

# Continuous-time Revenue Management in Carparks

## Part Two: Refining the PDE

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Abstract: In this paper, we study optimal revenue management applied to carparks, with the primary objective to maximize revenues under a continuous-time framework. This work is an extension to (Papayiannis et al., 2012) where the authors developed a Partial Differential Equation (PDE) model that could solve for the rate at which cash is generated through an infinitesimal time. However, in practice, carpark managers charge customers per day or per hour which is a finite period of time. Unfortunately, this situation was currently not captured by this previous work. Therefore, our current work attempts to reformulate the existing PDE in a way that it does capture the revenue that is generated within any finite time interval of length  $\Delta T$ . The new model is compared against the Monte Carlo (MC) approach for several choices of  $\Delta T$ ; the results are remarkable as the improvement in computation speed and efficiency are significant. Since, the algorithm in the PDE still does not solve the 'exact' problem, a method is proposed to marry the benefits of the PDE with those of the MC approach. Our results are prominent as the optimal values generated in this case have shown to be extremely close to the MC ones while the computation times are kept to a minimum.

## 1 INTRODUCTION

The primary objective of this paper is to study revenue management applied to carparks, in order to optimally manage the expected revenues of the carpark under a continuous time framework. There has been an increasing interest on car parking problems within the last two decades. Many researchers have worked on traffic congestion problems, among them are (Teodorović and Vukadinović, 1998), (Arnott and Rowse, 1999) and (Zhao et al., 2010), to just list a few. An extensive review on urban car parking models can be found in (Young et al., 1991). In the context of revenue management we refer to (Teodorović and Lučić, 2006) who proposed an intelligent parking inventory control system based on fuzzy logic theory. Moreover, (Onieva et al., 2011) have formulated and solved a Linear Programming (LP) problem in both a deterministic and a stochastic environment. In this paper, however, we aim to build upon the framework laid down by (Papayiannis et al., 2012) and extend the Partial Differential Equation (PDE) approach to a more realistic discrete time framework. In (Papayiannis et al., 2012) the authors present two approaches of modelling such a continuous-time stochastic optimization problem; the Monte-Carlo (MC) approach and the PDE approach. The first one solves the prob-

lem by first setting up the selling horizon and then discretising the horizon into finite intervals of time  $\Delta T$ . Bookings in both cases are modelled using fixed time-invariant Poisson distribution for the number of bookings with exponential waiting times for time between booking and length of stay exponential arrival times have also been employed in (Onieva et al., 2011; Teodorović and Lučić, 2006). The carpark is then managed by assigning an accept/reject decision to the bookings in each time interval given the current state of the carpark (i.e. number of spaces remaining and time until stay). This decision was optimally taken so that a parking slot never sells for less than what we might expect to receive for it in the future. For the PDE approach, the authors derive analytical formulae for the probability distributions of bookings so that they can formulate an expression for the expected value of all the available spaces at time  $t$  given we know the value of spaces remaining at time  $t + dt$ . Since the process is essentially Markov they can evoke the Hamilton-Jacobi-Bellman principle as in (Gallego and van Ryzin, 1994) to express this as a dynamic programming problem. The resulting PDE was in fact solving for the *rate* at which the value is generated through an infinitesimal time, rather than over some discrete time as in the case of the Monte Carlo method. Thus, a comparison on the optimal so-

lution between the MC and the PDE methodologies could only be made in the limit i.e. as the time intervals in the MC tend to zero  $\Delta T \rightarrow 0$ ; the optimal values however were showed to be remarkably close to one another.

On the one hand, the pricing structure of most carparks dictates that spaces are sold to customers in slots, typically an integer number of fixed periods of time, such as day or hour over which the space will be reserved (see (Teodorović and Lučić, 2006; Onieva et al., 2011)). On the other hand, (Bitran and Caldentey, 2003) argue that “the explosive growth of the Internet and e-commerce make the continuous-time model much more suitable in practice”. Moreover, the results of (Papayiannis et al., 2012) have shown the superiority of the PDE method over the MC method with respect to both efficiency and computational speed, but the PDE method presented there could not capture discrete time intervals. These then provide the motivation for the current study to extend the PDE model so that it can solve for the rate at which value is generated within any time period of finite size  $\Delta T$ .

The remainder of this report is organized as follows. In section 2 we explain the steps taken in order to reformulate the problem. Section 3 illustrates some numerical results along with discussion. All results which are presented in this paper are obtained using an Intel Xeon(R) CPU E5450 @ 3.00GHz with 16GB of RAM. Finally our conclusions can be found in section 4.

## 2 PROBLEM FORMULATION

The derived PDE in (Papayiannis et al., 2012) is based on the probability densities  $P_s$  and  $g$  which are used to calculate the rate at which bookings turn up and stay as of time  $t$  for the infinitesimal period  $T > t$ , and the rate of cash-flow running through that period. However, we note that the continuous time effect of the PDE is on the booking decision rather than the actual time spent in the carpark.

Therefore, we can still have the bookings arriving in a continuous-time but rather calculate the associated revenue rates within a discrete time interval in the future, the size of which can be of any finite length  $\Delta T$ . In other words, we can still consider that bookings are made instantaneously but, also adjust the probability distributions in such a way that we rather capture the probability of a customer being present within the interval  $[T, T + \Delta T]$  which takes place between  $[\tau, \tau + \Delta T]$  days after the booking is made.

Now within the time-invariant framework, we introduce  $\tilde{g}(\tau)$  (the symbol ‘ $\sim$ ’ will be used in the definitions throughout this report to distinguish the terms that involve the finite interval  $\Delta T$ ) to be the fraction of customers who made their bookings to be present between  $\tau$  and  $\tau + \Delta T$  days later. It is not difficult to show that this may be written as

$$\tilde{g}(\tau) = P_a(\tau + \Delta T) - P_a(\tau). \quad (1)$$

Similarly, we introduce  $\tilde{f}(\tau)$  to be the average intensity for bookings made so that they are present between  $[\tau, \tau + \Delta T]$  days later. This can be expressed as

$$\tilde{f}(\tau) = \lambda_b \tilde{g}(\tau). \quad (2)$$

Moreover, we define the probability density of a customer staying  $\xi$  days given that he/she is present between  $[\tau, \tau + \Delta T]$  days after the booking made by  $\tilde{\rho}_s(\xi|\tau)$  which can be expressed as,

$$\tilde{\rho}_s(\xi|\tau) = \frac{\rho_s(\xi) [P_a(\tau + \Delta T) - P_a(\{\tau - \xi, 0\}^+)]}{\tilde{g}(\tau)}. \quad (3)$$

It should be made clear at this stage that  $\rho_s(\xi|\tau) \subseteq \tilde{\rho}_s(\xi|\tau)$ .

Consequently, the cumulative probability of a customer staying no more than  $\xi$  days given that he/she is present between  $[\tau, \tau + \Delta T]$  after the booking,  $\tilde{P}_s(\xi|\tau)$ , is given by

$$\tilde{P}_s(\xi|\tau) = \int_0^\xi \tilde{\rho}_s(s|\tau) ds. \quad (4)$$

It is important to mention that although we have modified the probability distributions, the customers are still booking in continuous time and request to park for any length of stay which is again still a continuous quantity. For the case of maximising the cash-flows within a finite time period, any bookings that happen to be present at any time within this interval do contribute to the solution. In (Papayiannis et al., 2012) PDE model we looked at all bookings that are present during the same infinitesimal and given that bookings could only be distinguished by their length of stay (the larger the length of stay, the less the price to be paid per day) we imposed a rule to reject those of length greater than  $\xi^*$ . In our new formation, the idea is similar with the only difference that we are now looking at all bookings that are present at any time within a finite time interval. Unfortunately, this increases the complexity of the problem as bookings present in the same period, although being of same length of stay, they might have pay a different price rate according to how many time periods  $\Delta T$  each falls into in total (this depends on the exact time point the bookings lie when being within the time period). Therefore, we suggest a simple procedure to estimate

the number of time periods ( $n$ ) for which the customers are likely to occupy the slot, which will in turn enable us to determine a better estimate for the price rates that should be applied. We note that  $n$  should strongly depend on the size of the time period,  $\Delta T$ .

In particular, we assume that the required length of stay  $\xi$  is between  $k$  and  $k + 1$  times larger than the length of the interval  $\Delta T$ , i.e.  $k\Delta T \leq \xi \leq (k + 1)\Delta T$ , where  $k \in \mathbb{Z}$ . For simplicity, we assume that customers arrival times follow a uniform distribution,  $u$ . Figure 1 illustrates the situation; In order for customers who stay for  $\xi$  days to be accounted within the time interval  $[T, T + \Delta T]$ , they must have arrived no more than  $\xi$  days in advance and no later than  $T + \Delta T$ . Thus the feasible region, within which customers contribute to the solution, is  $T - \xi \leq t \leq T + \Delta T$ . Therefore, the uniform distribution's endpoints become  $0 \leq u \leq \Delta T + \xi$ .

Regarding the number of periods ( $n$ ) the customers are likely to occupy a slot for, there are only two possible scenarios;

1. the customer stays  $k + 1$  periods when his/her required duration of stay covers  $k$  periods plus a fraction of an additional period either before or after this interval (his/her arrival time lies on a red segment of figure 1),
2. or stays  $k + 2$  periods when his/her then required duration of stay covers the same  $k$  periods plus a fraction before and after this interval (his/her arrival time lies on a green segment of figure 1).

In particular, we can show that the associated probability in each case is given by,

$$P(n = k + 1) = (k + 1) \left( \frac{(k + 1)\Delta T - \xi}{\xi + \Delta T} \right) \quad (5)$$

while,

$$P(n = k + 2) = 1 - P(n = k + 1) \quad (6)$$

Therefore,  $n$  is given by

$$n = \begin{cases} k + 1 & \text{w.p } P_1 = (k + 1) \left( \frac{(k + 1)\Delta T - \xi}{\xi + \Delta T} \right) \\ k + 2 & \text{w.p } P_2 = \frac{(k + 2)\xi - (1 - (k + 1)^2)\Delta T}{\xi + \Delta T} \end{cases}$$

Consequently, the expected length of stay ( $E[n]$ ) becomes

$$E[n] = (k + 1)P_1 + (k + 2)P_2. \quad (7)$$

Finally, we can replace the exact duration of stay,  $\xi$ , by the expected number of time periods (of length  $\Delta T$ ) the customer is staying for,  $E[n]$ , and then we can calculate the required price rate per period as,

$$\Psi(\xi) \rightarrow \tilde{\Psi}(E[n]) \equiv \psi_1 + \psi_2 e^{-\mu E[n]\Delta T}. \quad (8)$$

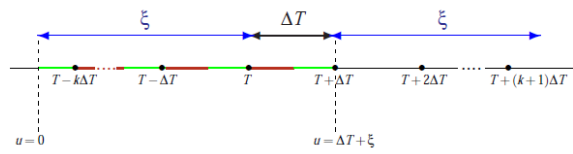


Figure 1: Uniform arrival distribution  $u(0, \Delta T + \xi)$  for customers that stay for  $k\Delta T \leq \xi \leq (k + 1)\Delta T$  days and they are present within the interval  $[T, T + \Delta T]$ . Customers who arrive anywhere on the red segments will occupy  $k + 1$  intervals while those that arrive anywhere on the green segments will occupy  $k + 2$  intervals.

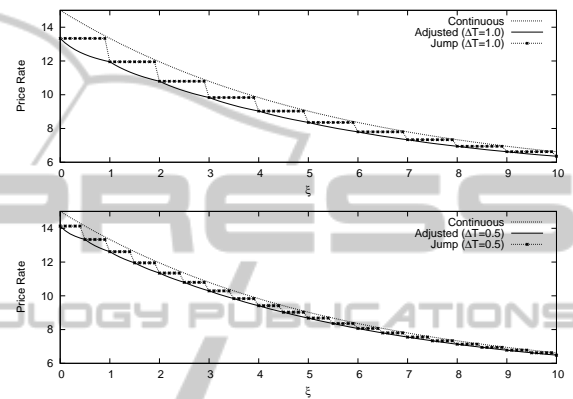


Figure 2: Plots of the adjusted price rate function used in the reformulated PDE and the discrete-jump price function used in the MC. Upper figure compares the two for  $\Delta T = 1.0$  while the lower figure plots these for  $\Delta T = 0.5$ . The continuous pricing function (as in (Papayiannis et al., 2012)) is shown as well.

with parameters  $\psi_1, \psi_2, \mu$  identical to those in (Papayiannis et al., 2012). Figure 2 compares the adjusted price rate function (8) with the corresponding discrete-jump price function that has been used in the MC; the upper figure illustrates this comparison for  $\Delta T = 1.0$  while the lower figure for  $\Delta T = 0.5$ . In each figure the continuous pricing function (as in (Papayiannis et al., 2012)) is plotted as well. The adjusted price rate function seems a reasonable approximation to the discrete-jump price function; the two functions approach one another as  $\Delta T$  decreases and, in fact, they become equal in the limit (as  $\Delta T \rightarrow 0$ ). In the limiting case both pricing functions are equal to the continuous pricing function (8). Finally, we define  $\tilde{V}(Q, t; T)$  to the rate at which revenue is generated in the carpark with  $Q$  spaces remaining for the future period  $[T, T + \Delta T]$  as of time  $t$ . Thus, within the time-invariant framework we may define  $\tilde{V}(Q, \tau)$  as the rate at which revenue is generated from cars present over the period interval which is formed between  $[\tau, \tau + \Delta T]$  days later.

Therefore, the modified PDE can be written as

$$\frac{\partial \tilde{V}}{\partial \tau} = \max_{\xi^*} \left[ \tilde{p}_s(\xi^*|\tau) \tilde{f}(\tau) (\tilde{V}(Q-1, \tau) - \tilde{V}(Q, \tau)) + \tilde{f}(\tau) \int_0^{\xi^*} \tilde{p}_s(\xi|\tau) \tilde{\Psi}(\xi) d\xi \right], \quad (9)$$

with the same boundary conditions as before

$$\tilde{V} = 0 \quad \text{when} \quad \tau = 0 \quad (10)$$

$$\tilde{V} = 0 \quad \text{when} \quad Q = 0. \quad (11)$$

The solution to this system gives the rate at which the value is generated in the future time period  $[T, T + \Delta T]$  which takes place between  $\tau$  and  $\tau + \Delta T$  days after the current time  $t$  and the values  $\xi^* = \xi^*(Q, \tau)$  that achieve the supremum construct the optimal rejection policy<sup>1</sup>.

The optimal rejection policy can alternatively be expressed in terms of the ‘Added Values’ of the spaces (Papayiannis et al., 2012); in particular the optimal policy satisfies

$$\tilde{\Psi}(\xi^*) = \tilde{V}(Q, \tau) - \tilde{V}(Q-1, \tau) \quad \forall Q \forall \tau.$$

In the revenue management literature this quantity is usually referred to as the *opportunity cost* of selling a unit of capacity when the time left is  $\tau$  and the available inventory is  $Q$ , or simply as the *expected marginal value of capacity* at time  $\tau$  (see (Talluri and van Ryzin, 2004)).

### 3 NUMERICAL RESULTS

#### 3.1 Numerical Solution of the PDE

In this section we present some numerical results to compare the PDE scheme in (9) with the MC approach<sup>2</sup>. We have used a time horizon  $T = 30$  and carpark capacities  $1 \leq Q \leq 100$ . Regarding the booking patterns, we use exponential distributions with constant intensities and two distinct booking classes as in (Papayiannis et al., 2012).

Below, we present the expected revenues generated within the period  $[T, T + \Delta T]$  as of time  $t = 0$ , calculated by the MC approach first and then from our adjusted PDE scheme. By assuming time-invariance,

<sup>1</sup>Note that the optimal durations  $\xi^*$  are now applied to entire period intervals of length  $\Delta T$ .

<sup>2</sup>In the MC approach we obtain the correct solution using an iteration scheme for which the number of paths (simulations) increase in each iteration. The iterations terminate once the frobenious norm is minimised and the converged solution forms the optimal solution.

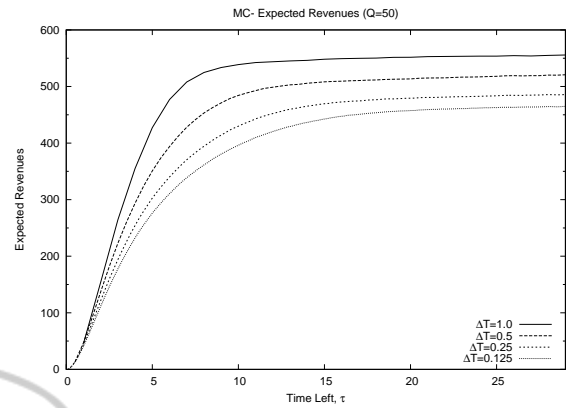


Figure 3: Expected Revenues using the MC approach as a function time left  $\tau$ . The capacity is fixed to  $Q = 50$  and the finite period is allowed to vary,  $\Delta T = 1.0, 0.5, 0.25, 0.125$ .

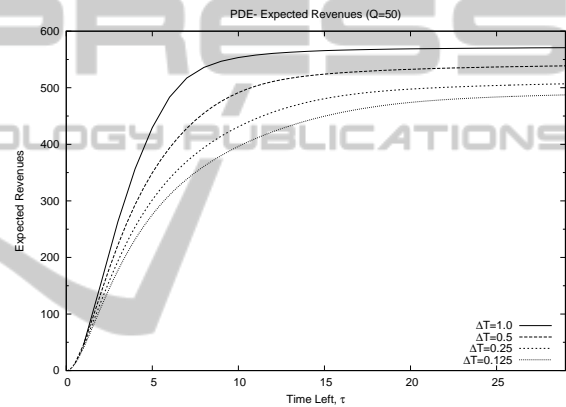


Figure 4: Expected Revenues using the reformulated PDE as a function time left  $\tau$ . The capacity is fixed to  $Q = 50$  and the finite period is allowed to vary,  $\Delta T = 1.0, 0.5, 0.25, 0.125$ .

these can be interpreted as the value of the carpark with capacity  $Q$  as of  $[\tau, \tau + \Delta T]$  time before all remaining spaces must be occupied. Figure 3 shows results from the MC approach when the capacity is 50 and  $\Delta T$  is allowed to change, while figure 4 shows the corresponding results using the PDE method. The expected revenues of the two approaches are remarkably close to each other for all choices of  $\Delta T$ . Despite its simplicity, the PDE in (9) proves capable of approximating the revenues, as these are derived from the MC approach, for any given  $\Delta T$ . We notice that the expected revenues increase as the time left increases; this implies that as the selling horizon increases the carpark manager faces a larger population of potential customers and therefore he/she can target the available capacity to the most lucrative ones (see (Bitran and Caldentey, 2003), (Gallego and van Ryzin, 1994)). In addition, the expected revenues increase as  $\Delta T$  increases; this is because when the pric-



Table 1: Computation Times and Convergence for the MC approach.

$\Delta T$	# of Paths	Iterations	Comput. Times
1.0	12672	15	126.7s
0.5	25342	17	325.8s
0.25	35839	18	621.1s
0.125	50683	19	1334.6s

ing policy of the carparks management is to sell the slots per day rather than per hour, for instance, a customer is forced to pay the price rate for the entire day even though he/she might only be staying for one hour.

Table 1 illustrates the statistics obtained in the process of calculating the entire set of optimal values (the entire matrix for all capacities and times). The results in table 1 indicate that the MC method (on its own) is infeasible in practice; for carparks that operate with slots that can be booked for a minimum of 3 hours ( $\Delta T = 0.125$ ) the MC approach is very poor as a customer would have to wait an unrealistically long time (more than 22 minutes!) until a decision would be made as to whether he/she is allowed to park or not. In the PDE, the time it takes to obtain the optimal solution is tiny (never more than 25 seconds) and it does not depend on  $\Delta T$  at all. That it is independent of  $\Delta T$  means that the PDE meets the needs of any carparking management since it fits to any given price policy, while the speed advantage of the PDE over the MC renders it ideal for being used in a web reservation engine.

However, in accordance to (Papayiannis et al., 2012) the PDE in (9) generates slightly higher revenues as it still does not solve the ‘exact’ problem; the optimal decision algorithm still looks at each booking length as separate days and solves for each day individually, whereas in the MC approach the rejection algorithm regards each request as a ‘group’ of days and, thus, the optimal decision is based on the entire booking length with a booking being accepted/rejected as a whole (which is the situation in a real carpark). Therefore, to exploit the speed advantage of the PDE while preserving the correct optimisation algorithm (of the MC approach) the PDE could be used jointly with a Monte-Carlo method.

### 3.2 Using the PDE and MC Methods Jointly

The MC approach in section 3.1 did calculate the optimal solution to the ‘exact’ problem in question but it failed to do that in a reasonable time; table 1 showed the large number of iterations and paths required for the MC approach to converge to the optimal solution.

By contrast, the PDE methodology (9) calculated the optimal solution in significantly less time but this solution did not correspond to the ‘exact’ problem; the optimal decision was found after solving for each period of day individually whereas in the MC the optimal solution takes into account the inter-dependence within days see (Papayiannis et al., 2012). Therefore, our aim is develop a new improved methodology that combines the best elements of the PDE (speed and efficiency) with the best ones of the MC (correct rejection algorithm that regards each request as a ‘group’ of days).

**The ‘Joint’ Method.** The optimal rejection policy (‘Added Values’) is derived using the PDE in (9). Then, this policy is employed in a Monte-Carlo procedure, where booking simulations are taken and requests are allowed/denied service as a ‘group’. The expected revenues are approximated by averaging over ten thousand paths. We note that this procedure does not require any iterations, as the optimal rejection policy would have already been derived by the PDE. The results are recorded and compared against the true results that have previously been calculated by running the MC approach solely.

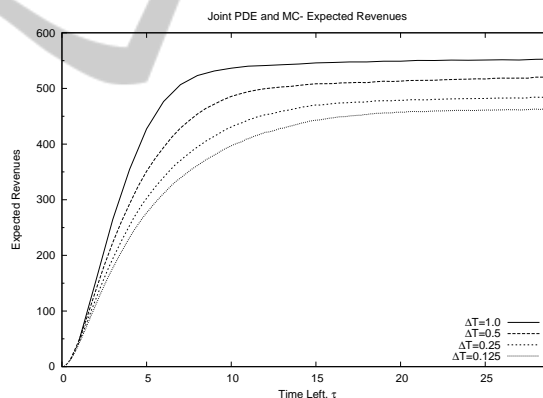


Figure 5: Expected Revenues using the joint approach as a function time left  $\tau$ . The capacity is fixed to  $Q = 50$  and the finite period is allowed to vary,  $\Delta T = 1.0, 0.5, 0.25, 0.125$ .

Figure 5 shows the expected revenues generated by the ‘joint’ method when  $Q = 50$  as functions of time, for varying time intervals  $\Delta T$ . It is clear that the curves are now outstandingly close to those in figure 3. Table 2 is provided in regards to the *maximum* relative percentage errors obtained (by considering all  $\tau$ ) for different carpark sizes and different time intervals  $\Delta T$ . from this one may notice that the maximum relative error decreases with the size of the interval  $\Delta T$ . Indeed, this is the case because the optimal policy comes from the PDE scheme and we know that the PDE solution converges to the MC solution

Table 2: Relative (%) Errors for varying capacities.

$\Delta T$	Capacity, $Q$				
	10	20	40	60	80
1.0	6.67	3.15	1.68	0.91	0.55
0.5	1.83	0.64	0.48	0.51	0.51
0.25	1.58	1.17	1.46	1.46	1.46
0.125	1.65	0.87	0.58	0.58	0.58

Table 3: Computation Times for the ‘joint’ method.

$\Delta T$	Comput. Times	Relative Comput. Times
1.0	45.5s	46.7%
0.5	56.8s	17.5%
0.25	71.6s	11.5%
0.125	102.4s	7.7%

as  $\Delta T \rightarrow 0$ . To justify this further the reader is referred to figure 2 where it shows that the price rate functions used in the two methods would converge as  $\Delta T \rightarrow 0$ . Moreover, the maximum relative error decreases with capacity. This observation has more to do with the nature of the carpark rather than the choice of the method, as relatively big carparks might require little or even no optimization at all, which implies that the role of any optimal policy is reduced and thus both methods would be even closer to each other. Lastly, table 3 presents the computation times needed for the ‘joint’ method to calculate the full set of optimal revenues (the entire matrix for all capacities and times) along with a relative speed comparison against the MC approach. Our results show that the reduction in computation time is significant; in particular, for  $\Delta T = 1$ , the ‘joint’ algorithm takes only 46% of the time it would take the MC approach to calculate the optimal solution. When  $\Delta T = 0.125$  the statistics are even more impressive as it only needs less than 8% of the original time. This monotonic increase in speed is explained by the fact that the time the PDE takes to calculate the optimal policy is independent of the size of  $\Delta T$ , while in the MC the computation times increase linearly.

## 4 CONCLUSIONS

In this paper, we have worked with the continuous-time PDE model and MC model proposed in (Papayiannis et al., 2012). Since the PDE could only solve for the rate at which cash is generated through an infinitesimal time we reformulated it in such a way that it solves for the cash that is generated within a time period of finite size  $\Delta T$  and matched that against the MC. The results validated our expectations as the optimal values and optimal rejection policies were

remarkably close to one another. Moreover, the optimal surfaces from the reformulated PDE were much smoother and the computation times have improved significantly. However, as the PDE could still not solve the ‘exact’ problem we have proposed a ‘joint’ method that calculates the optimal policy from the PDE and it then employs this to make optimal decisions by using the MC. This method performed exceptionally well as it produced revenues around 1% of the actual ones at most cases. For carparks that operate with slots that can be booked for a minimum of 3 hours ( $\Delta T = 0.125$ ) the improvement in computational speed was more than 92% (it only took 7.7% of the original time) which it renders it ideal for being used in a web reservation engine; for even smaller  $\Delta T$  the speed improvement is expected to be greater.

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