Simple Fuzzy Logic Models to Estimate the Global Temperature Change Due to GHG Emissions

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Abstract: Future scenarios (through 2100) developed by the Intergovernmental Panel on Climate Change (IPCC) indicate a wide range of concentrations of greenhouse gases (GHG) and aerosols, and the corresponding range of temperatures. These data, allow inferring that higher temperature increases are directly related to higher emission levels of GHG and to the increase in their atmospheric concentrations. It is evident that lower temperature increases are related to smaller amounts of emissions and, to lower GHG concentrations. In this work, simple linguistic rules are extracted from results obtained through the use of simple linear scenarios of emissions of GHG in the Magicc model. These rules describe the relations between the GHG, their concentrations, the radiative forcing associated with these concentrations, and the corresponding temperature changes. These rules are used to build a fuzzy model, which uses concentration values of GHG as input variables and gives, as output, the temperature increase projected for year 2100. A second fuzzy model is presented on the temperature increases obtained from the same model but including a second source of uncertainty: climate sensitivity. Both models are very attractive because their simplicity and capability to integrate the uncertainties to the input (emissions, sensitivity) and the output (temperature).

1 INTRODUCTION

There is a growing scientific consensus that increasing emissions of greenhouse gases (GHG) are changing the Earth's climate. The IPCCs Fourth Assessment Report (IPCC, 2007) states that warming of the climate system is unequivocal and notes that eleven of the last twelve years (1995-2006) rank among the twelve warmest years of recorded temperatures (since 1850). The projections of the IPCCs Third Assessment Report (TAR) (IPCC, 2001) regarding future global temperature change ranged from 1.4 to 5.8 °C. More recently, the projections indicate that temperatures would be in a range spanning from 1.1 to 4 °C, but that temperatures increases of more than 6 °C could not be ruled out (IPCC; 2007). This wide range of values reflects the uncertainty in the production of accurate projections of future climate change due to different potential pathways of GHG emissions. There are other sources of the uncertainty preventing us from obtaining better precision. One of them is related to the computer models used to project future climate change. The global climate is a highly complex system due to many physical, chemical, and biological processes that take place among its subsystems within a wide range of space and time scales.

Global circulation models (GCM) based on the fundamental laws of physics try to incorporate those known processes considered to constitute the climate system and are used for predicting its response to increases in GHG (IPCC, 2001). However, they are not perfect representations of reality because they do not include some important physical processes (e.g. ocean eddies, gravity waves, atmospheric convection, clouds and small-scale turbulence) which are too small or fast to be explicitly modeled. The net impact of these small scale physical processes is included in the model by means of parameterizations (Schmidt, 2007). In addition, more complex models imply a large number of parameterized processes and different models use different parameterizations. Thus, different models, using the same forcing produce different results.

One of the main sources of uncertainty is,

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however, the different potential pathways for anthropogenic GHG emissions, which are used to drive the climate models. Future emissions depend on numerous driving forces, including population growth, economic development, energy supply and use, land-use patterns, and a variety of other human activities (Special Report on Emissions Scenarios, SRES). Future temperature scenarios have been obtained with the emission profiles corresponding to the four principal SRES families (A1, A2, B1, and B2) (Nakicenovic et al., 2000). From the point of view of a policy-maker, the results of the 3rd and 4th IPCC's assessments regarding the projection of global or regional temperature increases are difficult to interpret due to the wide range of the estimated warming. Nevertheless this is an aspect of uncertainty that scientists and ultimately policy-makers have to deal with. Furthermore, most of the available methodologies that have been proposed for supporting decision-making under uncertainty do not take into account the nature of climate change's uncertainty and are based on classic statistical theory that might not be adequate. Climate change's uncertainty is predominately epistemic and, therefore, it is critical to produce or adapt methodologies that are suitable to deal with it and that can produce policy-useful information. The lack of such methodologies is noticeable in the IPCC's AR4 Contribution of the Working Group I, where the proposed best estimates, likely ranges and probabilistic scenarios are produced using statistically questionable devices (Gay and Estrada, 2009).

Two main strategies have been proposed for dealing with uncertainty: trying to reduce it by improving the science of climate change a feat tried in the AR4 of the IPCC, and integrating it into the decision-making processes (Schneider, 2003). There are clear limitations regarding how much of the uncertainty can be reduced by improving the state of knowledge of the climate system, since there remains the uncertainty about the emissions which is more a result of political and economic decisions that do not necessarily obey natural laws.

Therefore, we propose that the modern view of climate modelling and decision-making should become more tolerant to uncertainty because it is a feature of the real world (Klir and Elias, 2002). Choosing a modelling approach that includes uncertainty from the start tends to reduce its complexity and promotes a better understanding of the model itself and of its results. Science and decision-making have always had to deal with uncertainty and various methods and even branches of science, such as Probability, have been developed for this matter (Jaynes, 1957). Important efforts have been made for developing approaches that can integrate subjective and partial information, being the most successful ones Bayesian and maximum entropy methods and more recently, fuzzy set theory where the concept of objects that have not precise boundaries was introduced (Zadeh, 1965). Fuzzy logic provides a meaningful and powerful representation of uncertainties and is able to express efficiently the vague concepts of natural language (Zadeh, 1965). These characteristics could make it a very powerful and efficient tool for policy makers due to the fact that the models are based on linguistic rules that could be easily understood.

In this paper two fuzzy logic models are proposed for the global temperature changes (in the year 2100) that are expected to occur in this century. The first model incorporates the uncertainties related to the wide range of emission scenarios and illustrates in a simple manner the importance of the emissions in determining future temperatures. The second incorporates the uncertainty due to climate sensitivity that pretends to emulate the diversity in modelling approaches. Both models are built using the Magicc (Wigley, 2008) model and Zadeh´s extension principle for functions where the independent variable belongs to a fuzzy set. Magicc is capable of emulating the behaviour of complex GCMs using a relative simple one dimensional model that incorporates different processes e.g. carbon cycle, earth-ocean diffusivity, multiple gases and climate sensitivity. In our second case we intend to illustrate the combined effects of two sources of uncertainty: emissions and model sensitivity. It is clear that we are leaving out of this paper other important sources of uncertainty whose contribution would be interesting to explore. The GCMs are, from our point of view, useful and very valuable tools when it is intended to study specific aspects or details of the global temperature change. Nevertheless, when the goal is to study and to test global warming policies, simpler models much easier to understand become very attractive. Fuzzy models can perform this task very efficiently.

2 FUZZY LOGIC MODEL OBTAINED FROM IPCC DATA

The Fourth Assessment Report of the IPCC shows estimates of emissions, concentrations, forcing and temperatures through 2100 (IPCC, 2007). Although there are relationships among these variables, as those reflected in the figure 1 (upper panel), it would be useful to find a way to relate emissions directly with increases of temperature. A more physical relation is established between concentrations and temperature because the latter depends almost directly upon the former through the forcing terms. Concentrations are obtained integrating over time the emissions minus the sinks of the GHGs. One way of relating directly emissions and temperature, could be achieved if the emission trajectories were linear and non-intersecting as illustrated in figure 1 (upper panel). Here, we perform this task by means of a fuzzy model, which is based on the Magicc model (Wigley, 2008) and Zadeh´s extension principle (see Appendix).

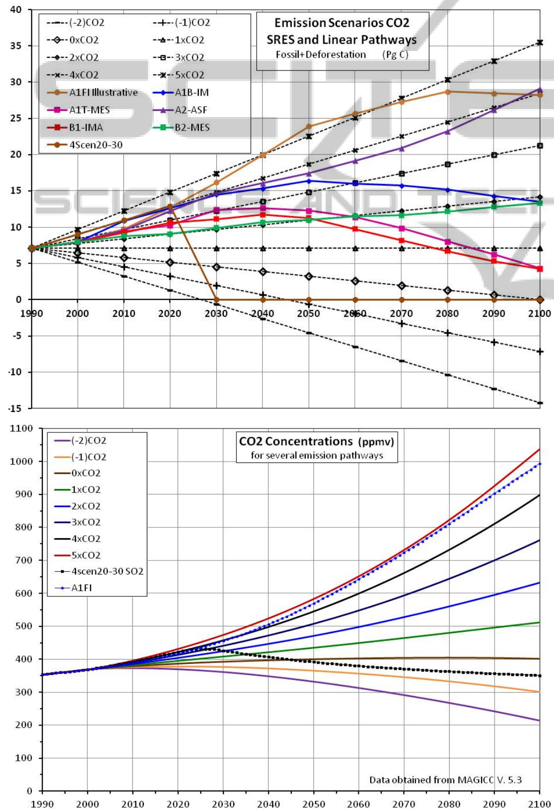

Figure 1: Upper panel: Emissions scenarios CO2, Illustrative SRES and Linear Pathways. (-2) CO2 means -2 times the emission (fossil + deforestation) of CO2 of 1990 by 2100 and so for -1, 0, 1, to 5 CO2. All the linear pathways contain the emission of non CO2 GHG as those of the A1FI. 4scen20-30 scenario follows the pathway of 4xCO2 but at 2030 all gases drop to 0 emissions or minimum value in CH4, N2O and SO2 cases. Lower panel: CO2 Concentrations for linear emission pathways, 4scen20-30 SO2 and A1FI are shown for reference. Data calculated using Magicc V. 5.3.

Using as input for the Magicc model the emissions shown in the previous figure we calculate the resulting concentrations (figure 1 lower panel); forcings (figure 2 upper panel) and global mean temperature increments (figure 2 lower panel).

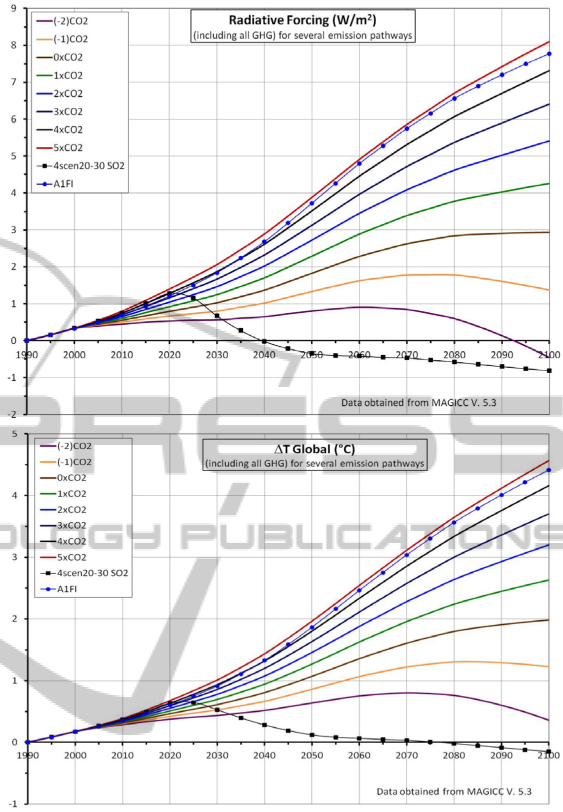

Figure 2: Upper panel: Radiative forcings (all GHG included) for linear emission pathways and A1FI SRES illustrative, the 4scen20-30 SO2 only include SO2. Lower panel: Global temperature increments for linear emission pathways, 4scen20-30 SO2 and A1FI as calculated using Magicc V. 5.3.

The set of emissions shown in figure 1 (upper panel) has been simplified to linear functions of time that reach by the year 2100 values from minus two times to 5 times the emissions of 1990. The trajectories labelled 5CO2 and (-2) CO2 contain the trajectories of the SRES. We observe that the concentrations corresponding to the 5CO2 and the A1FI trajectories, by year 2100 are very close. The choice of linear pathways allows us to associate emissions to concentrations to forcings and temperatures in a very simple manner. We can say than any trajectory of emissions contained within two of the linear ones will correspond, at any time with a temperature that falls within the interval delimited by the temperatures corresponding to the linear trajectories. This is illustrated for the A1FI trajectory, in figure 2 (lower panel) that falls within the temperatures of the 5CO2 and 4CO2 trajectories. We decided to find emission paths that would lead to temperatures of two degrees or less by the year 2100, this led us to the -2CO2, -1CO2 and 0CO2. The latter is a trajectory of constant emissions equal to the emissions in 1990 that gives us a temperature of two degrees by year 2100.

From the linear representation, it is easily deduced that *very high* emissions correspond to *very large* concentrations, forcings and *large* increases of temperature. It is also possible to say that large concentrations correspond to large temperature increases etc. This last statement is very important because in determining the temperature the climate models directly use the concentrations which are the time integral of sources and sinks of the green house gases (GHG). Therefore the detailed history of the emissions is lost. Nevertheless the statement, to large concentrations correspond large temperature increases still holds. These simple observations allows us to formulate a fuzzy model, based on linguistic rules of the **IF-THEN** form, which can be used to estimate increases of temperature within particular uncertainty intervals. Fuzzy logic provides a meaningful and powerful representation of measurement of uncertainties, and it is able to represent efficiently the vague concepts of natural language, of which the climate science is plagued. Therefore, it could be a very useful tool for decision makers. The basic concepts of fuzzy logic are presented in Appendix.

The first fuzzy model one input one output defined for the global temperature change is (quantities between parenthesis were used with Zadeh's principle to generate the fuzzy model, the number 1 means the membership value (μ) of the input variables used in formulating the fuzzy model):

```
1. If (concentration is very low (about -2CO2)) then (deltaT is very low (1)
2. If (concentration is low (about -1CO2)) then (deltaT is low (1)
3. If (concentration is medium-low (about 0CO2) then (deltaT is medium-low (1)
4. If (concentration is medium (about 1CO2)) then (deltaT is medium (1)
5. If (concentration is medium-high (about 2CO2)) then (deltaT is medium-high (1)
6. If (concentration is high (about 3CO2)) then (deltaT is high (1)
7. If (concentration is very high (about 4CO2)) then (deltaT is very high (1)
8. If (concentration is extremely high (about 5CO2)) then (deltaT is extremely high (1)
```

The 8 rules for concentration are based on 8 adjacent triangular membership functions (the simplest form) corresponding to linear emission trajectories (-2CO2 to 5CO2). The concentrations were obtained from Magicc model and cover the entire range (210 to 1045 ppmv). The apex of each membership function (μ=1) corresponds with the base (μ=0) of the adjacent one, as we show below:

```
                    (μ=0, μ=1, μ=0)
1. -2CO2            (210, 213, 300)
2. -1CO2            (213, 300, 401)
3.  0CO2            (300, 401, 513)
4.  1CO2            (401, 513, 633)
5.  2CO2            (513, 633, 762)
6.  3CO2            (633, 762, 899)
7.  4CO2            (762, 899, 1038)
8.  5CO2            (899, 1038, 1045)
```

The global temperature changes were obtained through Zadeh's extension principle applied to data from Magicc model.

From the point of view of a policy maker, a fuzzy model as the one represented by the previous rules is a very useful tool to study the effect of different policies on the increases of temperature.

The fuzzy rules model can be evaluated by means of the fuzzy inference process in such a way that each possible concentration value is mapped into an increase of temperature value by means of the Mamdani's defuzzification process (see Appendix). The resulting increases of temperature at year 2100 for each possible concentration (emission in the case of our linear model) value (solid line) are shown in the upper panel of figure 3.

The lower panel illustrates the formulation of the rules by showing the fuzzy set associated with the different classes of concentrations, the antecedent of the fuzzy rule, the IF part and the consequent fuzzy set temperature, the THEN part. The figure 3 lower panel also illustrates the uncertainties of one estimation: If the concentration is of 401 ppmv (it fires rule number 3) within an uncertainty interval of (300 to 513 ppmv) 4 then the temperature increment is 1.95 degrees within an uncertainty interval of (1.23 to 2.63 deg C) in this case the temperatures will have uncertainties of one or two times the intervals defined by the expert or the researcher.

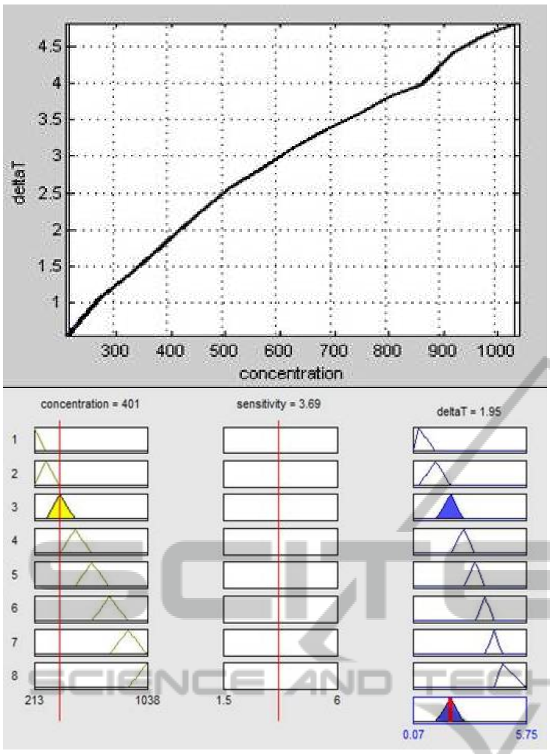

Figure 3: Fuzzy model based on linguistic rules and Zadeh´s principle. Upper panel: increases of temperature at year 2100 for each possible concentration (emission in the case of our linear model) value (solid line). Lower panel: Fuzzy rules associated with the different classes of concentrations. (Calculated with MATLAB).

The fuzzy model is simpler and obviously less computationally expensive than the set of GCM's reported by the IPCC. The most important benefit, however, is its usefulness for policy-makers. For example, if the required increase of temperature should be *very low* or *low (-2CO2, -1CO2)*, then the policy-maker knows, on the basis of this model, that concentrations should not exceed the class *small*.

3 A SIMPLE CLIMATE MODEL AND ITS CORRESPONDING FUZZY MODEL

Here, we again use the Magicc model but this time we introduce a second source of uncertainty, the climate sensitivity. The purpose is to illustrate the effects of the combination of two sources of uncertainty on the resulting temperatures. The climate model is driven by our linear emission paths. The relationship between concentrations and sensitivity and increases of temperature at year 2100 is then used to construct a fuzzy model following the extension principle of the fuzzy logic approach (see Appendix).

The set of fuzzy rules obtained in this case is the following.

1. If (concentration is very very low) and (sensitivity is low) then (deltaT is low) (1)
2. If (concentration is very low) and (sensitivity is low) then (deltaT is low) (1)
3. If (concentration is very low) and (sensitivity is high) then (deltaT is med) (1)
4. If (concentration is medium-low) and (sensitivity is low) then (deltaT is low) (1)
5. If (concentration is medium-low) and (sensitivity is high) then (deltaT is high) (1)
6. If (concentration is medium) and (sensitivity is low) then (deltaT is med) (1)
7. If (concentration is medium) and (sensitivity is high) then (deltaT is high) (1)
8. If (concentration is medium-high) and (sensitivity is low) then (deltaT is med) (1)
9. If (concentration is medium-high) and (sensitivity is high) then (deltaT is high) (1)
10. If (concentration is high) and (sensitivity is low) then (deltaT is med) (1)
11. If (concentration is high) and (sensitivity is high) then (deltaT is high) (1)
12. If (concentration is medium-low) and (sensitivity is med) then (deltaT is med) (1)

Note that we have used the same nomenclature as before and the very high and extremely high concentrations are not considered.

And the fuzzy sets for the temperature and sensitivity are shown in figure 4. We used this figure to build the rule above in combination with 6 fuzzy sets for concentration similar to those from our first model described in section 2:

	(μ=0, μ=1, μ=0)
1. -2CO2	(100, 213, 300)
2. -1CO2	(213, 300, 401)
3. 0CO2	(300, 401, 513)
4. 1CO2	(401, 513, 633)
5. 2CO2	(513, 633, 762)
6. 3CO2	(633, 762, 899)

For sensitivity we built 3 triangular fuzzy sets corresponding to sensitivity values of 1.5, 3 and 6 deg C/W/m2, showed below:

	(μ=0, μ=1, μ=0)
1. 1.5 (low)	(1.5, 1.5, 3.0)
2. 3.0 (medium)	(1.5, 3.0, 6.0)
3. 4.5 (high)	(3.0, 6.0, 6.0)

Similarly, for global temperature change we have 3 triangular fuzzy sets built with data obtained from Magicc model and Zadeh's extension principle for each sensitivity value; the apex of each fuzzy set is the value of global temperature change for the 0CO2 linear emission path according to the value of sensitivity, the base of the fuzzy sets range from -2CO2 to 3CO2 (assuming global temperature changes below 6 deg C) for each sensitivity value (see figure 4):

	(μ=0, μ=1, μ=0)
1. Low	(0.07, 1.07, 2.13)
2. Medium	(0.36, 1.98, 3.70)
3. High	(0.92, 3.27, 5.75)

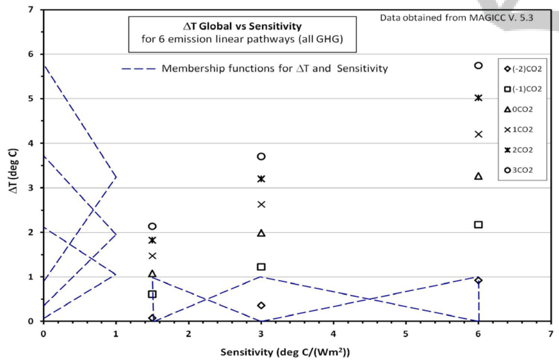

Figure 4: ΔT Global and sensitivity fuzzy sets for six linear emission pathways at 2100. The dashed lines show the membership functions.

The Mamdani's fuzzy inference method is used also here as the defuzzification method to compute the increase of temperature values. The results are shown in figure 5. The upper panel of figure 5 shows the surface resulting from the defuzzification process. The lower panel illustrates again that for the case of a concentration of 401 ppmv and a sensitivity of 3 (medium sensitivity) the temperature is about 2 degrees within an uncertainty interval of (0.36 to 3.70 deg C) where the membership value is different from 0. When we compare our previous result with this one we find that the answers are very close in fact the fall within the uncertainty intervals of both. The uncertainty of concentrations and sensitivity are respectively (300 to 513 ppmv) and (1.5 to 6 deg C/W/m2). The result is to be expected since in our first experiment we used the Magicc model with default value for the sensitivity and this turns to be of 3.

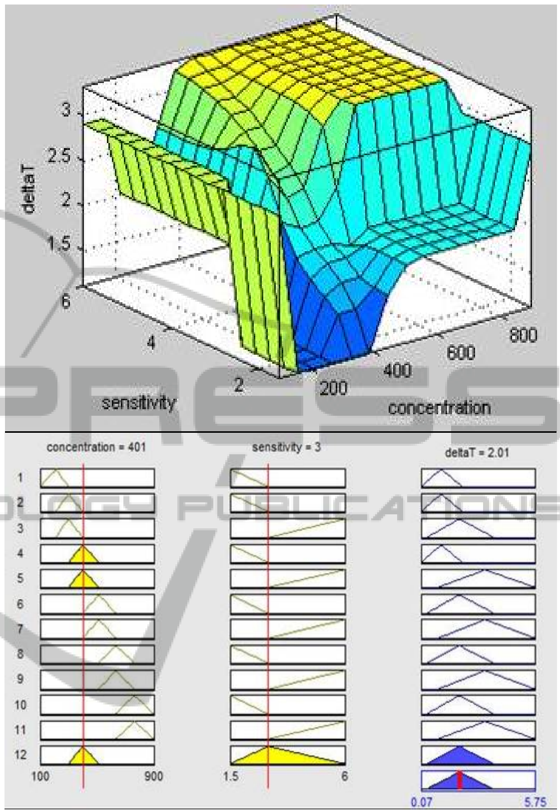

Figure 5: Fuzzy model for concentrations and sensitivities. Upper panel: ΔT surface. Lower panel: Fuzzy rules. (Calculated with MATLAB).

4 DISCUSSION AND CONCLUSIONS

In this work, simple linguistic fuzzy rules relating concentrations and increases of temperatures are extracted from the application of the Magicc model. The fuzzy model uses concentration values of GHG as input variable and gives, as output, the increase of temperature projected at year 2100. A second fuzzy model based on linguistic rules is developed based on the same Magicc climate model introducing a second source of uncertainty coming from the different sensitivities used by the Magicc to emulate more complicated GCMs used in the IPCC reports. These kind of fuzzy models are very useful due to their simplicity and to the fact that include the uncertainties associated to the input and output variables. Simple models that, however, could contain all the information that is necessary for policy makers, these characteristics of the fuzzy models allow not only the understanding of the problem but also the discussion of the possible options available to them. For example going back to the question of stabilizing global temperatures at about 2 degrees or less, we can see the fuzziness of the proposition; we could estate that we should stay well below 400 ppmv by year 2100. The observed emission pictured in figure 6 where the IPCC scenarios are also shown are contained within A1F1 and the A1B therefore we could say that they point to a temperature increase that will surpass the two degrees. In fact to keep temperatures under 2 degrees we have already stated we should remain under 400 ppmv and we are very very close (fuzzy concept) to this concentration.

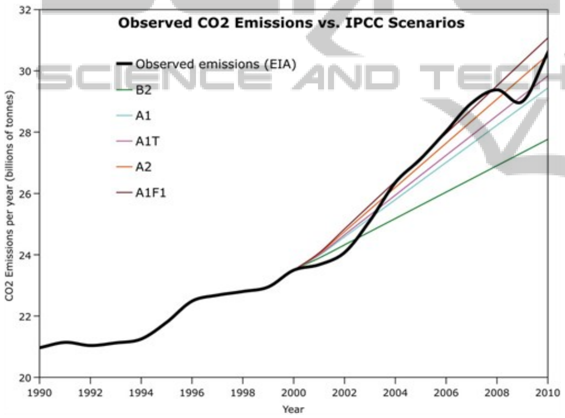

Figure 6: Observed CO2 emissions against IPCC AR4 scenarios (taken form http://www.treehugger.com/clean-technology/iea-co2-emissions-update-2010-bad-news-very-bad-news.html).

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APPENDIX

A.1 Fuzzy Logic Basic Concepts

As Klir stated in his book (Klir and Elias, 2002), the view of the concept of uncertainty has been changed in science over the years. The traditional view looks to uncertainty as undesirable in science and should be avoided by all possible means. The modern view is tolerant of uncertainty and considers that science should deal with it because it is part of the real world. This is especially relevant when the goal is to construct models. In this case, allowing more uncertainty tends to reduce complexity and increase credibility of the resulting model. The recognition by the researchers of the important role of uncertainty mainly occurs with the first publication of the fuzzy set theory, where the concept of objects that have not precise boundaries (fuzzy sets) is introduced (Zadeh, 1965).

Fuzzy logic, based on fuzzy sets, is a superset of conventional two-valued logic that has been extended to handle the concept of partial truth, i.e. truth values between completely true and completely false.

In classical set theory, when A is a set and x is an object, the proposition "x is a member of A" is necessarily true or false, as stated on equation 1,

$$A(x) = \begin{cases} 1 & for \ \ x \in A \\ 0 & for \ \ x \notin A \end{cases} \tag{1}$$

whereas, in fuzzy set theory, the same proposition is not necessarily either true or false, it may be true only to some degree. In this case, the restriction of classical set theory is relaxed allowing different degrees of membership for the above proposition, represented by real numbers in the closed interval [0,1], i.e. $A : X \rightarrow [0,1]$. Figure A.1 presents this concept graphically.

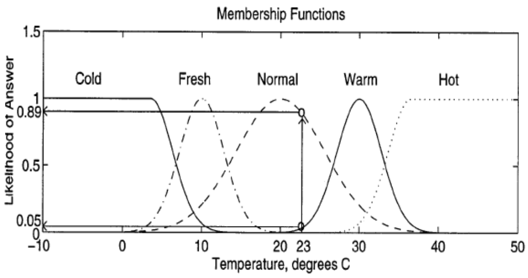

Figure A.1: Gaussian membership functions of a quantitative variable representing ambient temperature.

Figure A.1 illustrates the membership functions of the classes: cold, fresh, normal, warm, and hot, of the ambient temperature variable. A temperature of 23°C is a member of the class normal with a grade of 0.89 and a member of the class warm with a grade of 0.05. The definition of the membership functions may change with regard to who define them. For example, the class normal for ambient temperature variable in Mexico City can be defined as it is shown in figure A.1. The same class in Anchorage, however, will be defined more likely in the range from -8°C to -2°C. It is important to understand that the membership functions are not probability functions but subjective measures. The opportunity that brings fuzzy logic to represent sets as degrees of membership has a broad utility. On the one hand, it provides a meaningful and powerful representation of measurement uncertainties, and, on the other hand, it is able to represent efficiently the vague concepts of natural language. Going back to the example of figure A.1, it is more common and useful for people to know that tomorrow will be hot than to know the exact temperature grade.

At this point, the question is, once we have the variables of the system that we want to study described in terms of fuzzy sets, what can we do with them? The membership functions are the basis of the fuzzy inference concept. The compositional rule of inference is the tool used in fuzzy logic to perform approximate reasoning. Approximate reasoning is a process by which an imprecise conclusion is deduced from a collection of imprecise premises using fuzzy sets theory as the main tool.

The compositional rule of inference translates the modus ponens of the classical logic to fuzzy logic. The generalized modus ponens is expressed by:

Rule: If X is A then Y is B
Fact: X is A'
Conclusion: Y is B'

where, X and Y are variables that take values from the sets X and Y, respectively, and A, A' and B, B' are fuzzy sets on X and Y, respectively. Notice that the Rule expresses a fuzzy relation, R, on X x Y.

Then, if the fuzzy relation, R, and the fuzzy set A' are given, it is possible to obtain B' by the compositional rule of inference, given in equation 2,

$$B'(y) = \sup_{x \in X} \min\left[A'(x), R(x,y)\right] \tag{2}$$

where **sup** stands for supremum (least upper bound) and **min** stands for minimum. When sets X and Y are finite, **sup** is replaced by the maximum operator, **max**. Figure A.2 illustrates in a simplified way the compositional rule of inference graphically.

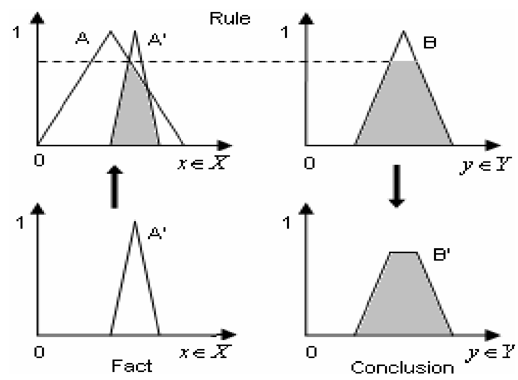

Figure A.2: Simplified graphical representation of the compositional rule of inference.

The compositional rule of inference is also useful in the general case where a set of rules, instead of only one, define the fuzzy relation, R.

A.2 Extension Principle

Zadeh says that rather than regarding fuzzy theory as a single theory, we should regard the conversion process from binary to membership functions as a methodology to generalize any specific theory from a crisp (discrete) to a continuous (fuzzy) form. The extension principle enables us to extend the domain of a function on fuzzy sets, i.e., it allows us to determine the fuzziness in the output given that the input variables are already fuzzy. Therefore, it is a particular case of the compositional rule of inference. Figure A.3 gives a first idea of the extension principle showing an example of two input variables with 3 fuzzy sets each.

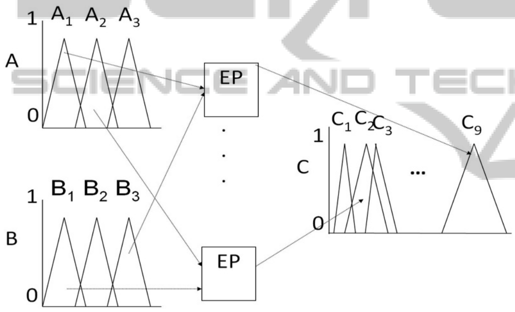

Figure A.3: Extension principle example for two input fuzzy variables A and B with 3 fuzzy sets each.

The extension principle is applied to transform each fuzzy pair (A_i, B_j), in a fuzzy set of the C output variable. Notice that in the example of figure A.3 we have 9 pairs of fuzzy input sets and, therefore, 9 fuzzy sets are obtained representing the conclusion as shown in the right hand side of figure A.3. The extension principle when two input variables are available is presented in equation 3. C_k is the k^{th} output fuzzy set extended from the two input fuzzy sets A_i and B_j. In the example at hand, as illustrated in figure A.3, the extension principle is applied 9 times, to obtain each of the output fuzzy sets associated to each fuzzy input pair.

$$C_k = \max_{C_k = f(A_i, B_j)} \min \left[A_i, B_j \right] \qquad (3)$$

For instance, the output fuzzy set C_9, is obtained when using the extension principle of equation 3 with the input fuzzy sets A_1 and B_3 (Klir and Elias, 2002); (Dubois and Prade, 1980); (Ross, 2004).