Modal Semirings with Operators for Knowledge Representation

Kim Solin
Dept. of Philosophy, Uppsala University, Uppsala, Sweden

Keywords: Modal Semirings, Modal Algebra, Boolean Algebra with Operators, Modal Semirings with Operators, Knowledge Representation, Dynamic Epistemic Logic.

Abstract: Modal semirings are combined with modal algebra (Boolean algebra with operators) to form *modal semirings with operators*. In turn, these are extended with a revision operator and used for knowledge representation.

1 INTRODUCTION

There are several ways of representing and reasoning about changing knowledge. Two of these stand out. First, one has the semiformal approach of Alchourron, Gärdenfors and Makinson (Alchourrón et al., 1985), which became a landmark in this field, and which has spurred a colossal amount of research. Second, one has a way of reasoning about changing knowledge that is fully formal: dynamic epistemic logic, see (van Ditmarsch et al., 2007) with references. But this paper will be about a third style, namely, the algebraic. If one takes this approach, one strives to reason about knowledge with the aid of algebra, see (Baltag et al., 2007; Baltag et al., 2005; Baltag and Sadrzadeh, 2006; Möller, 2008; Solin, 2010; Panangaden and Sadrzadeh, 2010). Why would one take the algebraic approach? The answer is that there are many advantages to algebra, in particular clear-cut reasoning, and efficient automation and mechanisation (Aboul-Hosn and Kozen, 2006; Höfner and Struth, 2007; Foster and Struth, 2012).

The algebraic approach can be further subdivided into a concrete and an abstract approach. The concrete-algebraic approach means that one reasons in a concrete algebra of, for example, relations. The abstract-algebraic approach means that one via axioms defines an abstract algebra suited for one's purposes; the approach could also be called axiomatic-algebraic. I shall focus on the abstract-algebraic development that has its roots in Dexter Kozen's work on Kleene algebra (Kozen, 1994).

1.1 Background

Dexter Kozen conceived of Kleene algebra with tests in the late 1990s (Kozen, 1997). This structure, which is an idempotent semiring extended with Kleene star, and embedded with a Boolean subalgebra of tests, was later developed into Kleene algebra with domain by Desharnais, Möller and Struth (Desharnais et al., 2006). The last-mentioned structure allows the introduction of modal operators in the sense of dynamic logic (Harel et al., 2000), so it is also called modal Kleene algebra.

Modal Kleene algebra was further augmented with a revision operator (Solin, 2010) that allows one to reason abstract-algebraically about changing knowledge in the style of dynamic epistemic logic (van Ditmarsch et al., 2007). The tests were then conceived of as actions that check whether the agent believes some proposition or not. But since the tests form a Boolean algebra, this had the unfortunate consequence that $\neg \mathsf{B} \neg p$ by double negation reduces to p. In addition, one could not distinguish between the agent's beliefs and truth, since they were both modelled by the same structure. Although this might be all right in some circumstances, in others it is certainly not. The aim of this paper is to remedy that situation by introducing the belief operator as an algebraic operator in its own right, and to incorporate that operator into modal Kleene algebra. I will focus on the semiring reduct, that is, on modal semirings.

Since one has a Boolean subalgebra at hand in any modal semiring, it is suitable to model the belief operator with the aid of a Boolelan algebra with operators. Boolean algebras with operators were introduced by (Jónsson and Tarski, 1951; Jónsson and Tarski, 1952) and – independently, it seems – by (Lemmon, 1966a; Lemmon, 1966b). Lemmon called them modal algebras. The work in this paper will build mainly on Lemmon's results. Indeed, I will

Solin K..
Modal Semirings with Operators for Knowledge Representation.
DOI: 10.5220/0004181001970202
In *Proceedings of the 5th International Conference on Agents and Artificial Intelligence* (ICAART-2013), pages 197-202
ISBN: 978-989-8565-39-6

propose that for modelling the belief operator one can use what Lemmon calls a normal epistemic algebra.

The focus of this paper is the abstract algebra, and not some specific set-theoretic model – such as relations or hypertheories (Segerberg, 1999) – that could satisfy the axioms of the algebra; for appreciating the paper, it is pivotal that the reader constantly bears this in mind. The axioms that will be presented are motivated only by common sense, and there will not be any model-theoretic *justifications* for them. The abstract algebra is here considered as a first-class conceputal tool, and not merely as a calucaltional aid for a model.[1] Nevertheless, we shall throughout this paper consider a simple relational model that will be given below. It is a model with which many are familiar and that is useful for getting a more tangible technical-mathematical understanding of the axioms. But I want to emphasise, that this model is only a heuristic aid and a technical tool. I shall only use the model to prove technical results, such as the fact that the tentative axioms do not contradict each other.

1.2 Structure of the Paper

The paper has the following structure. First, the theory leading up to modal semirings is reviewed. Then the parts of Lemmon's work on modal algebra that are needed for this paper are presented, upon which – the main contribution of this paper – modal semirings and modal algebra are combined; this structure we'll call a *modal semiring with operators*. After combining the two algebraic structures, a revision operator is also proposed, yielding *dynamic epistemic modal algebra*. Then some basic properties in the algebra are considered, and the paper concludes with a number of interesting themes for further investigation, theoretical as well as practical.

2 MODAL SEMIRINGS

An *idempotent semiring* is a structure over the signature $(+,;,0,1)$ that satisfies the following axioms (a,b and c in the carrier set, and ; left implicit):

[1]In contrast, (Solin, 2010) viewed the abstract algebra as something of a calculational tool intended for existing models. The models were the driving force behind the axioms of the algebra. The perspective in this paper is the opposite: the abstract algebra here speaks for itself, the axioms are motivated only by common sense, and the various models are considered secondary. – There will of course be *relations* to set-theoretic models, but these will not be investigated in this paper. And, needless to say, the algebraic formulation can itself be seen as a sort of axiomatically defined set-theoretic model.

$$a+(b+c) = (a+b)+c, \quad (1)$$
$$a+b = b+a, \quad (2)$$
$$a+0 = a, \quad (3)$$
$$a+a = a, \quad (4)$$
$$a(bc) = (ab)c, \quad (5)$$
$$1a = a = a1, \quad (6)$$
$$0a = 0 = a0, \quad (7)$$
$$a(b+c) = ab+ac \text{ and} \quad (8)$$
$$(a+b)c = ac+bc. \quad (9)$$

This means that $(+,0)$ is a commutative and idempotent monoid, and that $(;,1)$ is a monoid such that ; distributes over $+$, and that 0 annihilates to left and right. The operator ; will often be left implicit. The *canonical order* $\leq$ on a semiring is defined by $a \leq b \Leftrightarrow_{\mathsf{def}} a+b=b$ for all a and b in the carrier set. By this, 0 is the least element, $+$ is join, and ; and $+$ are both monotone (isotone).

A *test semiring* (Desharnais et al., 2006; Kozen, 1997) is a two-sorted algebra

$$(S, \mathsf{test}(S), +, ;, \neg, 0, 1)$$

such that

(i) $(S,+,;,0,1)$ is an idempotent semiring,

(ii) $(\mathsf{test}(S),+,;,\neg,0,1)$ is a Boolean algebra (BA) and

(iii) $\mathsf{test}(S) \subseteq S$.

So join and meet in $\mathsf{test}(S)$ are $+$ and ;, respectively, and the complement is given by $\neg$. Naturally, 0 is the least and 1 is the greatest element. For general semiring elements it is the custom to use the notation $a,b,\ldots$, and for test elements $p,q,\ldots$ is the convention. On a test semiring one can axiomatise a *domain operator* $\ulcorner : S \to \mathsf{test}(S)$ by

$$a \leq \ulcorner a \, ; a, \quad (10)$$
$$\ulcorner(pa) \leq p \text{ and} \quad (11)$$
$$\ulcorner(a \ulcorner b) \leq \ulcorner(ab), \quad (12)$$

for all $a,b \in S$ and $p \in \mathsf{test}(S)$ (Desharnais et al., 2006). It has been proved that (10) and (12) can be strengthened to equalities.

A relational interpretation of domain and the other operators is given below, and an epistemic interpretation will be given later (Section 5.1).

As mentioned in the introduction, one can define modal operators with the aid of the domain operator (Desharnais et al., 2006):

$$\langle a \rangle p =_{\mathsf{def}} \ulcorner(ap) \quad \text{and} \quad [a]p =_{\mathsf{def}} \neg\langle a \rangle \neg p \,. \quad (13)$$

This is why test semirings with a domain operator are also called *modal semirings*. The diamond should be read "it is possible to successfully perform a so that p will hold" and the box as "after every successful way of performing a it will be the case that p holds." In the terminology of Dijkstra (Dijkstra, 1976), the box is the weakest liberal precondition of a with respect to p: if a terminates, p will hold. What is here called 'successful' corresponds to what Dijkstra calls 'terminating'.

Let us list some basic properties of this structure (Desharnais et al., 2006). One has stability of tests,

$$\ulcorner p = p \,, \tag{14}$$

and additivity,

$$\ulcorner(a+b) = \ulcorner a + \ulcorner b \,. \tag{15}$$

Of course, isotony of domain is immediate from additivity. If one lets the operator $p \to q$ be the syntactic equivalent of $\neg p + q$, then

$$(p \to q = 1) \Leftrightarrow p \leq q$$

obtains.

2.1 A Relational Model

Below follows a concrete algebra that forms a relational model for the above axioms.

Let $S, T \subseteq \Sigma \times \Sigma$, and define the binary operators ; and $+$, and the unary operator $\ulcorner$ as follows:

$$\begin{aligned} (x,y) \in (S;T) &\Leftrightarrow_{\mathsf{def}} \exists z \in \Sigma \bullet (x,z) \in S \\ &\qquad \text{and } (z,y) \in T, \\ (x,y) \in (S+T) &\Leftrightarrow_{\mathsf{def}} (x,y) \in S \text{ or } (x,y) \in T, \\ (x,x) \in \ulcorner S &\Leftrightarrow_{\mathsf{def}} (x,y) \in S \text{ for some } y \in \Sigma. \end{aligned}$$

Now let the identity relation be denoted by Δ and the empty relation by $\emptyset$, and let $\neg$ be the complement relative to Δ. Then the structure

$$(\wp(\Sigma \times \Sigma), \wp(\Delta), +, ;, \neg, \ulcorner, \emptyset, \Delta)$$

is a modal semiring (Desharnais et al., 2006).

The following example from (Solin, 2010) is informative when figuring out how the modal operators work in this heuristic model. But keep in mind that the model is *only a technical tool* in this paper, since the axioms are motivated primarily by the commonsense interpretation found in Section 5.1.

Example 2.1. To the letter from (Solin, 2010). Let $\Sigma = \{1,2,3,4,5\}$, let

$$S = \{(1,2),(2,2),(2,5),(4,3)\}$$

and let

$$P = \{(2,2),(3,3)\}.$$

Think of Σ as a set of states, S as a state-changing action and P as a predicate determining the states 2 and 3. According to the definitions above we then have

$$\langle S \rangle P = \{(1,1),(2,2),(4,4)\}$$

and

$$[S]P = \{(1,1),(3,3),(4,4),(5,5)\}.$$

Note that both $\langle S \rangle P$ and $[S]P$ are predicates. It is thus possible to use the action S to reach one of the states prescribed by P if the action is performed from one of the states 1, 2 or 4; but it is not guaranteed since from state 2 we could also end up in 5. From the states 1, 3, 4 and 5 we are guaranteed to end up in a state prescribed by P as long as the action is performed successfully – but from states 3 and 5 there is no succesful way of performing the action. $\diamond$

3 MODAL ALGEBRA

A modal algebra is defined by (Lemmon, 1966a) in the following way.

Definition 3.1. *A structure* $(M, \cup, \cap, \neg, 0, 1, \mathsf{P})$ *is a* modal algebra *iff M is a set of elements closed under operations* $\cup, \cap, \neg$, *and* P *such that:*

(i) M is a Boolean algebra with respect to $\cup, \cap, \neg, 0, 1$*;*

(ii) for $x, y \in M$, $\mathsf{P}(x \cup y) = \mathsf{P}x \cup \mathsf{P}y$.

$\lhd$

This is a Boolean algebra with operators in Jónsson and Tarski's sense, when the Boolean algebra is endowed with only one function (Jónsson and Tarski, 1951). The operator P may be read as "possible", and with it one can define the operator N for "necessary" by $\mathsf{N}x =_{\mathsf{def}} \neg\mathsf{P}\neg x$.

Lemmon calls a modal algebra that satisfies the postulate

(iii) $\mathsf{P}1 = 1$

a *deontic algebra*, and a modal algebra that satisfies the postulate

(iv) $x \leq \mathsf{P}x$

an *epistemic algebra*. A modal algebra is *normal* if it satisfies the postulate

(v) $\mathsf{P}0 = 0$.

As Lemmon shows, it is easy to establish the following proposition.

Proposition 1. All epistemic algebras are deontic, and in a normal epistemic algebra it is the case that

$$(\mathsf{N}x \to \mathsf{P}x) = 1 \tag{16}$$

and

$$(\mathsf{N}x \to x) = 1 \qquad (17)$$

hold. □

It is a normal epistemic algebra that will be used later.

3.1 The Model

In the relational model, one can simply take the Boolean algebra of

$$(\wp(\Delta), \cup, \cap, \neg, \emptyset, \Delta)$$

and let P be the identity function. This does not do anything for intuition, but that is also not the intention, as explained above. Letting P be the identity function is useful for proving the soundness of the proposed axiomatisation, which will be made clear below.

4 THE TWO COMBINED: MODAL SEMIRINGS WITH OPERATORS

The structure

$$(M, \mathsf{test}(M), +, \,;\, , \neg, ^{\ulcorner}, \mathsf{P}, 0, 1),$$

where

$$(M, \mathsf{test}(M), +, \,;\, , \neg, ^{\ulcorner} 0, 1)$$

is a modal semiring, and

$$(\mathsf{test}(M), +, \,;\, , \neg, \mathsf{P}, 0, 1)$$

is a modal algebra, we shall call a *modal semiring with one operator*.

This structure can be generalised to form *modal semirings with multiple operators*, which in turn can be extended to *modal Kleene algebra with multiple operators*. If a modal operator is normal, epistemic or deontic, respectively, we shall use the terminology modal semiring with a normal, epistemic, or deontic operator, respectively.[2]

[2]The mathematical properties of these structures are, to the best of my knowledge, completely uninvestigated, and deserve proper attention along the lines of (Jónsson and Tarski, 1951; Jónsson and Tarski, 1952; Lemmon, 1966a; Lemmon, 1966b). I hope to return to this in future work.

5 ADDING A REVISION OPERATOR

In this section we add a belief-revision operator to a modal semiring with a normal epistemic operator. As will be shown, this makes reasoning about dynamic epistemic properties possible.

5.1 Intended Interpretation

This is how the constituents of the below algebra are intended to be understood. The semiring axioms in Section 2 are all to be interpreted against this background; the reader might want to have the axioms at hand. The elements of the carrier set should be seen as actions upon which the operators (a sort of meta-actions) work. The operator $+$ has the agent choose between performing either the left or the right action, and the operator ; has the agent perform the left action first, and then the right action. The constant 0 always fails (creates a disaster) and the idle action 1 leaves everything as it is. Tests are seen as actions checking if some proposition is true. So 0 checks if the contradiction holds, which it never does, so it fails, and 1 checks if the tautology holds, which it always does, so why bother. The domain operator applied to an action gives a test that is true whenever it is possible to successfully perform the action. The modal operators are interpreted as above (page 2).

A belief operator B will be given as the modal operator of a normal epistemic algebra. For saying that some proposition holds, one writes $p = 1$. For saying that the agent believes that a proposition holds, one writes $\mathsf{B}p = 1$. In (Solin, 2010) it was not possible to make this distinction, which amongst other things meant that $\neg\mathsf{B}\neg p$ couldn't be expressed properly. Finally, $\circledast p$ is an action that revises the agent's beliefs by p.

5.2 Dynamic Epistemic Modal Algebra

We are now ready to state the formal definition of a dynamic epistemic modal algebra. It should be noted that the axioms for the revision operator are only tentative and that, depending on one's goals, they can be modified at will. The main point of this definition is to show what something like a dynamic epistemic modal algebra should look like. The axioms are inspired by those of (Solin, 2010).

Definition 5.1. *A* dynamic epistemic modal algebra *(*DEMA*) is a two-sorted algebra*

$$(D, \mathsf{test}(D), +, \,;\, , \neg, \circledast, ^{\ulcorner}, \mathsf{B}, 0, 1)$$

such that

$$(D, \mathsf{test}(D), +, ;, \neg, \ulcorner, \mathsf{B}, 0, 1)$$

is a modal semiring with a normal epistemic operator, and $\circledast : \mathsf{test}(D) \to D$ *satisfies*

$$\mathsf{B}p \leq \circledast p, \quad (18)$$

$$\circledast p; \circledast p \leq \circledast p, \quad (19)$$

$$\circledast p; \mathsf{B}q \leq \circledast(p;q), \quad (20)$$

$$\circledast(p+q); \mathsf{B}p = \circledast p, \quad (21)$$

$$\circledast p \leq \circledast q \Rightarrow p \leq q, \quad (22)$$

$$\ulcorner \circledast p = 1, \text{ when } p \neq 0. \quad (23)$$

This concludes the definition. ◁

The axioms for the revision operator can all be justified according to the interpretation given in Section 5.1.[3]

5.3 The Model and Soundness

We can now use our relational model to show that the above axioms will not contradict each other, and the that the axiomatisation is sound in that sense. In the relational model, let $\circledast P =_{\mathsf{def}} \nabla; P$, for any relation $P \in \wp(\Delta)$, where $\nabla = \Sigma \times \Sigma$. Now, if one lets all the operators of the modal semiring be interpreted as suggested above, and lets the modal operator B be interpreted as the identity, then the resulting relational structure is exactly that of (Solin, 2010, Section 4). From this, it follows that the above axiomatisation is sound, in the sense that the axioms will not lead to contradictions (granted that the relational structure is noncontradictory, which shouldn't be controversial). A completeness theorem for this model would not be very interesting in our context, since the model is not what justifies the axioms.

6 A FEW CLASSICAL PROPERTIES

In the algebra, it is easy to prove some classical properties that stem from formal reasoning about belief change (Alchourrón et al., 1985; Segerberg, 1999; Cantwell, 2000; van Ditmarsch et al., 2007). The proofs closely follow those for dynamic epistemic semirings (Solin, 2010), but the modal operator B must be taken into account.

The next proposition is one of Segerberg's *success conditions* (Segerberg, 1999). After successfully revising by p, the agent believes p.

Proposition 2. In any DEMA the property

$$[\circledast p]\mathsf{B}p = 1 \quad (24)$$

holds.

The following property is an encoding of one of the AGM axioms (Alchourrón et al., 1985; Cantwell, 2000). If the agent believes q after successfully revising by p, then the agent believes that q follows from p.

Proposition 3. In any DEMA the property

$$[\circledast p]\mathsf{B}q \leq \mathsf{B}(p \to q) \quad (25)$$

holds.

The proposition below, which also relates to an AGM axiom, says that a successful revision will never have the agent believe a contradiction.

Proposition 4. In any DEMA the property

$$[\circledast p]\neg\mathsf{B}0 = 1 \quad (26)$$

holds.

From the proof of the last proposition, it is evident that a successful revision (or any successul action) will never stop the agent from believing the tautology.

7 CONCLUSIONS

This paper opens up a wealth of interesting themes to investigate.

7.1 Theoretical Themes

On the purely mathematical side, the general modal semirings with operators deserve proper attention along the lines of (Jónsson and Tarski, 1951; Jónsson and Tarski, 1952; Lemmon, 1966a; Lemmon, 1966b). Another interesting theme is the investigation of different forms of dynamic epistemic modal algebras, in which, for example, a different axiomatisation of revision is given, or in which only a deontic operator is embedded. As in (Solin, 2010), iteration could be introduced as Kleene star. And it would be interesting to sort out the connections between modal semirings, modal algebras and the systems employed by (Baltag et al., 2005; Baltag et al., 2007; Baltag and Sadrzadeh, 2006; Panangaden and Sadrzadeh, 2010).

How these algebras relate to various set-theoretic models for belief revision and the alike could also be investigated, following Lemmon's investigation of correspondences between his algebras and the standard models for modal logic – but as noted in earlier, such an investigation is in no way a prerequisite for using the abstract algebra as a conceptual tool.

[3] Cf. also (Solin, 2010, p. 598).

7.2 Practical Themes

As shown by (Höfner and Struth, 2007; Foster and Struth, 2012), the level of abstraction that semirings and Kleene algebra provide is very well-suited for automation. It would therefore be very interesting to mechanise and automate the algebraic theory of this paper. With the aid of such a mechanisation, more elaborate applications of the algebra would become feasible.

I have emphasised earlier, that the abstract algebra is the pivot of the framework presented in this paper – not a model (a conrete algebra). The axioms were justified by common sense, and the relational model was only used for proving technical results. Reasoning in this framework is therefore genuinely about pushing symbols, especially since the operators in Kleene algebra and semirings all have a low, fixed arity.[4] This means that a robot, say, that would be constructed on the basis of the theory in this paper, would not have a sole, involved model as its theoretical underpinning, but just the axioms of a very simple abstract algebra, and with them the whole associated class of models (such as sets with only a few members, the relational model, etc.). In combination with the tools available for automated proof and counterexample generation, this could make for very efficient and pragmatic knowledge engineering.

ACKNOWLEDGEMENTS

Most of this work was done at the Department of Philosophy, Uppsala University, but some was done at the Computing Department, Macquarie University, Sydney, and a significant part at the University of Queensland. A grant from *Sven och Dagmar Saléns Stiftelse* made the stay in Sydney possible. The work in Queensland was supported by the Australian Research Council (ARC) Discovery Grant DP0987452.

REFERENCES

Aboul-Hosn, K. and Kozen, D. (2006). KAT-ML: an interactive theorem prover for Kleene algebra with tests. *Journal of Applied Non-Classical Logics*, 16(1-2):9–34.

Alchourrón, C., Gärdenfors, P., and Makinson, D. (1985). On the logic of theory change: partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50(2):510–530.

Baltag, A., Coecke, B., and Sadrzadeh, M. (2005). Algebra and sequent calculus for epistemic actions. *Electronic Notes in Theoretical Computer Science*, 126:27–52.

Baltag, A., Coecke, B., and Sadrzadeh, M. (2007). Epistemic actions as resources. *Journal of Logic and Computation*, 17(3):555–585.

Baltag, A. and Sadrzadeh, M. (2006). The algebra of multi–agent dynamic belief revision. *Electronic Notes in Theoretical Computer Science*, 157(4):37–56.

Cantwell, J. (2000). *Non-Linear Belief Revision: Foundations and Applications*. Acta Universitatis Upsaliensis. Dissertation.

Desharnais, J., Möller, B., and Struth, G. (2006). Kleene algebra with domain. *ACM Transactions on Computational Logic*, 7(4):798–833.

Dijkstra, E. W. (1976). *A Discipline of Programming*. Prentice-Hall.

Foster, S. and Struth, G. (2012). Automated analysis of regular algebra. In Gramlich, B., Miller, D., and Sattler, U., editors, *IJCAR*, volume 7364 of *Lecture Notes in Computer Science*, pages 271–285. Springer.

Harel, D., Kozen, D., and Tiurun, J. (2000). *Dynamic Logic*. MIT Press.

Höfner, P. and Struth, G. (2007). Automated reasoning in Kleene algebra. In Pfenning, F., editor, *CADE*, volume 4603 of *Lecture Notes in Computer Science*, pages 279–294. Springer.

Jónsson, B. and Tarski, A. (1951). Boolean algebra with operators. Part I. *American Journal of Mathematics*, 73(4):891–939.

Jónsson, B. and Tarski, A. (1952). Boolean algebra with operators. Part II. *American Journal of Mathematics*, 74(1):127–939.

Kozen, D. (1994). A completeness theorem for Kleene algebras and the algebra of regular events. *Inf. Comput.*, 110(2):366–390.

Kozen, D. (1997). Kleene algebra with tests. *ACM Transactions on Programming Languages and Systems*, 19(3):427–443.

Lemmon, E. J. (1966a). Algebraic semantics for modal logics I. *The Journal of Symbolic Logic*, 31(1):46–65.

Lemmon, E. J. (1966b). Algebraic semantics for modal logics II. *The Journal of Symbolic Logic*, 31(1):191–218.

Möller, B. (2008). Knowledge and games in modal semirings. In Berghammer, R., Möller, B., and Struth, G., editors, *RelMiCS*, volume 4988 of *Lecture Notes in Computer Science*, pages 320–336. Springer.

Panangaden, P. and Sadrzadeh, M. (2010). Learning in a changing world, an algebraic modal logical approach. In Johnson, M. and Pavlovic, D., editors, *AMAST*, volume 6486 of *Lecture Notes in Computer Science*, pages 128–141. Springer.

Segerberg, K. (1999). Two traditions in the logic of belief: bringing them together. In Ohlback, H. and Reyle, U., editors, *Logic, Language and Reasoning*, pages 134–147. Kluwer.

Solin, K. (2010). A sketch of a dynamic epistemic semiring. *Inf. Comput.*, 208(5):594–604.

van Ditmarsch, H., van der Hoek, W., and Kooi, B. (2007). *Dynamic Epistemic Logic*. Springer.

[4]This stands in contrast to a quantale, for instance, that has an operator over a possibly infinite index set; an operator that sometimes requires careful thought when applied.