Non-commutative Fuzzy Logic psMTL
An Alternative Proof for the Standard Completeness Theorem

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Keywords: Non-commutative Fuzzy Logic, psMTL Logic, Standard Semantics.

Abstract: In (Jenei and Montagna, 2003) was proved that the non-commutative psMTL logic introduced in (Hájek, 2003b) is the logic of left-continuous non-commutative t-norms or, equivalently, that the logic psMTL enjoys standard completeness. In the present paper we provide an alternative proof for the standard completeness theorem for the logic psMTL and we furthermore show that this result can be obtained also for finite theories.

1 INTRODUCTION

In general, a non-commutative logic is a logic equipped with a non-commutative conjunction &, i.e. the formula $\varphi \& \psi$ is not equivalent with the formula $\psi \& \varphi$. Motivations for considering non-commutative logics appear in different areas, within and also outside mathematics. Some situations that give rise to non-commutative conjunctions are the semantics of parallel programming, quantum mechanics which make use of non-commutative observables, Lambek calculus (Lambek, 1958) modelling sentence structures using concatenation as a non-commutative operation, logics based on formal matrix multiplication, linear logics studied in (Girard, 1995), quantales that have the non-commutative "and" given by closure of the product subspace (Rosenthal, 1990), (Mulvey and Nawaz, 1995). On the other hand, in ordinary language a commutative "and" between clauses indicates independence, while a non-commutative "and" indicates a pragmatic dependence as pointed out by (Schmerling, 1975). Consider, for example, statements from natural language such as "he lost weight and he got sick" vs. "he got sick and he lost weight".

T-norms were introduced by (Schweizer and Sklar, 1960) in order to formulate properly the triangle inequality in probabilistic metric spaces. Since then, t-norms have been applied in various other mathematical disciplines including game theory, theory of non-additive measures and integrals, theory of measure-free conditioning, theory of aggregation operations, fuzzy set theory, fuzzy logic, fuzzy control, preference modeling and decision analysis, artificial intelligence. The role of t-norms in the theory of many-valued logics is extremely important. Many-valued t-norm based logics are non-classical logical systems that use t-norms as truth-functions for the conjunction connective and their residua as truth-functions for the implicative connective. The semantics of many-valued logics given by t-norms is usually called standard semantics.

Hájek introduced in his influential monograph (Hájek, 1998) a very general many-valued logic, called BL logic (BL stands for 'Basic Logic'), with the idea to formalize the many-valued logics induced by continuous t-norms and their residua; this fact was proved later in (Cignoli et al., 2000). On the other hand, Esteva and Godo observed that the minimal condition for a t-norm to have a residuum, and therefore to determine a logic, is left-continuity, not continuity. Therefore, in (Esteva and Godo, 2001) they proposed a weaker logic, called MTL logic (MTL stands for 'Monoidal T-norm based Logic'). In (Jenei and Montagna, 2002) was proved that MTL is indeed the logic of left-continuous t-norms and their residua.

The non-commutative counterparts of t-norms were investigated by Flondor, Georgescu and Iorgulescu in (Flondor et al., 2001) under the name of pseudo-t-norms. Every continuous pseudo-t-norm is commutative, but there are left-continuous pseudo-t-norms which are not commutative. As a consequence, the non-commutative counterpart of the logic BL cannot provide a fruitful logic, since the standard semantics do not have a non-commutative nature; despite this weakness, the non-commutative BL logic was developed in (Hájek, 2003a).

On the other hand, the non-commutative MTL logic introduced in (Hájek, 2003b) under the name of

Diaconescu D..
Non-commutative Fuzzy Logic psMTL - An Alternative Proof for the Standard Completeness Theorem.
DOI: 10.5220/0004151603500356
In *Proceedings of the 4th International Joint Conference on Computational Intelligence* (FCTA-2012), pages 350-356
ISBN: 978-989-8565-33-4

psMTL logic (psMTL stands for 'pseudo Monoidal T-norm based Logic') enjoys standard completeness with respect to left-continuous pseudo-t-norms, as proved in (Jenei and Montagna, 2003).

The goal of the present paper is to provide an alternative method to prove the standard completeness theorem for the logic psMTL and to show that this result holds even for finite theories, i.e. we prove the finite strong standard completeness theorem for psMTL logic.

This paper is organized as follows: in Section 2 we recall the definitions and the basic properties on psMTL logic; Section 3 is the innovative part of this paper, where we give an alternative proof for the finite strong standard completeness theorem for psMTL logic; Section 4 contains some concluding remarks.

2 THE LOGIC PSMTL

Hájek introduced the non-commutative logic psMTL in (Hájek, 2003b). In this section we recall this logical system in a somewhat different formulation but equivalent with that of Hájek, following the line of (Diaconescu, 2012).

The fact that the logic psMTL has a non-commutative conjunction & leads to the existence of two implications. Therefore, the language of the propositional calculus psMTL consists of denumerable many propositional variables, whose set is denoted by Var, the primitive connectives $\to$, $\rightsquigarrow$, &, $\wedge$, $\vee$ and the constant $\overline{0}$. From this primitive connectives, further connectives and constants are defined by:

$$
\begin{array}{rcl}
\neg\varphi & \text{is} & \varphi \to \overline{0}, \\
\sim\varphi & \text{is} & \varphi \rightsquigarrow \overline{0}, \\
\overline{1} & \text{is} & \overline{0} \to \overline{0}, \\
\varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi)\&(\psi \to \varphi), \\
\varphi \leftrightsquigarrow \psi & \text{is} & (\varphi \rightsquigarrow \psi)\&(\psi \rightsquigarrow \varphi).
\end{array}
$$

The formulas are defined by structural induction as usual and we denote by $\mathrm{Form}_{\mathrm{psMTL}}$ the set of all formulas of the logic psMTL. For any formula φ of psMTL, we define the formula $\varphi^{\bullet}$ obtained by interchanging the two implications $\to$ and $\rightsquigarrow$ and reversing the arguments of &. Notice that $(\varphi^{\bullet})^{\bullet} = \varphi$. The axioms of psMTL are:

I. a formula which has one of the following forms is an axiom:

$(A1)$ $(\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

$(A2)$ $(\varphi\&\psi) \to \varphi$

$(A3)$ $(\varphi \wedge \psi) \to \varphi$

$(A4)$ $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

$(A5)$ $((\varphi \to \psi)\&\varphi) \to (\varphi \wedge \psi)$

$(A6a)$ $(\varphi \to (\psi \to \chi)) \to ((\varphi\&\psi) \to \chi)$

$(A6b)$ $((\varphi\&\psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

$(A7)$ $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

$(A8a)$ $(\varphi \vee \psi) \to (((\varphi \rightsquigarrow \psi) \to \psi) \wedge ((\psi \rightsquigarrow \varphi) \to \varphi))$

$(A8b)$ $(((\varphi \rightsquigarrow \psi) \to \psi) \wedge ((\psi \rightsquigarrow \varphi) \to \varphi)) \to (\varphi \vee \psi)$

$(A9)$ $\overline{0} \to \varphi$

II. if φ is an axiom of the form $(A1)$, $(A2)$, $(A5)$, $(A6a)$, $(A6b)$, $(A7)$, $(A8a)$ or $(A8b)$, then $\varphi^{\bullet}$ is also an axiom.

The deduction rules of psMTL are:

$$
\text{(MP1)}\ \frac{\varphi \quad \varphi \to \psi}{\psi} \qquad \text{(Impl1)}\ \frac{\varphi \to \psi}{\varphi \rightsquigarrow \psi}
$$
$$
\text{(MP2)}\ \frac{\varphi \quad \varphi \rightsquigarrow \psi}{\psi} \qquad \text{(Impl2)}\ \frac{\varphi \rightsquigarrow \psi}{\varphi \to \psi}
$$

A theory over psMTL logic is any set of formulas. Provability in a theory T is defined in the obvious way, using all the deduction rules. We note by $T \vdash \varphi$ the fact that φ is a T-theorem and by $\mathrm{Theor}_{\mathrm{psMTL}}(T)$ the set of all T-theorems (we use the notation $\mathrm{Theor}_{\mathrm{psMTL}}$ when T is the empty set).

Proposition 2.1. *For any formula* φ *of* psMTL *and any theory* T, $T \vdash \varphi$ *iff* $T \vdash \varphi^{\bullet}$.

The algebraic models for psMTL logic were introduced by Flondor, Georgescu and Iorgulescu:

Definition 2.1. (Flondor et al., 2001). A *psMTL-algebra* is a structure $\mathrm{A} = (A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ such that the following conditions are fulfilled:

(i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(ii) $(A, \odot, 1)$ is a monoid,

(iii) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$,

(iv) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

Condition (iii) is called the *adjointness property*, while (iv) is known as the *prelinearity condition*. We say that a psMTL-algebra is a *chain* if the underlying lattice order is linear.

For any psMTL-algebra A, an *A-evaluation of propositional variables* is a mapping $e : \mathrm{Var} \to A$. Any A-evaluation of propositional variables e can be uniquely extended to an *A-evaluation* of all formulas (denoted also by e) using the operations on A as truth functions, i.e. $e(\overline{0}) = 0$, $e(\varphi\&\psi) = e(\varphi) \odot e(\psi)$, and $e(\varphi \circ \psi) = e(\varphi) \circ e(\psi)$, where $\circ \in \{\vee, \wedge, \to, \rightsquigarrow\}$.

If T is a theory of the logic psMTL, A is a psMTL-algebra and e is an A-evaluation such that $e(\psi) = 1$, for any $\psi \in T$, then we call e an A-*model* of T.

Kühr introduced a particular class of psMTL-algebras, namely representable psMTL-algebras. The variety of psMTL-algebras do not enjoy the subdirect representation property (i.e. each algebra is a subalgebra of a direct product of chains), but it was proved that a psMTL-algebra is subdirectly representable if and only if it is a representable psMTL-algebra.

Definition 2.2. (Kühr, 2003). A *representable psMTL-algebra* (*psMTL^r-algebra*, for short) is a psMTL-algebra in which the following identities are valid:

(R1) $(y \to x) \vee (z \rightsquigarrow ((x \to y) \odot z)) = 1$,

(R2) $(y \rightsquigarrow x) \vee (z \to (z \odot (x \rightsquigarrow y))) = 1$.

Conditions (R1) and (R2) are also known as *Kühr's identities*.

Hájek reflected this problem in the logical framework by introducing in (Hájek, 2003b) the logic psMTL^r as an extension of psMTL by the axioms:

$(A10)$ $(\varphi \to \psi) \vee (\chi \rightsquigarrow ((\psi \to \varphi) \& \chi))$

$(A10^\bullet)$ $(\varphi \rightsquigarrow \psi) \vee (\chi \to (\chi \& (\psi \rightsquigarrow \varphi)))$

Axioms $(A10)$ and $(A10^\bullet)$ are just the logical reflection of Kühr's identities. Due to the properties of psMTL^r-algebras, a chain completeness theorem for psMTL^r was proved by Hájek:

Theorem 2.1. *(Hájek, 2003b). Let T be a theory over psMTL^r and φ be a formula. T proves φ iff for each psMTL^r-chain L and each L-model e of T, $e(\varphi) = 1$.*

The non-commutative generalizations of t-norms were studied by Flondor, Georgescu and Iorgulescu:

Definition 2.3. (Flondor et al., 2001). A *pseudo-t-norm* is a binary relation $\otimes$ on the real unit interval $[0,1]$ that is associative, non-decreasing in both arguments and satisfying $x \otimes 1 = 1 \otimes x = x$, for all $x \in [0,1]$.

A pseudo-t-norm $\otimes$ is left-continuous if $\bigvee_{i\in I}(a_i \otimes b) = (\bigvee_{i\in I} a_i) \otimes b$ and $\bigvee_{i\in I}(b \otimes a_i) = b \otimes (\bigvee_{i\in I} a_i)$. Moreover, any left-continuous pseudo-t-norm $\otimes$ has a left residuum $\to$ and a right residuum $\rightsquigarrow$ given by

$$\begin{aligned} a \to b &= \sup\{c \mid c \otimes a \leq b\}, \\ a \rightsquigarrow b &= \sup\{c \mid a \otimes c \leq b\}. \end{aligned}$$

The structure $([0,1], \max, \min, \otimes, \to, \rightsquigarrow, 0, 1)$ is a psMTL^r-chain, called a *standard psMTL-algebra*. Notice that every standard psMTL-algebra is representable.

It was proved in (Flondor et al., 2001) that any continuous pseudo-t-norm is commutative, but there are left-continuous pseudo-t-norms which are not commutative as shown by the following example:

Example 2.1. (Flondor et al., 2001).
Let $0 < a_1 < a_2 < b_2 < 1$ and $T : [0,1] \times [0,1] \to [0,1]$ defined by

$$T(x,y) = \begin{cases} a_1, & a_1 < x \leq a_2,\ a_1 < y \leq b_2 \\ \min(x,y), & \text{otherwise} \end{cases}.$$

Then T is a left-continuous pseudo-t-norm which is not commutative.

3 STANDARD COMPLETENESS

This section is the main contribution of this paper.

In (Jenei and Montagna, 2003) it was proved that the logic psMTL^r is complete with respect to left-continuous pseudo-t-norms and their residua, by generalizing to the non-commutative case the proof for the standard completeness theorem for MTL logic from (Jenei and Montagna, 2002).

In this section we provide an alternative proof for the standard completeness theorem for psMTL logic. Moreover, we prove a strong standard completeness theorem with respect to finite theories, namely:

Theorem 3.1 (Finite Strong Standard Completeness). *Let T be a finite theory over* psMTL^r *and φ be a formula. T proves φ iff for each standard psMTL-algebra L and each L-model e of T, $e(\varphi) = 1$.*

This theorem is proved along this section. The idea behind the proof is to adapt to the non-commutative case the proof given by (Horčik, 2007) for the standard completeness theorem for MTL logic.

Let us consider a finite theory T and a formula φ over the logic psMTL^r such that $T \nvdash \varphi$. Thus, by Theorem 2.1, there is a psMTL^r-chain

$$\mathrm{L} = (L, \star_L, \to_L, \rightsquigarrow_L, \leq, 0, 1)$$

and an L-model e_L of T such that $e_L(\varphi) < 1$. We will show that L can be embedded into a standard psMTL-algebra. As in (Horčik, 2007), we define the following set:

$$G = \{e_L(\psi) \mid \psi \text{ is a subformula of } \chi \in T \cup \{\varphi\}\}.$$

Let S be the submonoid of L generated by the set G, i.e. $\mathrm{S} = (S, \star, \leq, 0, 1)$, where $\star$ denotes the restriction of $\star_L$ to S. Observe that S is finitely generated since G is finite.

Lemma 3.1. *The monoid* S *is countable and inversely well ordered, i.e. each subset of S has a maximum.*

Proof. Let $g_1, \ldots, g_n$ be the generators of S and M a subset of S. For any element of $m \in M$ we have $|m|_{g_i} = k_i$, for each $i \in \{1, \ldots, n\}$, where $|m|_{g_i}$ represents the number of occurrences of g_i in m. Therefore we can assign to each element of M an n-tuple $(k_1, \ldots, k_n) \in \mathbb{N}^n$. Let us denote by $m_{(k_1,\ldots,k_n)}$ the element from M associated with the n-tuple $(k_1, \ldots, k_n)$. Thus there is a subset $H \subseteq \mathbb{N}^n$ such that $(k_1, \ldots, k_n) \in H$ iff $m_{(k_1,\ldots,k_n)} \in M$. Moreover, if $(k_1, \ldots, k_n) \leq (t_1, \ldots, t_n)$, we have $m_{(k_1,\ldots,k_n)} \geq m_{(t_1,\ldots,t_n)}$ since $\star$ is order-preserving. Since H has only finitely many minimal elements, one of them must correspond to the maximum of M, thus S is inversely well ordered. □

Due to the previous lemma, we can introduce the following residua on S:

$$\begin{array}{rcl} a \to b & = & \max\{z \in S \mid z \star a \leq b\}, \\ a \rightsquigarrow b & = & \max\{z \in S \mid a \star z \leq b\}. \end{array}$$

Therefore we can prove the following result:

Proposition 3.1. *The enriched monoid*

$$\mathrm{S} = (S, \star, \to, \rightsquigarrow, \leq, 0, 1)$$

is a psMTL^r-chain and there exists an evaluation e_S of T such that $e_S(\varphi) < 1$.

Proof. We know that S is an integral totally ordered monoid and that $\leq$ is compatible with $\star$. The fact that $a \star b \leq c$ iff $a \leq b \to c$ iff $b \leq a \rightsquigarrow c$ follows immediately from the definitions of $\to$ and $\rightsquigarrow$. Hence S is an psMTL^r-chain.

Let us define an evaluation $e_S(p) = e_L(p)$, for every propositional variable p appearing in any $\chi \in T \cup \{\varphi\}$ and arbitrarily, otherwise. Let $\chi \in T \cup \{\varphi\}$. We show by induction on the complexity of χ that $e_S(\psi) = e_L(\psi)$, for each subformula ψ of χ. Almost all cases are trivial, therefore we treat only the case when $\psi = \psi_1 \rightsquigarrow \psi_2$. By definition we have $e_S(\psi) = e_L(\psi_1) \rightsquigarrow e_L(\psi_2) = \max\{z \in S \mid e_L(\psi_1) \star z \leq e_L(\psi_2)\}$. Since ψ is a subformula of χ, we have $e_L(\psi) \in S$. Consequently, $e_L(\psi_1) \star e_L(\psi) = e_L(\psi_1) *_L e_L(\psi) \leq e_L(\psi_2)$. Thus $e_L(\psi) \leq e_S(\psi)$. Now suppose there is an element $z' \in S$ such that $z' > e_L(\psi)$ and $e_L(\psi_1) \star z' \leq e_L(\psi_2)$. Since $z' \in L$, we get $z' \leq e_L(\psi_1) \rightsquigarrow e_L(\psi_2) = e_L(\psi)$ (contradiction). Hence $e_S(\psi) = e_L(\psi)$.

Since $e_S(\tau) = e_L(\tau) = 1$, for any $\tau \in T$, e_S is an S-model of T. Moreover, $e_S(\varphi) = e_L(\varphi) < 1$. □

Therefore we have the psMTL^r-chain S which is countable and inversely well ordered, and the evaluation e_S on S such that $e_S(\varphi) < 1$. The next step is to build a new psMTL^r-chain S$'$ order-isomorphic to $[0,1]$ in which S can be embedded.

In order to define such a psMTL^r-chain, we will use a similar construction as in the original proof of the standard completeness theorem from (Jenei and Montagna, 2003). We define the new universe by:

$$S' = \{(a,x) \mid a \in S \setminus \{0\},\ x \in (0,1]\} \cup \{(0,1)\}.$$

The order $\leq'$ on S' is the lexicographic order, i.e.

$$(a,x) \leq' (b,y) \text{ iff } a < b \text{ or } (a = b \text{ and } x \leq y).$$

Let I be the set of all idempotents of S, i.e $x \star x = x$. We define the following monoidal operation on S':

$$(a,x) \star' (b,y) = \begin{cases} (a \star b, 1), & a \star b < a \wedge b \\ (a, xy), & a = b,\ a \in I \\ \min\{(a,x),(b,y)\}, & \text{otherwise} \end{cases}$$

where xy stands for the usual product of reals. Note that in any case, the first coordonate always equals $a \star b$.

The proof of the following result follows closely the proof of Lemma 6 from (Horčik, 2007), but differences do appear since we are dealing with a non-commutativity operation $\star'$.

Lemma 3.2. *The structure*

$$\mathrm{S}' = (S', \star', \leq', (0,1), (1,1))$$

is a totally ordered integral monoid, where $(1,1)$ is the neutral element and the top element as well, $(0,1)$ is the bottom element and $\star'$ is monotone with respect to $\leq'$ on both arguments.

Proof. It is obvious that $(1,1)$ is the neutral element and the top element and that $(0,1)$ is the bottom element. Let us prove that $\star'$ is associative, i.e.

$$(a,x) \star' ((b,y) \star' (c,z)) = ((a,x) \star' (b,y)) \star' (c,z).$$

We denote the left-hand side (right-hand side, respectively) of the above equation by L (R, respectively). Let $P(a,b)$ denote the following property of $a,b \in S'$:

$$P(a,b): \ a = b \text{ and } a \in I.$$

Clearly, if $P(a,b)$ holds, then $P(b,a)$ holds as well. We must analyse several cases:

1. Suppose that none of $P(a,b)$, $P(b,c)$, $P(a, b \star c)$, $P(a \star b, c)$ is valid. We distinguish two cases:
   (i) $a \star b \star c = a \wedge b \wedge c$. Then both L and R are equal to $\min\{(a,x),(b,y),(c,z)\}$ and therefore associativity of $\star'$ holds.
   (ii) $a \star b \star c \neq a \wedge b \wedge c$. We claim that in this case both L and R are equal to $(a \star b \star c, 1)$. Let us consider the case of L, since for R we have a similar proof. If $b \star c = b \wedge c$, then $a \star (b \star c) \neq a \wedge (b \star c)$, therefore $\mathrm{L} = (a \star b \star c, 1)$. If $b \star c \neq b \wedge c$, then $(b,y) \star' (c,z) = (b \star c, 1)$ and $a \star b \star c \neq a$. Thus either $a \star b \star c \neq b \star c$ and then $\mathrm{L} = (a \star b \star c, 1)$, or $a \star b \star c = b \star c$ and then $\mathrm{L} = (a,x) \wedge (b \star c, 1) = (b \star c, 1) = (a \star b \star c, 1)$.
2. Suppose that $P(b,c)$ holds. Then $b = c$, b is idempotent and $\mathrm{L} = (a,x) \star' (b, yz)$. We obtain that

$$\mathrm{L} = \begin{cases} (b, xyz), & \text{if } a = b \\ (a \star b, 1), & \text{if } a \star b < a \wedge b \\ (a,x), & \text{if } a \star b = a \wedge b, a < b \\ (b, yz), & \text{if } a \star b = a \wedge b, a > b \end{cases}.$$

For R we have the following cases:

– if $a = b$, then $\mathrm{R} = (b,xy) \star' (c,z) = (b \star c, xyz) = (b,xyz)$;
– if $a \star b < a \wedge b$, then $(a \star b) \star b = a \star b = (a \star b) \wedge b$. Moreover $a \star b < a \wedge b \leq b$. Thus $\mathrm{R} = (a \star b, 1) \star' (b,z) = (a \star b, 1)$;
– if $a \star b = a \wedge b$ and $a < b$, then $\mathrm{R} = (a,x) \star' (b,z) = (a,x)$;
– if $a \star b = a \wedge b$ and $b < a$, then $\mathrm{R} = (b,y) \star' (b,z) = (b,yz)$.

3. Suppose that $P(a,b)$ holds. The proof follows similarly with the case 2.
4. Suppose that $P(a, b \star c)$ holds and none of $P(a,b)$, $P(b,c)$ is valid. Then $a = b \star c$ which implies $a \leq b, c$. Moreover $a < b$, since $a = b$ implies $P(a,b)$. From $a < b$, we obtain $a = a \star a < a \star b$, but $a \star b \leq a$, thus $a \star b = a$. Similar we get that $a = b \star a = a \star c = c \star a$. Therefore we have:
$$\mathrm{R} = (a,x) \star' (c,z) = \begin{cases} (a,xz), & \text{if } a = c \\ (a,x), & \text{if } a < c \end{cases}.$$
If $a = c$, then $\mathrm{L} = (a,x) \star' ((b,y) \star' (a,z)) = (a,x) \star' (a,z) = (a,xz)$. On the other hand, if $a < c$, then $a < b \wedge c$ since $a < b$. As $b \star c = a < b \wedge c$, we get $\mathrm{L} = (a,x) \star' (b \star c, 1) = (a,x)$.
5. Suppose that $P(a \star b, c)$ is valid and none of $P(a,b)$, $P(b,c)$ is valid. Similarly with the previous case, we can show that $\mathrm{L} = \mathrm{R}$.

Finally, let us prove that $\star'$ is monotone, i.e. $(a,x) \leq' (b,y)$ implies $(a,x) \star' (c,z) \leq' (b,y) \star' (c,z)$ and $(c,z) \star' (a,x) \leq (c,z) \star' (b,y)$. We prove only the first implication, since the other has a similar proof. We distinguish several cases:

1. Suppose that none of $P(a,c)$, $P(b,c)$ holds. We have the following cases:
– $a \star c = a \wedge c$ and $b \star c = b \wedge c$. We have that $(a,x) \star' (c,z) = \min\{(a,x),(c,z)\} \leq' \min\{(b,y),(c,z)\} = (b,y) \star' (c,z)$.
– $a \star c = a \wedge c$ and $b \star c < b \wedge c$. Notice that $a \star c \leq b \star c$ implies $(a,x) \star' (c,z) = \min\{(a,x),(c,z)\} \leq' (b \star c, 1) = (b,y) \star' (c,z)$.
– $a \star c < a \wedge c$ and $b \star c = b \wedge c$. We have that $a \star c < a \wedge c \leq b \wedge c = b \star c$. Therefore $(a,x) \star' (c,z) = (a \star c, 1)$ and the first component of $(b,y) \star' (c,z)$ is $b \star c = b \wedge c > a \star c$. Thus $(a,x) \star' (c,z) \leq' (b,y) \star' (c,z)$.
– $a \star c < a \wedge c$ and $b \star c < b \wedge c$. Notice that $a \star c \leq b \star c$ and $(a,x) \star' (c,z) = (a \star c, 1) \leq' (b \star c, 1) = (b,y) \star' (c,z)$.
2. Suppose that $P(a,c)$ holds. Then $a \star b = a$, since $a \leq b$ and a is idempotent. We have $(a,x) \star' (c,z) = (a,xz)$. Moreover,
$$(b,y) \star' (c,z) = \begin{cases} (a,yz), & \text{if } a = b \\ (a,z), & \text{if } a < b \end{cases}$$
If $a = b$, then $x \leq y$ and $(a,xz) \leq' (a,yz)$ since the usual product of reals is monotone. If $a < b$, then $(a,xz) \leq' (a,z)$ since $xz \leq z$.
3. Suppose that $P(b,c)$ holds. Moreover, suppose that $P(a,c)$ is not valid. Then $b = c$ and $a < b$. Thus $(b,y) \star' (c,z) = (b,yz)$ and
$$(a,x) \star' (c,z) = \begin{cases} (a \star b, 1), & \text{if } a \star b < a \\ (a,x), & \text{if } a \star b = a \end{cases}.$$
Since $a \star b \leq b$, we get $(a,x) \star' (c,z) \leq' (b,y) \star' (c,z)$.

□

Moreover, we can endow S' with two additional operations such that it becomes a psMTL^r-chain:

Lemma 3.3. *The structure*
$$\mathrm{S}' = (S', \star', \rightarrow', \rightsquigarrow', \leq', (0,1), (1,1))$$
is a psMTL^r-chain, where
$$(a,x) \rightarrow' (b,y) = \max\{(c,z) \mid (c,z) \star' (a,x) \leq' (b,y)\},$$
$$(a,x) \rightsquigarrow' (b,y) = \max\{(c,z) \mid (a,x) \star' (c,z) \leq' (b,y)\}.$$
Moreover, the mapping $\psi : \mathrm{S} \to \mathrm{S}'$ defined by $\psi(x) = (x,1)$ is an embedding of psMTL-algebras.

Proof. By Lemma 3.2, it is enough to show that $\star'$ has a left and a right residuum. Therefore we must show that each set of the form $M_1 = \{(c,z) \mid (c,z) \star' (a,x) \leq' (b,y)\}$ or $M_2 = \{(c,z) \mid (a,x) \star' (c,z) \leq' (b,y)\}$ has a maximum. Let us consider the case of M_1. Since S is inversely well ordered by Lemma 3.1, $\pi_1(M_1)$ has a maximum c_{M_1}, where π_1 is the projection on the first component. Thus there is an element of the form $(c_{M_1}, z) \in M_1$. If $c_{M_1} \star a < b$, then $(c_{M_1}, 1)$ is the maximum of M_1. Thus suppose that $c_{M_1} \star a = b$. We distinguish several cases:

1. Suppose $c_{M_1} \star a < c_{M_1} \wedge a$. Then, for any z, we have $(c_{M_1}, z) \star' (a,x) = (c_{M_1} \star a, 1)$. Thus $(c_{M_1}, 1)$ must be the maximum of M_1.
2. Suppose that $P(c_{M_1}, a)$ holds. Then we have $(c_{M_1}, z) \star' (a,x) = (c_{M_1}, zx)$. If $x \leq y$, then $(c_{M_1}, 1)$ is the maximum of M_1. If $x > y$, then the maximum of M_1 is $(c_{M_1}, y/x)$.
3. Suppose that $c_{M_1} \star a = c_{M_1} \wedge a$. Moreover, let us assume that $P(c_{M_1}, a)$ is not valid. Then $(c_{M_1}, z) \star' (a,x) = \min\{(c_{M_1}, z), (a,x)\}$. The have several cases:
· if $a = c_{M_1}$, then $\min\{(c_{M_1}, z), (a,x)\} = (c_{M_1}, x \wedge z)$ and the maximum of M_1 is either $(c_{M_1}, 1)$ when $x \leq y$, or (c_{M_1}, y) otherwise;

- if $a < c_{M_1}$, then $\min\{(c_{M_1}, z), (a,x)\} = (a,x)$ and $(c_{M_1}, 1)$ is the maximum of M_1;
- if $a > c_{M_1}$, then $\min\{(c_{M_1}, z), (a,x)\} = (c_{M_1}, z)$ and (c_{M_1}, y) is the maximum of M_1.

It follows immediately from the definitions of $\rightarrow'$ and $\rightsquigarrow'$ that the following hold:

$$(a,x) \star' (b,y) \leq' (c,z) \text{ iff } (a,x) \leq' (b,y) \rightarrow' (c,z) \text{ iff } (b,y) \leq' (a,x) \rightsquigarrow (c,z).$$

Finally, it can be easily verified that the mapping ψ satisfies the following equalities:
$\psi(a) \star' \psi(b) = (a,1) \star' (b,1) = (a \star b, 1) = \psi(a \star b)$,
$\psi(a) \rightarrow' \psi(b) = \max\{(c,z) | (c,z) \star' (a,1) \leq' (b,1)\} = (a \rightarrow b, 1) = \psi(a \rightarrow b)$,
$\psi(a) \rightsquigarrow' \psi(b) = \max\{(c,z) | (a,1) \star' (c,z) \leq' (b,1)\} = (a \rightsquigarrow b, 1) = \psi(a \rightsquigarrow b)$.
Therefore the proof is completed. □

The remaining step is to show that S$'$ is order-isomorphic to $[0,1]$. As shown in (Hrbacek and Jech, 1999), any totally ordered set X is order-isomorphic to $[0,1]$ if it satisfies the following properties: X is complete, X has a maximum and a minimum and X has a countable subset D which is dense in X, i.e. for each $x, y \in X$ such that $x < y$, there is $z \in D$ such that $x < z < y$.

Thus it is enough to prove that S$'$ satisfies all the above mentioned conditions. Clearly, S$'$ has a maximum and a minimum. Further, the subset

$$\{(a,x) \mid a \in S,\ x \in \mathbb{Q} \cap [0,1]\}$$

of S' is countable and dense in S'. Finally, given $X \subseteq S'$, $X \neq \emptyset$, let $Z = \pi_1(X)$. Then $Z \subseteq S$ and $Z \neq \emptyset$, hence Z has a maximum a_0, since S is inversely well ordered by Lemma 3.1. Now let

$$\alpha = \sup\{x \in (0,1] \mid (a_0, x) \in X\}.$$

Then $(a_0, \alpha) = \sup(X)$. In conclusion S' is complete and we have the following result:

Lemma 3.4. *The set S' is order-isomorphic to $[0,1]$, i.e. there is a bijection $\Phi : S' \rightarrow [0,1]$ such that $(a,x) \leq' (b,y)$ implies $\Phi(a,x) \leq \Phi(b,y)$.*

Let us define the following operations on $[0,1]$:

$$a \odot b = \Phi(\Phi^{-1}(a) \star' \Phi^{-1}(b)),$$

$$a \rightarrow_{\odot} b = \Phi(\Phi^{-1}(a) \rightarrow' \Phi^{-1}(b)),$$

$$a \rightsquigarrow_{\odot} b = \Phi(\Phi^{-1}(a) \rightsquigarrow' \Phi^{-1}(b)).$$

Then the structure

$$[0,1]_{\odot} = ([0,1], \odot, \rightarrow_{\odot}, \rightsquigarrow_{\odot}, \leq, 0, 1)$$

is a standard psMTL-algebra and we have an $[0,1]_{\odot}$-model of T such that $\Phi(\psi(e_S(\varphi))) < 1$. Thus the proof of Theorem 3.1 is finished.

4 CONCLUSIONS

In this paper we proved the finite strong standard completeness theorem for the non-commutative logic psMTL. For proving this result, we used a construction of standard psMTL-algebras which is interesting on its own. Our proof can also be seen as an alternative proof for the standard completeness theorem for psMTL logic given in (Jenei and Montagna, 2003).

As a future research, it would be interesting to extend our proof for the finite strong standard completeness theorem for psMTL for some schematic extensions of psMTL logic. We mention that in (Diaconescu, 2012), the proof for the standard completeness theorem for psMTL logic given in (Jenei and Montagna, 2003) was extended for the extensions psSMTL and psIMTL of psMTL logic.

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