necessary. Fuzzy set theory, introduced by Zadeh in 1965, provide the flexibility needed to represent the uncertainty resulted from the lack of knowledge.

In the context of congestion games, the outcomes faced by the players cannot be known in a precise manner as human judgements are often vague. In other words, it is difficult to have strict values of payoffs, because players are sometimes not able to analyze certain data of a game and as a result, their information is not complete. In this work, inspired by the theory of ordinal singleton congestion games and the fact that the decisions made by rational players may be imprecise because of players' doubts, we model fuzzy singleton congestion games. The payoff functions of players are represented as triangular fuzzy numbers - which allow to consider the uncertainty associated with the mapping of human perception on a number - and the importance weights of various criteria are assessed by means of linguistic variables. *But*

Alexandra L. and Samir S..
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the question is how we can establish a preference ordering of alternatives knowing that the payoff functions are fuzzy. To do so, we use the 'Technique for order performance by similarity to ideal solution' (TOPSIS) method with fuzzy data, introduced by Chen in 2000. In order to provide a complete analysis of such games, we firstly rank the choice of alternatives of all players and afterwards, we examine the existence of equilibria.

This paper is arranged into five sections. In section 2 we present some basic definitions and notations concerning singleton congestion games and fuzzy sets. The intent of section 3 is to introduce fuzzy singleton congestion games and to provide the TOPSIS procedure with fuzzy data which leads us to the presentation of fuzzy Nash equilibria in this kind of games. A numerical example is used to illustrate the feasibility of our model in section 4. The last section concludes.

2 PRELIMINARIES

2.1 Background of Singleton Congestion Games

Monotone singleton congestion games (singleton congestion games for short) are defined by a tuple $\Gamma(N,R,(d_{i,r})_{i\in N,r\in R})$, where $N=\{1,\ldots,n\}$ is a set of n players, $R=\{1,\ldots,m\}$ is a set of m resources and $d_{i,r}:\{1,\ldots,m\}\to\mathbb{R}$ is a non-increasing payoff function associated with resource r. Let S_i be a finite set of strategies available to player i and $\sigma=(\sigma_i)_{i\in N}$ its elements, called strategy profiles. For a profile σ and a resource r, the congestion on r, i.e. the number of players using r, is defined by $n_r(\sigma)=|\{i\in N: r\in\sigma_i\}|$. The vector $(n_1(\sigma),\ldots,n_m(\sigma))$ is the congestion vector corresponding to σ.

In this kind of games each player is allowed to choose any resource from R but must choose only one. Thus, a player's strategy consists of a single resource in R. Since the utility an anonymous player derives from selecting a single resource depends only on the number of the players doing the same choice, the utility function is a mapping $u: R\times N\to\mathbb{R}$, with $(r,n_r(\sigma))\mapsto u(r,n_r(\sigma))$, where u decreases with the number of players sharing the same resource r. The utility of player i for a profile σ is given by $u_i(\sigma)=d_{i,r}(n_r(\sigma))$, with $r\in\sigma_i$.

We should not forget that in a singleton congestion game all data is given in crisp values and the aim is *to obtain a decreasing order of preferences* for each player or, in other words, a ranking of payoffs, which leads us to identify all possible solutions of the game. The approach of Milchtaich (Milchtaich, 1996) enables to do so and leads to the conclusion that such games admit at least one Nash equilibrium.

However, crisp data are inadequate to model real-life situations since it is impossible to have a precise estimation of the payoff functions obtained by the players and the human perception of criteria is often expressed linguistically.

Thus, a fuzzy approach is provided in this article to examine such cases. More precisely, we introduce a new class of games namely, the fuzzy singleton congestion games where the data are fuzzy when players have to make a decision on the congestion they have to choose. Before going over our analysis, we need the following basic definitions.

2.2 Concepts from Fuzzy Set Theory

The earliest formulation of the concepts of fuzzy sets is due to Zadeh (Zadeh, 1965) who generalized the idea of a crisp set by extending the range of its characteristic function. Actually, he considered that the latter can take any value in the interval $[0,1]$.

Let X be a set of objects whose generic elements are denoted by x. The membership in a crisp subset of X is a characteristic function μ_Φ from X to $\{0,1\}$ such that:

$$\mu_\Phi=\begin{cases}1 & \text{if and only if } x\in\Phi,\\ 0 & \text{otherwise,}\end{cases}$$

where $\{0,1\}$ is called a valuation set. If the latter is the interval $[0,1]$, Φ is called a fuzzy set and is denoted by $\widetilde{\Phi}$. We say that $\mu_{\widetilde{\Phi}}(x)$ is the degree of membership of x in $\widetilde{\Phi}$.

Definition 1. If $\widetilde{\Phi}$ is a fuzzy set, then $\widetilde{\Phi}$ is characterized by the set of ordered pairs: $\widetilde{\Phi}=\{(x,\mu_{\widetilde{\Phi}}(x))|x\in X\}$.

Definition 2. A fuzzy set $\widetilde{\Phi}$ of the universe of discourse X is convex if and only if for all x_1,x_2 in X, $\mu_{\widetilde{\Phi}}(\lambda x_1+(1-\lambda)x_2)\geq min(\mu_{\widetilde{\Phi}}(x_1),\mu_{\widetilde{\Phi}}(x_2))$, where $\lambda\in[0,1]$.

Definition 3. The height of a fuzzy set is the largest membership grade attained by any element in that set. A fuzzy set $\widetilde{\Phi}$ in the universe of discourse X is called normalized when the height of $\widetilde{\Phi}$ is equal to 1, i.e. $\exists x\in X$ such that $\mu_{\widetilde{\Phi}}(x)=1$.

Definition 4. A fuzzy number is a fuzzy subset in the universe of discourse X that is both convex and normal.

Definition 5. The α-cut of a fuzzy number $\widetilde{a}$ is defined $\widetilde{a}^{\alpha} = \{x | \mu_{\widetilde{a}}(x) \geq \alpha, x \in X\}$, where $\alpha \in [0,1]$.

Definition 6. A triangular fuzzy number $\widetilde{a}$ is defined by a triplet (a_1, a_2, a_3) shown in Figure 1. The membership function $\mu_{\widetilde{a}}(x)$ is defined as :

$$\mu_{\widetilde{a}}(x) = \begin{cases} 0, & x < a_1, \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ \dfrac{x - a_3}{a_2 - a_3}, & a_2 \leq x \leq a_3, \\ 0, & x > a_3. \end{cases}$$

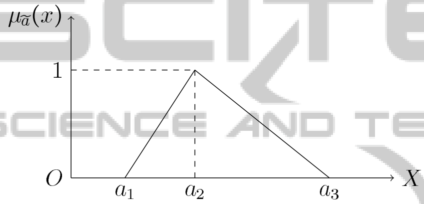

Figure 1: Fuzzy triangular number.

Definition 7. The graded mean integration representation of a given a triangular fuzzy number $\widetilde{a} = (a_1, a_2, a_3)$ is defined as : $P(\widetilde{a}) = \dfrac{a_1 + 4a_2 + a_3}{6}$.

In this paper, without loosing integrity and just to simplify the calculations, we assume that the fuzzy triangular numbers are symmetric and their corresponding crisp value is given by the graded mean integration representation.

Definition 8. Let $\widetilde{a} = (a_1, a_2, a_3)$ and $\widetilde{b} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. Then the distance between them can be calculated by using the vertex method : $\delta(\widetilde{a}, \widetilde{b}) = \sqrt{\frac{1}{3}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2]}$.

Definition 9. A linguistic variable is a variable whose values are expressed in linguistic terms.

The concept of a linguistic variable is very useful in dealing with situations, which are too complex or not well defined to be reasonably described in conventional quantitative expressions. For example, 'weight' is a linguistic variable whose values are : very low, low, medium, high, very high, etc. These values can also be represented by fuzzy numbers.

3 FUZZY SINGLETON CONGESTION GAMES

Fuzzy noncooperative games were first developed by Butnariu (Butnariu, 1978) and later refined by Billot (Billot, 1992). In their formulation each player's beliefs about the actions of the other players are modeled as fuzzy sets. Inspired by their approach, we define fuzzy singleton congestion games by a tuple $\widetilde{\Gamma}(N, R, (\widetilde{d}_{i,r})_{i \in N, r \in R})$, where $N = \{1, \ldots, n\}$ is a set of players, $R = \{1, \ldots, m\}$ is a set of resources and $\widetilde{d}_{i,r}$ is a fuzzy non-increasing payoff function associated with resource r, expressed by a triangular fuzzy number $(a_{1_{ir}}, a_{2_{ir}}, a_{3_{ir}})$ and such that its defuzzified value is given by $\dfrac{a_{1_{ir}} + 4a_{2_{ir}} + a_{3_{ir}}}{6}$.

As for ordinal singleton congestion games, the utility function decreases with the number of players sharing the same resource and for a given profile σ is given by $\widetilde{u}_i(\sigma) = \widetilde{d}_{i,r}(n_r(\sigma))$, with $r \in \sigma_i$.

A fuzzy singleton congestion game can be represented by the following fuzzy decision matrix:

	1	2	...	m
C_1	$\widetilde{d}_{11}$	$\widetilde{d}_{12}$	...	$\widetilde{d}_{1m}$
C_2	$\widetilde{d}_{21}$	$\widetilde{d}_{22}$	...	$\widetilde{d}_{2m}$
$\vdots$				
C_n	$\widetilde{d}_{n1}$	$\widetilde{d}_{n2}$	...	$\widetilde{d}_{nm}$

with $\widetilde{W} = [\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_n]$.

This means that there are m resources among which a player has to choose. Each resource is evaluated through n criteria C_i $(i = 1, \ldots, n)$ which correspond to the congestion vector $n_r(\sigma)$, i.e. the number of players choosing a resource. The element $\widetilde{d}_{ir}$ represents the rating of resource r with respect to criterion C_i, that is to say the payoff function associated with r.

Each player in the game may judge how important a criterion is. The weighting vector $\widetilde{W} = [\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_n]$, with $\widetilde{w}_i = \{\widetilde{w}_i^1, ..., \widetilde{w}_i^m\}$, represents the relative importance of n criteria C_i. In this paper, the importance weights of various criteria are considered as linguistic variables. These variables can be expressed in triangular fuzzy numbers as shown in Table 1.

Given the fuzzy decision matrix and the fuzzy weighting vector $\widetilde{W}$, the objective is to rank all the resources with respect to all criteria, so as to obtain a decreasing order of preferences for each player and identify possible solutions of fuzzy singleton congestion games. At this point, the following question is raised. *How can we rank fuzzy data?* The best way to do so in our context is to apply the TOPSIS procedure with fuzzy data proposed by Chen (Chen, 2000).

Table 1: Linguistic variables for the importance weight of each criterion.

Very Low (VL)	(0, 0, 0.2)
Low (L)	(0.1, 0.2, 0.3)
Medium Low (ML)	(0.2, 0.35, 0.5)
Medium (M)	(0.4, 0.5, 0.6)
Medium High (MH)	(0.5, 0.65, 0.8)
High (H)	(0.7, 0.8, 0.9)
Very High (VH)	(0.8, 1, 1)

3.1 Topsis Procedure with Fuzzy Data

At the beginning, resources should be generated and criteria should be identified. Then, the following steps are carried out.

Step 1: Construct the fuzzy decision matrix $\widetilde{D}$. Let $\widetilde{d_{ir}} = (a_{1_{ir}}, a_{2_{ir}}, a_{3_{ir}})$ be a triangular fuzzy number such that its defuzzified value is given by $d_{ir} = \dfrac{a_{1_{ir}} + 4a_{2_{ir}} + a_{3_{ir}}}{6}$, for all $i = 1,2,\ldots,n$ and $r = 1,2,\ldots,m$.

Step 2: Choose the appropriate linguistic variables to evaluate the importance weight of each criterion[1]. These linguistic variables can be expressed in positive triangular fuzzy numbers as in Table 1.

Step 3: Construct the normalized fuzzy decision matrix $\widetilde{Z} = [\widetilde{z}_{ir}]_{n\times m}$, where, for B and C being the set of benefit and cost criteria, respectively, we have

$$\widetilde{z}_{ir} = \left(\frac{a_{1_{ir}}}{a_{3_i}}, \frac{a_{2_{ir}}}{a_{3_i}}, \frac{a_{3_{ir}}}{a_{3_i}}\right), \text{ with } a_{3_i} = \max_r a_{3_{ir}} \text{ when } i \in B$$

or

$$\widetilde{z}_{ir} = \left(\frac{a_{1_i}}{a_{3_{ir}}}, \frac{a_{1_i}}{a_{2_{ir}}}, \frac{a_{1_i}}{a_{1_{ir}}}\right), \text{ with } a_{1_i} = \min_r a_{1_{ir}} \text{ when } i \in C.$$

The normalization method mentioned above is designed to preserve the property in which the elements $\widetilde{z}_{ij}$ are normalized triangular fuzzy numbers that belong to $[0,1]$.

Step 4: Considering the different importance of each criterion, construct the weighted normalized fuzzy decision matrix $\widetilde{V} = [\widetilde{u}_{ir}]_{n\times m}$, $i = 1,2,\ldots,n$ and $r = 1,2,\ldots,m$, where $\widetilde{u}_{ir} = \widetilde{z}_{ir} \cdot \widetilde{w}_i$.

[1] In our context, the utility is decreasing with the number of players sharing a resource, i.e. the criteria, and so the respective importance weights.

Step 5: All elements $\widetilde{u}_{ir}$, for all $1 \le i \le n$ and $1 \le r \le m$, are normalized positive triangular fuzzy numbers and their ranges belong to $[0,1]$. So, we can determine the fuzzy positive and fuzzy negative ideal solutions as:

$$\widetilde{V}^+ = (\widetilde{u}_1^+, \ldots, \widetilde{u}_n^+) \text{ and } \widetilde{V}^- = (\widetilde{u}_1^-, \ldots, \widetilde{u}_n^-),$$

where $\widetilde{u}_i^+ = (1,1,1)$ and $\widetilde{u}_i^- = (0,0,0)$, $i = 1,2,\ldots,n$.

Step 6: Calculate the distance of each resource from the fuzzy positive and fuzzy negative ideal solution, respectively, as :

$$\widetilde{\delta}_{ir}^+ = \widetilde{\delta}(\widetilde{u}_{ir}, \widetilde{u}_r^+) \text{ and } \widetilde{\delta}_{ir}^- = \widetilde{\delta}(\widetilde{u}_{ir}, \widetilde{u}_r^-),$$

using the vertex method to compute the distance between two fuzzy numbers (Definition 8).

Step 7: Calculate the relative closeness coefficient of each resource to the ideal solution defined as : $CC_{ir} = \dfrac{\widetilde{\delta}_{ir}^-}{\widetilde{\delta}_{ir}^+ + \widetilde{\delta}_{ir}^-}$. Since $\widetilde{\delta}_{ir} \ge 0$, clearly $CC_{ir} \in [0,1]$.

Step 8: According to the closeness coefficient, the resources are ranked in descending order. The best resource is the one with the shortest distance to the fuzzy positive ideal solution and with the longest distance to the fuzzy negative ideal solution.

Therefore, in a fuzzy singleton congestion game, each player is capable to determine which resource maximizes his satisfaction with respect to multiple criteria (number of players sharing the same resource).

3.2 Equilibrium Analysis

It is the nature of multi-criteria decision making problems (MCDM) to have conflicting attributes (Hwang and Yoon, 1981). We should not forget that each alternative can be characterized by a large number of attributes, chosen by the decision maker's conception of criteria. That is why, usually, there is no optimal solution to a MCDM problem. However, there may exist many possible efficient solutions as, in real-life situations, players make a choice with a random behavior and the attributes are not deterministic but rather fuzzy/imprecise. Such solutions correspond to fuzzy Nash equilibria.

Definition 10. A strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_i^*, \ldots, \sigma_n^*)$ is a fuzzy Nash equilibrium of a fuzzy singleton congestion game $\widetilde{\Gamma}(N, R, (\widetilde{d}_{i,r})_{i\in N, r\in R})$ if and only if

$$\forall i \in N, \forall \sigma_i, \sigma_i^* \in S_i, \ \widetilde{u}_i(\sigma_i^*, \sigma_{-i}^*) \ge \widetilde{u}_i(\sigma_i, \sigma_{-i}^*)$$

where σ^*_{-i} refers to all strategies except those chosen by the player i.

In our setting, fuzzy Nash equilibria can be obtained (Sbabou et al., 2012) by the ranking of resources at the end of the TOPSIS procedure[2].

In the coming section, an example is introduced to illustrate the TOPSIS procedure and to show the computation process of fuzzy Nash equilibria.

4 ILLUSTRATIVE EXAMPLE

Consider the following *symmetric*[3] singleton congestion game, applied in a routing problem: there are three drivers (players) who want to reach a destination and three roads (resources). Each driver chooses the path between his origin and destination in such a way that his utility is maximized. He must choose only one resource and all drivers have the same utility function which is non-increasing and depends on the number of drivers making the same choice. The payoff matrix is common for all drivers and the payoff of each of them is determined through the principle: 'I earn much more when I am the only person to take this road'.

In reality, however, it is very difficult to have the traffic situation at one's finger tips. Human perception and intuitive judgement play an important role in route choices. Therefore, it is impossible to have a precise estimation of the payoff of each driver.

Example 1. Let $\widetilde{\Gamma}(N,R,(\widetilde{u}_i)_{i\in N})$ be a symmetric fuzzy singleton congestion game with $N=\{1,2,3\}$, $R=\{a,b,c\}$ and such that $3a \prec 2a \prec a$, $3b \prec 2b \prec b$, $3c \prec 2c \prec c$.

The drivers' payoff functions can be estimated, but not very precisely. Suppose that they have the following conception of the game : 'If I choose road a my payoff is as much as 10, if I choose b my utility and approximately 8 and if I opt for c my utility is no less than 9. Once I am alone on the road, it is great for me (VH). However, sharing the road with someone else is less nice (H). If I share road a with one more person my payoff is approximately 7, if both of us choose b my payoff is at least 6, otherwise my utility is no less than 7. Finally, if we are three to go from road a my payoff is as much as 1, if we choose b my payoff is at most 2 and if we select resource c my utility is as much as 5. In this case my trip is of medium quality (M).'

It can be easily understood that in this context the payoff values cannot be denoted by real numbers. However, fuzzy numbers can describe this kind of fuzzy information as well as the importance weights of various criteria. In Section 3 we raised the question of how we can rank such fuzzy data and the answer was by applying the TOPSIS procedure with fuzzy data. The proposed method is currently applied to the game $\widetilde{\Gamma}$ and the computational procedure is diagrammed below.

Step 1: Construct the fuzzy decision matrix.

Driver 1/2/3	a	b	c
C_1	$(5,10,15)$	$(6,8,10)$	$(7,9,11)$
C_2	$(5,7,9)$	$(2,6,10)$	$(5,7,9)$
C_3	$(0.5,1,1.5)$	$(1,2,3)$	$(3,5,7)$

Step 2: The drivers evaluate the importance weight of each criterion in linguistic terms and present it below. Then, the linguistic evaluation is converted into triangular fuzzy numbers according to Table 1[4].

	Driver 1/2/3
C_1	VH
C_2	H
C_3	M

Step 3: Construct the normalized fuzzy decision matrix, common for all drivers[5].

	a	b	c
C_1	$(0.33,0.67,1)$	$(0.4,0.53,0.67)$	$(0.47,0.6,0.73)$
C_2	$(0.5,0.7,0.9)$	$(0.2,0.6,1)$	$(0.5,0.7,0.9)$
C_3	$(0.07,0.14,0.21)$	$(0.14,0.29,0.43)$	$(0.43,0.71,1)$

Step 4: Construct the weighted normalized fuzzy decision matrix.

	a	b	c
C_1	$(0.26,0.67,1)$	$(0.32,0.53,0.67)$	$(0.38,0.6,0.73)$
C_2	$(0.35,0.56,0.81)$	$(0.14,0.48,0.9)$	$(0.35,0.56,0.81)$
C_3	$(0.03,0.07,0.13)$	$(0.06,0.15,0.26)$	$(0.17,0.36,0.6)$

Step 5: Determine the fuzzy positive and fuzzy negative ideal solutions as : $\widetilde{V}^+ = [(1,1,1),(1,1,1),(1,1,1)]$ and $\widetilde{V}^- = [(0,0,0),(0,0,0),(0,0,0)]$.

Step 6 & **7:** Calculate the distance of each resource from the fuzzy positive (DFP) and fuzzy negative

[2]For detailed information about fuzzy Nash equilibria in noncooperative games, please refer to (Billot, 1992).

[3]In a nonsymmetric singleton congestion game the procedure works in the same way.

[4]If the game was nonsymmetric each player would have his own decision matrix and importance weight of each criterion.

[5]In this example we deal with benefit criteria as we make use of utilities.

(DFN) ideal solution, respectively, as well as the relative closeness coefficient (CC) of each resource to the ideal solution:

(DFP)	(DFN)	(CC)
$\widetilde{\delta}^+_{1a} = 0.467$	$\widetilde{\delta}^-_{1a} = 0.71$	$CC_{1a} = 0.6$
$\widetilde{\delta}^+_{2a} = 0.466$	$\widetilde{\delta}^-_{2a} = 0.6$	$CC_{2a} = 0.56$
$\widetilde{\delta}^+_{3a} = 0.94$	$\widetilde{\delta}^-_{3a} = 0.087$	$CC_{3a} = 0.08$
$\widetilde{\delta}^+_{1b} = 0.51$	$\widetilde{\delta}^-_{1b} = 0.527$	$CC_{1b} = 0.5$
$\widetilde{\delta}^+_{2b} = 0.58$	$\widetilde{\delta}^-_{2b} = 0.594$	$CC_{2b} = 0.5$
$\widetilde{\delta}^+_{3b} = 0.85$	$\widetilde{\delta}^-_{3b} = 0.18$	$CC_{3b} = 0.17$
$\widetilde{\delta}^+_{1c} = 0.45$	$\widetilde{\delta}^-_{1c} = 0.588$	$CC_{1c} = 0.56$
$\widetilde{\delta}^+_{2c} = 0.466$	$\widetilde{\delta}^-_{2c} = 0.6$	$CC_{2c} = 0.56$
$\widetilde{\delta}^+_{3c} = 0.65$	$\widetilde{\delta}^-_{3c} = 0.41$	$CC_{3c} = 0.39$

Step 8: Rank the resources in descending order according to the closeness coefficient.

$$3a \prec 3b \prec 3c \prec 2b \sim b \prec 2c \sim c \sim 2a \prec a$$

The fuzzy Nash equilibria of the game, provided by the ranking, are $(2a,c)$ and $(2c,a)$.

Remark: If the game were crisp, that is if we had a common order of preferences for all players represented by $3a \prec 3b \prec 3c \prec 2b \prec 2c \sim 2a \prec b \prec c \prec a$, then the Nash equilibrium of the game would be (a,b,c). As in the proposed method payoffs are fuzzy and we also include the players' viewpoint of a given situation, through the notion of the weight, Nash equilibria are fuzzy and we deal with them randomness, fuzziness and uncertainties.

5 CONCLUSIONS

In this work, we have studied fuzzy singleton congestion games with fuzzy payoff values. Our research was inspired by the fact that until today, the theory of congestion games, as most of the economic theories, has been based on classical set theory, two-valued logic and classical theory of additive measures, which is not realistic in economics. Human reasoning and decision making in natural language, which play a crucial role in economic situations, involve a kind of uncertainty that cannot be modeled with the use of classical mathematics. Fuzzy set theory facilitate the modeling of such situations but it seems that after almost fifty years is still at its beginning, perhaps because of the difficult incentive issues raised!

Our approach seems to have an important theoretical and applicable value in many other domains such as soft computing and resource allocations. As Klir and Yuan mention (Klir and Yuan, 1995), for example, in soft computing the aim is to develop computational methods that produce acceptable *approximate solutions* at *low cost* - fuzzy hardware and computer architectures for approximate reasoning. And the capability of communicating with robots, so as to allocate them miscellaneous tasks that have to be done as *fast* as possible, involves *natural language*, that is fuzzy logic.

Obviously, there are still many open questions and problems left for further research, especially as it is the first time that a study on fuzzy congestion games is provided. Firstly, the case where the utility decreases with the number of players sharing the same resource but the importance weight of each criterion goes to the opposite direction needs to be examined. Secondly, an experimental survey of our result would be important in order to determine the choice of an itinerary. Finally, we hope to extend our result to standard congestion games, network design problems and resource allocations.

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