Forecasting for Discrete Time Processes based on Causal Band-limited Approximation

Nikolai Dokuchaev
Department of Mathematics & Statistics, Curtin University, GPO Box U1987, Perth, 6845 Western Australia, Australia

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Abstract: We study causal dynamic smoothing of discrete time processes via approximation by band-limited discrete time processes. More precisely, a part of the historical path of the underlying process is approximated in Euclidean norm by the trace of a band-limited process. We analyze related optimization problem and obtain some conditions of solvability and uniqueness. An unique extrapolation to future times of the optimal approximating band-limited process can be interpreted as an optimal forecast.

1 INTRODUCTION

We study causal dynamic smoothing of discrete time processes via approximation by band-limited discrete time processes. More precisely, a part of the historical path of the underlying process is approximated in Euclidean norm by the trace of a band-limited discrete time process. Since an unique extrapolation to future times of the optimal approximating band-limited process can be interpreted as an optimal forecast, this task has many practical applications. It is well known that it is not possible to find an ideal low-pass causal linear time-invariant filter. In continuous time setting, it is known that the distance of the set of ideal low-pass filters from the set of all causal filters is positive (Almira and Romero, 2008) and that the optimal approximation of the ideal low-pass filter is not possible (Dokuchaev, 2012c). Our goal is to substitute the solution of these unsolvable problems by solution of an easier problem in discrete time setting such that the filter is not necessary time invariant. Our motivation is that, for some problems, the absence of time invariancy for a filter can be tolerated. For example, a typical approach to forecasting in finance is to approximate the known path of the stock price process by a process allowing an unique extrapolation that can be used as a forecast. This has to be done at current time; at future times, forecasting rule can be amended according to new data collected.

We suggest to approximate discrete time processes by the discrete time band-limited processes. More precisely, we suggest to approximate the known historical path of the process by the trace of a band-limited process. The approximating sequence does not necessary match the underlying process at sampling points. This is different from classical sampling approach; see, e.g., (Jerry, 1977). Our approach is close to the approach from (Ferreira, 1995b) and (Ferreira, 1995a), where the estimate of the error norm is given. The difference is that, in our setting, it is guaranteed that the approximation generates the error of the minimal Euclidean norm.

We obtain analyze existence and uniqueness of an optimal approximation. The optimal process is derived in time domain in a form of sinc series.The approximating band-limited process can be interpreted as a causal and linear filter that is not time invariant. The filter obtained is not time invariant; as a consequence, the coefficients of these series and have to be changed dynamically, to accommodate the current flow of observations. An unique extrapolation to future times of the optimal approximating band-limited process can be interpreted as an optimal forecast at any given time. This paper develops further the approach suggested in (Dokuchaev, 2011) where the continuous time setting was considered. We extend now this approach on discrete time processes. Some related results can be found in (Dokuchaev, 2012b) and (Dokuchaev, 2012d) for discrete time processes that are band-limited or close to band-limited.

2 DEFINITIONS

For a Hilbert space H, we denote by $(\cdot,\cdot)_H$ the cor-

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responding inner product. We use notation $\mathrm{sinc}(x) = \sin(x)/x$.

Let $\mathbb{Z}$ be the set of all integers, and let $\mathbb{Z}^+$ be the set of all positive integers. We denote by ℓ_r the set of all sequences $x = \{x(t)\}_{t\in\mathbb{Z}} \subset \mathbf{R}$, such that $\|x\|_{\ell_r} = \left(\sum_{t=-\infty}^{\infty} |x(t)|^r\right)^{1/r} < +\infty$ for $r \in [1,\infty)$ or for $r = +\infty$.

Let ℓ_r^+ be the set of all sequences $x \in \ell_r$ such that $x(t) = 0$ for $t = -1,-2,-3,\ldots$.

For $x \in \ell_1$ or $x \in \ell_2$, we denote by $X = \mathcal{Z}x$ the Z-transform

$$X(z) = \sum_{t=-\infty}^{\infty} x(t) z^{-t}, \quad z \in \mathbf{C}.$$

Respectively, the inverse Z-transform $x = \mathcal{Z}^{-1}X$ is defined as

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{i\omega}\right) e^{i\omega t} d\omega, \quad t = 0,\pm 1,\pm 2,\ldots.$$

If $x \in \ell_2$, then $X|_{\mathbb{T}}$ is defined as an element of $L_2(\mathbb{T})$.

Let $\theta,\tau \in \mathbb{Z}\cup\{+\infty\}$ and $\theta < \tau$. We denote by $\ell_2(\theta,\tau)$ the Hilbert space of complex valued sequences $\{x(t)\}_{t=\theta}^{\tau}$ such that $\|x\|_{\ell_2(\theta,\tau)} = \left(\sum_{t=\theta}^{\tau} |x(t)|^2\right)^{1/2} < +\infty$.

Let $\mathcal{U}_{\Omega,\infty}$ be the set of all mappings $X : \mathbb{T} \to \mathbf{C}$ such that $X\left(e^{i\omega}\right) \in L_2(-\pi,\pi)$ and $X\left(e^{i\omega}\right) = 0$ for $|\omega| > \Omega$. Note that the corresponding processes $x = \mathcal{Z}^{-1}X$ are said to be band-limited.

Let $\mathcal{U}_{\Omega,N}$ be the set of all $X \in U_{\Omega,\infty}$ such that there exists a sequence $\{y_k\}_{k=-N}^{N} \in \mathbf{C}^{2N+1}$ such that $X\left(e^{i\omega}\right) = \sum_{k=-N}^{N} y_k e^{ik\omega/\Omega}\mathbb{I}_{\{|\omega|\le\Omega\}}$, where $\mathbb{I}$ is the indicator function.

We assume that we are given $\Omega \in (\pi/2,\pi)$, $N \in \mathbb{Z}^+$, $s \in \mathbb{Z}$ and $q \in \mathbb{Z}$, such that $q < s$ and $s - q \ge 2N+1$.

Let $\mathcal{T} = \{t \in \mathbb{Z} :\ q \le t \le s\}$.

Let Z_N be the set of all integers k such that $|k| \le N$.

Let $\mathcal{Y}_N$ be the Hilbert space of sequences $\{y_k\}_{k=-N}^{N} \subset \mathbf{C}$ provided with the Euclidean norm, i.e., such that $\|y\|_{\mathcal{Y}_N} = \left(\sum_{k\in Z_N} |y_k|^2\right)^{1/2}$.

Consider the Hilbert spaces of sequences $\mathcal{X} = \ell_2$ and $\mathcal{X}_- = \ell_2(q,s)$.

Let $\mathcal{X}_{\Omega,N}$ be the subset of $\mathcal{X}_-$ consisting of sequences $\{x(t)\}|_{t\in\mathcal{T}}$, where $x \in \mathcal{X}$ are such that $x(t) = (\mathcal{Z}^{-1}X)(t)$ for $t \in \mathcal{T}$ for some $X\left(e^{i\omega}\right) \in \mathcal{U}_{\Omega,N}$.

Up to the end of this paper, we assume that the following condition is satisfied.

Condition 2.1. *The matrix* $\{\mathrm{sinc}(k\pi + \Omega m)\}_{k,m=-N}^{N}$ *is nondegenerate.*

Lemma 1. *Let* $\Omega_0 \in (\pi/2,\pi)$ *be selected such that there exists* $p \in (0,1)$ *such that*

$$\min_{k\in Z_N} |\mathrm{sinc}(\pi k - \Omega k)| \ge p,$$

$$\max_{k,m\in Z_N,\ t\ne -k} |\mathrm{sinc}(\pi k + \Omega m)| < \frac{p}{2N} \quad \text{for all} \quad \Omega \in [\Omega_0,\pi). \tag{1}$$

Then the matrix $\{\mathrm{sinc}(k\pi + \Omega m)\}_{k,m=-N}^{N}$ *is nondegenerate for all* $\Omega \in [\Omega_0,\pi)$.

Clearly, (1) holds for any Ω_0 that is close enough to π, since $\mathrm{sinc}(x) \to 1$ as $x \to 0$ and $\mathrm{sinc}(x) \to 0$ as $x \to \pi m$, where $m \in \mathbb{Z}$, $m \ne 0$. Therefore, Condition 2.1 can be satisfied with selection of Ω being close enough to π.

Lemma 2. *For any* $x \in \mathcal{X}_{\Omega,N}$, *there exists an unique* $X \in \mathcal{U}_{\Omega,N}$ *such that* $x(t) = (\mathcal{Z}^{-1}X)(t)$.

By Lemma 2, the future of even more "smooth" processes from $\mathcal{X}_{\Omega,N}$ is uniquely defined by a finite set of historical values that has at least $2N+1$ elements for any $N < +\infty$ and $\Omega \in [\Omega_0,\pi)$.

3 APPROXIMATION RESULTS

3.1 The Optimization Problem in the Time Domain

Let $x \in \mathcal{X}$ be a process. We assume that the sequence $\{x(t)\}_{t\in\mathcal{T}}$ represents available historical data. Let Hermitian form $F : \mathcal{X}_{\Omega,N} \times \mathcal{X}_- \to \mathbf{R}$ be defined as

$$F(\widehat{x},x) = \sum_{t=q}^{s} |\widehat{x}(t) - x(t)|^2.$$

Theorem 1. *(i) There exists an optimal solution* $\widehat{x}$ *of the minimization problem*

$$\text{Minimize} \qquad F(\widehat{x},x) \quad \text{over} \quad \widehat{x} \in \mathcal{X}_{\Omega,N}. \tag{2}$$

(ii) If $s - q \ge 2N+1$, *then the corresponding optimal process* $\widehat{x}$ *is uniquely defined.*

Remark 1. *By Proposition 2, there exists an unique extrapolation of the band-limited solution* $\widehat{x}$ *of problem (2) on the future times* $t > s$, *under the assumptions of Theorem 1. It can be interpreted as the optimal forecast (optimal given* Ω *and* N*).*

3.2 The Optimization Problem for Fourier Coefficients

To solve problem (2) numerically, it is convenient to expand Z-transform $X\left(e^{i\omega}\right)$ on the unit circle via Fourier series.

Consider the mapping $\mathcal{Q}:\mathcal{Y}_N\to\mathcal{X}_{\Omega,N}$ such that $\widehat{x}=\mathcal{Q}y$ is such that $\widehat{x}(t)=(\mathcal{Z}^{-1}\widehat{X})(t)$ for $t\in(q,s]$, where

$$\widehat{X}\left(e^{i\omega}\right)=\sum_{k\in Z_N}y_k e^{ik\omega/\Omega}\mathbb{I}_{\{|\omega|\le\Omega\}}. \qquad (3)$$

Clearly, this mapping is linear and continuous.

Let Hermitian form $G:\mathcal{Y}_N\times\mathcal{X}_-\to\mathbf{R}$ be defined as

$$G(y,x)=F(\mathcal{Q}y,x)=\sum_{t=q}^{s}|\widehat{x}(t)-x(t)|^2,$$
$$\widehat{x}=\mathcal{Q}y. \qquad (4)$$

Corollary 1. *There exists an unique solution y of the minimization problem*

$$\text{Minimize} \qquad G(y,x) \quad \text{over} \quad y\in\mathcal{Y}_N. \qquad (5)$$

Problem (2) can be solved via problem (5); its solution can be found numerically.

Let $\widehat{X}$ be defined by (3), where $\{y_k\}\in\mathcal{Y}_N$. Let $\widehat{x}=\mathcal{Z}^{-1}\widehat{X}$. We have that

$$\begin{aligned}\widehat{x}(t)&=\frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left(\sum_{k\in Z_N}y_k e^{ik\omega\pi/\Omega}\right)e^{i\omega t}d\omega\\&=\frac{1}{2\pi}\sum_{k\in Z_N}y_k\int_{-\Omega}^{\Omega}e^{ik\omega\pi/\Omega+i\omega t}d\omega\\&=\frac{1}{2\pi}\sum_{k\in Z_N}y_k\frac{e^{ik\pi+i\Omega t}-e^{-ik\pi-i\Omega t}}{ik\pi/\Omega+it}\\&=\frac{\Omega}{\pi}\sum_{k\in Z_N}y_k\mathrm{sinc}\,(k\pi+\Omega t).\end{aligned}$$

Hence

$$\begin{aligned}G(y,x)&=\sum_{t=q}^{s}|\widehat{x}(t)-x(t)|^2\\&=\sum_{t=q}^{s}\left|\frac{\Omega}{\pi}\sum_{k\in Z_N}y_k\mathrm{sinc}\,(k\pi+\Omega t)-x(t)\right|^2\\&=(y,Ry)_{\mathcal{Y}_N}-2\mathrm{Re}\,(y,rx)_{\mathcal{X}_-}+(\rho x,x)_{\mathcal{X}_-}. \qquad (6)\end{aligned}$$

Here $R:\mathcal{Y}_N\times\mathcal{Y}_N\to\mathcal{Y}_N$ is a linear bounded Hermitian operator, $r:\mathcal{X}_-\to\mathcal{Y}_N$ is a bounded linear operator, $\rho:\mathcal{X}_-\times\mathcal{X}_-\to\mathcal{X}_-$ is a linear bounded Hermitian operator.

It follows from the definitions that the operator R is non-negatively defined (it suffices to substitute $x(t)\equiv 0$ into the Hermitian form).

3.3 The Explicit Solution of the Optimization Problem

Since the space $\mathcal{Y}_N$ is finite dimensional, the operator R can be represented via a matrix $R=\{R_{km}\}\in\mathbf{C}^{2N+1,2N+1}$, where $R_{km}=R_{mk}$. In this setting, $(Ry)_k=\sum_{k=-N}^{N}R_{km}y_m$.

Theorem 2. *(i) The operator R is positively defined.*

(ii) Problem (5) has a unique solution $\widehat{y}=R^{-1}rx$.

(iii) The components of the matrix R can be found from the equality

$$R_{km}=\frac{\Omega^2}{\pi^2}\sum_{t=q}^{s}\mathrm{sinc}\,(m\pi+\Omega t)\mathrm{sinc}\,(k\pi+\Omega t). \qquad (7)$$

(iv) The components of the vector $rx=\{(rx)_k\}_{k=-N}^{N}$ can be found from the equality

$$(rx)_k=\frac{\Omega}{\pi}\sum_{t=q}^{s}\mathrm{sinc}\,(k\pi+\Omega t)x(t). \qquad (8)$$

Corollary 2. *Let $\widehat{y}$ be the vector calculated as in Theorem 2, $\widehat{y}=\{\widehat{y}_k\}_{k=-N}^{N}$. The process*

$$\widehat{x}(t)=\widehat{x}(t,q,s)=\frac{\Omega}{\pi}\sum_{k\in Z_N}y_k\mathrm{sinc}\,(k\pi+\Omega t)$$

represents the output of a causal filter that is linear but not time invariant.

The proofs of results given above can be found in the working paper (Dokuchaev, 2012a).

4 NUMERICAL EXPERIMENTS

In the numerical experiments described below, we have used MATLAB.

The experiments show that some eigenvalues of R are quite close to zero despite the fact that, by Theorem 2, $R>0$. Respectively, the error for the MATLAB solution of the equation $R\widehat{y}=rx$ does not vanish. Further, in our experiments, we found that the error E can be decreased by the replacing R in the equation $\widehat{x}=R^{-1}rx$ by $R_\varepsilon=R+\varepsilon I$, where I is the unit matrix and where $\varepsilon>0$ is small. We have used $\varepsilon=0.001$.

Figures 2 show examples of processes $x(t)$ and the corresponding band-limited processes $\widehat{x}(t)$ with approximating $x(t)$ with $N=15$ at times $t\in\{-25,\dots,15\}$ (i.e., with $q=-25$, $s=15$). The values of $\widehat{x}(t)$ for $t>15$ were calculated using $\{x(s)\}_{s\le 15}$ and can be considered as an optimal forecast of $x(t)$. Figure 2 shows the result for $\Omega=0.2$; Figure **??** shows the result for $\Omega=0.9$.

We have verified numerically that the matrix $\{\mathrm{sinc}\,(k\pi+\Omega m)\}_{k,m=-N}^{N}$ is nondegenerate. Therefore, Condition 2.1 is satisfied. In fact, we found that this matrix was nondegenerate in all experiments for all kinds of Ω and N.

By Remark 1, the extrapolation of the process $\widehat{x}\in\mathcal{X}_{\Omega,N}$ to the future times $t>s$ can be interpreted as the optimal forecast (optimal given Ω and N).

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APPENDIX

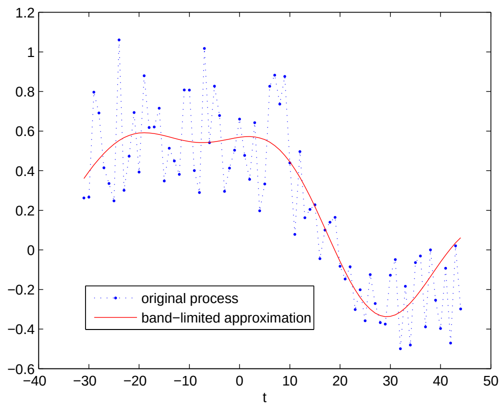

Figure 1: Example of $x(t)$ and band-limited process $\widehat{x}(t)$ approximating $x(t)$ for $t \in \{-25,..,15\}$, with $\Omega = 0.2$, and $N = 15$.

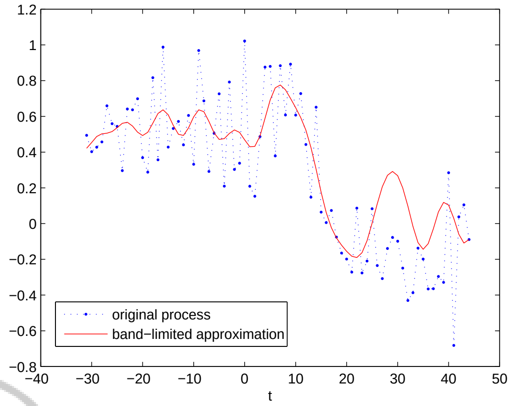

Figure 2: Example of $x(t)$ and band-limited process $\widehat{x}(t)$ approximating $x(t)$ for $t \in \{-25,..,15\}$, with $\Omega = 0.9$, and $N = 15$.