Symbol Error Rate as a Function of the Residual ISI Obtained by Blind Adaptive Equalizers

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Keywords: Symbol Error Rate (SER), Blind Adaptive Equalizers, Intersymbol interference (ISI).

Abstract: A non-zero residual intersymbol interference (ISI) causes the symbol error rate (SER) to increase where the achievable SER may not answer any more on the system's requirements. Thus, having a closed-form expression for the SER that takes into account the achievable performance of the chosen blind adaptive equalizer from the residual ISI point of view is important. In this paper, we propose a new expression for the SER valid for the two independent quadrature carrier case that depends on the step-size parameter, equalizer's tap length, input signal statistics, channel power and signal to noise ratio (SNR).

1 INTRODUCTION

We consider a system involving a blind adaptive equalizer where we wish to obtain the achievable SER as a function of the performance of the chosen blind adaptive equalizer from the residual ISI point of view. The equalization performance depends on the nature of the chosen equalizer, on the channel characteristics, on the added noise (SNR), on the step-size parameter used in the adaptation process, on the equalizer's tap length and on the input signal statistics. Fast convergence speed and reaching a residual ISI where the eye diagram is considered to be open are the main requirements from a blind equalizer. Fast convergence speed may be obtained by increasing the step-size parameter. But increasing the step-size parameter may lead to a higher residual ISI which may cause the achievable SER to increase beyond the system's requirements. Up to now, there is no closed-form expression for the achievable SER that takes into account the performance of the chosen blind adaptive equalizer from the residual ISI point of view.

In this paper, we propose for the two independent quadrature carrier case a closed-form approximated expression for the achievable SER as a function of the step-size parameter, equalizer's tap length, input signal power, SNR and channel power. The new obtained expression for the SER is based on the closed-form approximated expression for the residual ISI obtained by blind adaptive equalizers presented in (Pinchas, 2010b). Thus, it is applicable for type of blind adaptive equalizers where the error that is fed into the adaptive mechanism which updates the equalizer's taps can be expressed as a polynomial function of order three of the equalized output and where the gain between the source and equalized output signal is equal to one. It should be pointed out that Godard's (Godard, 1980) algorithm for example, belongs to the mentioned type of blind adaptive equalizers.

The paper is organized as follows: After having described the system under consideration in Section II, the closed-form approximated expression for the achievable SER is introduced in Section III . In Section IV simulation results are presented and the conclusion is given in Section V.

2 SYSTEM DESCRIPTION

The system under consideration is illustrated in Figure 1, where we make the following assumptions:
1. The input sequence $x(n)$ belongs to a two independent quadrature carrier case constellation input with variance σ_x^2 where $x_r(n)$ and $x_i(n)$ are the real and imaginary parts of $x(n)$ respectively and $\sigma_{x_r}^2$ is the variance of $x_r(n)$. In the following we denote $x_r(n)$ as x_r.
2. The unknown channel $h(n)$ is a possibly nonminimum phase linear time-invariant filter in which the transfer function has no "deep zeros", namely, the zeros lie sufficiently far from the unit circle.
3. The equalizer $c(n)$ is a tap-delay line.
4. The noise $w(n)$ is an additive Gaussian white noise

Pinchas M..
Symbol Error Rate as a Function of the Residual ISI Obtained by Blind Adaptive Equalizers.
DOI: 10.5220/0004236601560160
In *Proceedings of the 3rd International Conference on Pervasive Embedded Computing and Communication Systems* (PECCS-2013), pages 156-160
ISBN: 978-989-8565-43-3

with zero mean and variance $\sigma_w^2 = E[w(n)w^*(n)]$ where $(\cdot)^*$ and $E[\cdot]$ denote the conjugate and expectation operator on $(\cdot)$ and on $[\cdot]$ respectively.

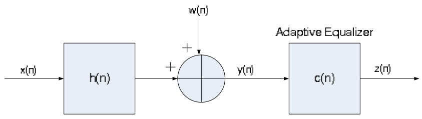

Figure 1: Block diagram of a baseband communication system.

The transmitted sequence $x(n)$ is sent through the channel $h(n)$ and is corrupted with noise $w(n)$. Therefore, the equalizer's input sequence $y(n)$ may be written as:

$$y(n) = x(n) * h(n) + w(n) \tag{1}$$

where "$*$" denotes the convolution operation. The equalized output signal can be written as:

$$z(n) = x(n) + p(n) + \tilde{w}(n) \tag{2}$$

where $p(n)$ is the convolutional noise, namely, the residual intersymbol interference (ISI) arising from the difference between the ideal equalizer's coefficients and those chosen in the system and $\tilde{w}(n) = w(n) * c(n)$. Next we turn to the adaptation mechanism of the equalizer which is based on a predefined cost function $F(n)$ that characterizes the intersymbol interference, see (Godard, 1980), (Pinchas, 2011), (Gi-Hong et al., 2009), (Lazaro et al., 2005) and (Shalvi and Weinstein, 1990). Minimizing this $F(n)$ with respect to the equalizer parameters will reduce the convolutional error. Minimization is performed with the gradient descent algorithm that searches for an optimal filter tap setting by moving in the direction of the negative gradient $-\nabla_c F(n)$ over the surface of the cost function in the equalizer filter tap space (Nandi, 1999). Thus the updated equation is given by (Nandi, 1999):

$$\begin{aligned} \underline{c}_{eq}(n+1) = \underline{c}_{eq}(n) + \mu \cdot \left(-\nabla_{c_{eq}} F(n)\right) = \\ \underline{c}_{eq}(n) - \mu \frac{\partial F(n)}{\partial z(n)} \underline{y}^*(n) \end{aligned} \tag{3}$$

where μ is the step-size parameter, $\underline{c}_{eq}(n)$ is the equalizer vector where the input vector is $\underline{y}(n) = [y(n) \ldots y(n-N+1)]^T$ and N is the equalizer's tap length. The operator $()^T$ denotes for transpose of the function $()$.

3 SER AS A FUNCTION OF EQUALIZER'S PERFORMANCE

In this section we derive the closed-form approximated expression for the SER as a function of the equalizer's performance from the residual ISI point of view.

Theorem. For the following (additional) assumptions:
1. The convolutional noise $p(n)$, is a zero mean, white Gaussian process with variance $\sigma_p^2 = E[p(n)p^*(n)]$. The real part of $p(n)$ is denoted as $p_r(n)$ and $E[p_r^2(n)] = m_p$.
2. The source signal $x(n)$ is a rectangular QAM (Quadrature Amplitude Modulation) signal (where the real part of $x(n)$ is independent with the imaginary part of $x(n)$) signal with known variance and higher moments.
3. The convolutional noise $p(n)$ and the source signal are independent. Thus, $\sigma_z^2 = E[z(n)z^*(n)] = E[(x(n) + p(n))(x(n) + p(n))^*] = E[x(n)x^*(n)] + E[p(n)p^*(n)]$.
4. $\frac{\partial F(n)}{\partial z(n)}$ can be expressed as a polynomial function of the equalized output namely as $P(z)$ of order three.
5. The gain between the source and equalized output signal is equal to one.
6. The convolutional noise $p(n)$ is independent with $\tilde{w}(n)$.

The achievable SER may be defined as:

$$SER_{QAM} = 4\frac{M-1}{M} Q\left(\frac{d}{\sigma_T}\right)\left(1 - \frac{M-1}{M} Q\left(\frac{d}{\sigma_T}\right)\right) \tag{4}$$

where $M = \sqrt{M_{QAM}}$ and M_{QAM} is the number of signal points for a M_{QAM}-ary QAM constellation, d is half the distance between adjacent $\sqrt{M_{QAM}}$- ary PAM signals.

$$\sigma_T = \sqrt{m_p + \sigma_{\tilde{w}_r}^2}; \qquad Q\left(\frac{d}{\sigma_T}\right) = \frac{1}{\sqrt{2\pi}} \int_{\frac{d}{\sigma_T}}^{\infty} e^{\frac{-u^2}{2}} du \tag{5}$$

and m_p, $\sigma_{\tilde{w}_r}^2$ are according to (Pinchas, 2010b):

$$\begin{gathered} \text{for} \quad Sol_1^{mp_1} > 0 \quad \text{and} \quad Sol_2^{mp_1} > 0 \\ m_p = \min\left[Sol_1^{mp_1}, Sol_2^{mp_1}\right] \\ \text{or} \\ \text{for} \quad Sol_1^{mp_1} \cdot Sol_2^{mp_1} < 0 \\ m_p = \max\left[Sol_1^{mp_1}, Sol_2^{mp_1}\right] \\ \text{where} \\ Sol_1^{mp_1} = \frac{-B_1 + \sqrt{B_1^2 - 4A_1C_1B}}{2A_1} \\ Sol_2^{mp_1} = \frac{-B_1 - \sqrt{B_1^2 - 4A_1C_1B}}{2A_1} \end{gathered} \tag{6}$$

$$\begin{aligned} A_1 = \Big(B\big(45\sigma_{x_r}^2 a_3^2 + 18\sigma_{x_r}^2 a_3 a_{12} + 6a_1a_3 + \\ 9\sigma_{x_r}^2 a_{12}^2 + 2a_1a_{12}\big) - 2\left(3a_3 + a_{12}\right)\Big) + \\ B\left(45a_3^2 + 18a_3a_{12} + 9a_{12}^2\right)\sigma_{\tilde{w}_r}^2 \end{aligned}$$

$$B_1 = \Big(B\Big(12\left(\sigma_{x_r}^2\right)^2 a_3 a_{12} + 6\left(\sigma_{x_r}^2\right)^2 a_{12}^2 + 12\sigma_{x_r}^2 a_1 a_3 + 4\sigma_{x_r}^2 a_1 a_{12} + a_1^2 + 15E\left[x_r^4\right] a_3^2 + 2E\left[x_r^4\right] a_3 a_{12} + E\left[x_r^4\right] a_{12}^2\Big) - 2\left(a_1 + 3\sigma_{x_r}^2 a_3 + \sigma_{x_r}^2 a_{12}\right)\Big) + B\left(45a_3^2 + 16a_3 a_{12} + 9a_{12}^2\right)\sigma_{\tilde{w}_r}^4 + \left(B\left(90a_3^2\sigma_{x_r}^2 + 36a_3 a_{12}\sigma_{x_r}^2 + 12a_1 a_3 + 18a_{12}^2\sigma_{x_r}^2 + 4a_1 a_{12}\right) - 2a_{12} - 6a_3\right)\sigma_{\tilde{w}_r}^2$$

$$C_1 = \Big(2\left(\sigma_{x_r}^2\right)^2 a_1 a_{12} + \sigma_{x_r}^2 a_1^2 + 2E\left[x_r^4\right]\sigma_{x_r}^2 a_3 a_{12} + E\left[x_r^4\right]\sigma_{x_r}^2 a_{12}^2 + 2E\left[x_r^4\right] a_1 a_3 + E\left[x_r^6\right] a_3^2\Big) + \left(15a_3^2 + 6a_3 a_{12} + 3a_{12}^2\right)\sigma_{\tilde{w}_r}^6 + \left(45a_3^2\sigma_{x_r}^2 + 18a_3 a_{12}\sigma_{x_r}^2 + 6a_1 a_3 + 9a_{12}^2\sigma_{x_r}^2 + 2a_1 a_{12}\right)\sigma_{\tilde{w}_r}^4 + \Big(a_1^2 + 12a_1 a_3\sigma_{x_r}^2 + 4a_1 a_{12}\sigma_{x_r}^2 + 15a_3^2 E\left[x_r^4\right] + 12a_3 a_{12}\left(\sigma_{x_r}^2\right)^2 + 2a_3 a_{12} E\left[x_r^4\right] + a_{12}^2 E\left[x_r^4\right] + 6a_{12}^2\left(\sigma_{x_r}^2\right)^2\Big)\sigma_{\tilde{w}_r}^2$$

$$B = \mu N\sigma_x^2 \textstyle\sum_{l=0}^{l=R-1} |h_l(n)|^2 + \frac{\mu N\sigma_x^2}{SNR} \tag{7}$$

$\sigma_{\tilde{w}_r}^2 \cong \frac{\sigma_{x_r}^2}{SNR\sum_{l=0}^{l=R-1}|h_l(n)|^2}$, $SNR = \frac{\sigma_x^2}{\sigma_w^2}$, R is the channel length and a_1, a_{12}, a_3 are properties of the chosen equalizer and given by (Pinchas, 2010b):

$$Re\left(\frac{\partial F(n)}{\partial z(n)}\right) = \left(a_1(z_r) + a_3(z_r)^3 + a_{12}(z_r)(z_i)^2\right) \tag{8}$$

where $Re(\cdot)$ is the real part of $(\cdot)$ and z_r, z_i are the real and imaginary parts of the equalized output $z(n)$ respectively.

Comments:
Assumptions 1 and 3 were also made in (Nikias and Petropulu, 1993), (Bellini, 1986), (Fiori, 2001) and in (Haykin, 1991). It should be noted that the described model for the convolutional noise $p(n)$ is applicable during the latter stages of the process where the process is close to optimality (Haykin, 1991). According to (Haykin, 1991), in the early stages of the iterative deconvolution process, the ISI is typically large with the result that the data sequence and the convolutional noise are strongly correlated and the convolutional noise sequence is more uniform than Gaussian (Godfrey and Rocca, 1981). However, satisfying equalization performance were obtained by (Fiori, 2001) and others (Pinchas and Bobrovsky, 2006) in spite of the fact that the described model for the convolutional noise $p(n)$ was used. These results (Fiori, 2001), (Pinchas and Bobrovsky, 2006) may indicate that the described model for the convolutional noise $p(n)$ can be used (maybe not in the optimum way) in the early stages where the "eye diagram" is still closed. It should be pointed out that we are interested in the SER where the blind adaptive equalizer has already converged and leaves the system with a residual ISI which is much lower compared with the initial ISI (which is the usual situation). Thus, we are far away from the case where the data sequence and the convolutional noise are strongly correlated.

Proof :
According to (Proakis, 1995), for rectangular QAM signal constellations in which $M_{QAM} = 2^k$, where k is even, the QAM signal constellation is equivalent to two PAM (Pulse Amplitude Modulation) signals on quadrature carriers, each having $\sqrt{M_{QAM}} = 2^{\frac{k}{2}}$ signal points. Since the signals in the phase-quadrature components can be perfectly separated at the demodulator, the probability of error for QAM is easily determined from the probability of error for PAM (Proakis, 1995). Specifically, the probability of a correct decision for the M_{QAM}-ary QAM system is (Proakis, 1995):

$$P_c = \left(1 - P_{\sqrt{M_{QAM}}}\right)^2 \tag{9}$$

where $P_{\sqrt{M_{QAM}}}$ is the probability of error of a $\sqrt{M_{QAM}}$-ary PAM with one-half the average power in each quadrature signal of the equivalent QAM system. Therefore, the probability of a symbol error for the M_{QAM}-ary QAM is (Proakis, 1995):

$$P_{M_{QAM}} = 1 - \left(1 - P_{\sqrt{M_{QAM}}}\right)^2 \tag{10}$$

According to (Thompson, 2005), for the $M-2$ inner points, the probability of error of a $\sqrt{M_{QAM}}$-ary PAM is:

$$\begin{aligned} P_{inner} = P\left(|p_r(n) + \tilde{w}_r(n)| > d\right) = \\ 2P\left((p_r(n) + \tilde{w}_r(n)) > d\right) \end{aligned} \tag{11}$$

where $\tilde{w}_r(n)$ is the real part of $\tilde{w}(n)$. For the two outer points, the probability error is (Thompson, 2005):

$$P_{outer} = P\left((p_r(n) + \tilde{w}_r(n)) > d\right) = \frac{P_{inner}}{2} \tag{12}$$

Therefore, by using (11) and (12), the overall SER of a $\sqrt{M_{QAM}}$-ary PAM (Thompson, 2005) is:

$$SER_{PAM} = \frac{M-2}{M}P_{inner} + \frac{2}{M}P_{outer} = \frac{M-1}{M}P_{inner} \tag{13}$$

Next we turn to calculate $P\left((p_r(n) + \tilde{w}_r(n)) > d\right)$. According to assumptions 1 and 6 from this section, we may approximate the probability density function (pdf) of $(p_r(n) + \tilde{w}_r(n))$ as:

$$\hat{f}(p_r(n) + \tilde{w}_r(n)) \simeq \frac{1}{\sqrt{2\pi}\sigma_T} e^{\frac{-(p_r(n)+\tilde{w}_r(n))^2}{2\sigma_T^2}} \tag{14}$$

By using (14) we may write:

$$P(t > d) \simeq \int_d^{\infty} \frac{1}{\sqrt{2\pi}\sigma_T} e^{\frac{-t^2}{2\sigma_T^2}} dt = Q\left(\frac{d}{\sigma_T}\right) \quad (15)$$

where $t = (p_r(n) + \tilde{w}_r(n))$. Now, by substituting (15) into (11) and the obtained expression for P_{inner} into (13) we obtain:

$$\begin{aligned} SER_{QAM} = 1 - \left(1 - P_{\sqrt{M_{QAM}}}\right)^2 = \\ 1 - \left(1 - 2\tfrac{M-1}{M} Q\left(\tfrac{d}{\sigma_T}\right)\right)^2 = \\ 4\tfrac{M-1}{M} Q\left(\tfrac{d}{\sigma_T}\right)\left(1 - \tfrac{M-1}{M} Q\left(\tfrac{d}{\sigma_T}\right)\right) \end{aligned} \quad (16)$$

This completes our proof.

4 SIMULATION

In this section we test our new proposed expression for the SER (4) for the 16QAM case (a modulation using $\pm\{1,3\}$ levels for in-phase and quadrature components) with Godard's algorithm (Godard, 1980) for various SNR and step-size values. It should be pointed out that the closed-form approximated expression for m_p was tested for various types of equalizers, step-size parameters, channel types, equalizer's tap length and input constellations in (Pinchas, 2010b) and (Pinchas, 2010a). Thus it is reasonable to show here only the performance for the 16QAM case with a specific channel type. The equalizer taps for Godard's algorithm (Godard, 1980) were updated according to:

$$\begin{aligned} c_m(n+1) = c_m(n) - \\ \mu_G\left(|z(n)|^2 - \frac{E\left[|x(n)|^4\right]}{E\left[|x(n)|^2\right]}\right) z(n) y^*(n-m) \end{aligned} \quad (17)$$

where μ_G is the step-size. The values for a_1, a_{12} and a_3 corresponding to Godards's (Godard, 1980) algorithm are given by:

$$a_1 = -\frac{E\left[|x(n)|^4\right]}{E\left[|x(n)|^2\right]}; \qquad a_{12} = 1; \qquad a_3 = 1 \quad (18)$$

The following channel was considered: **Channel1** (initial ISI $= 0.44$): The channel parameters were determined according to (Shalvi and Weinstein, 1990):
$h_n = (0 \quad \text{for} \quad n < 0; \quad -0.4 \quad \text{for} \quad n = 0$
$0.84 \cdot 0.4^{n-1} \quad \text{for} \quad n > 0)$.
The equalizer's tap length was set to 13. The equalizer was initialized by setting the center tap equal to one and all others to zero.

In the following we denote the SER performance according to (4) as "Calculated with Equalizer". In

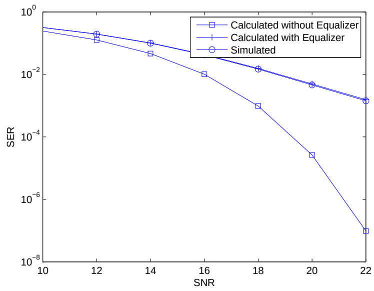

Figure 2: SER comparison with the following parameters: $d = 1$, the step-size parameter $\mu = 0.00005$, the averaged results were obtained in 100 Monte Carlo trials where 128000 symbols were produced for each trial.

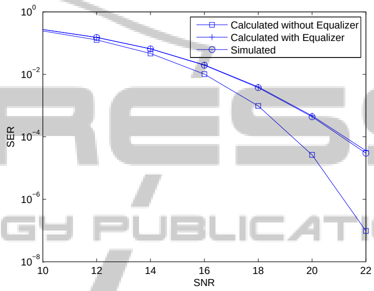

Figure 3: SER comparison with the following parameters: $d = 1$, the step-size parameter $\mu = 0.00002$, the averaged results were obtained in 100 Monte Carlo trials where 128000 symbols were produced for each trial.

addition we wish to show the SER performance for the case where the residual ISI is not taken into account. Therefore, we denote in the following the SER performance that does not take into account the residual ISI as "Calculated without Equalizer". Figure 2 and Figure 3 show the SER performance as a function of SNR of our proposed expression (4) compared with the simulated results and with those calculated results that do not take into account the residual ISI. According to Figure 2 and Figure 3, a high correlation is observed between the simulated and calculated results (4), while the opposite is seen by comparing the simulated and those calculated results that do not take into account the residual ISI. Figure 4 shows the SER performance as a function of the step-size parameter of our proposed expression (4) compared with the simulated results. According to Figure 4, a high correlation is observed between the simulated and calculated results (4).

5 CONCLUSIONS

In this paper, we propose a closed-form expression for the SER that takes into account the achievable

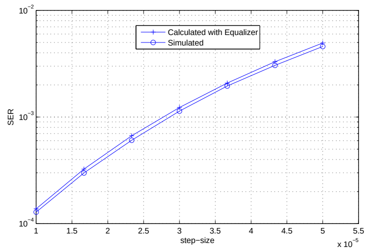

Figure 4: SER comparison with the following parameters: $d = 1$, $SNR = 20$ [dB], the averaged results were obtained in 100 Monte Carlo trials where 128000 symbols were produced for each trial.

performance of the chosen blind adaptive equalizer from the residual ISI point of view. Thus, this expression depends on the step-size parameter, equalizer's tap length, input signal statistics, channel power and SNR. According to simulation results, a high correlation exists between the simulated and calculated results.

ACKNOWLEDGEMENTS

I would like to thank the anonymous reviewers for their helpful comments.

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