Measuring Linearity of Curves

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Abstract: In this paper we define a new linearity measure which can be applied to open curve segments. The new measure ranges over the interval $(0,1]$, and produces the value 1 if and only if the measured line is a perfect straight line segment. Also, the new linearity measure is invariant with respect to translations, rotations and scaling transformations.

1 INTRODUCTION

Shape descriptors have been employed in many computer vision and image processing tasks (e.g. image retrieval, object classification, object recognition, object identification, etc). Different mathematical tools have been used to define the shape descriptors: algebraic invariants (Hu, 1962), geometric approaches (Žunić et al., 2010), Fourier analysis (Bowman et al., 2001), morphological operations (Ruberto and Dempster, 2000), integral transformations (Rahtu et al., 2006), statistical methods (Melter et al., 1993), fractal techniques (Imre, 2009), logic (Schweitzer and Straach, 1998), multiscale approaches (Direkoglu and Nixon, 2011), integral invariants (Manay et al., 2006), etc. Generally speaking, shape descriptors can be classified into two groups: area based descriptors and boundary based ones. Area based descriptors are more robust (i.e. less sensitive to noise or shape deformations) while boundary based descriptors are more sensitive. A preference for either type of descriptor depends on the application performed and the data available. For example low quality data would require robust descriptors (i.e. area based ones) while high precision tasks would require more sensitive descriptors (i.e. boundary based ones). In the literature so far, more attention has been paid to the area based descriptors, not only because of their robustness but also because they are easier to be efficiently estimated when working with discrete data. Due to the recent proliferation of image verification, identification and recognition systems there is a strong demand for shape properties that can be derived from their boundaries (Manay et al., 2006; Mio et al., 2007; Stojmenović and Žunić, 2008). It is worth mentioning that some objects, like human signatures for example, are linear by their nature and area based descriptors cannot be used for their analysis.

In this paper we deal with linearity measures that should indicate the degree to which an open curve segment differs from a perfect straight line segment. Several linearity measures for curve segments are already considered in the literature (Gautama et al., 2004; Gautama et al., 2003; Stojmenović et al., 2008; Žunić and Rosin, 2011).

Perhaps the simplest way to define the linearity measure of an open curve segment is to consider the ratio between the length of the curve considered and the distance between its end points. This is a natural and simple definition which is also called the *straightness index* (Benhamou, 2004). It satisfies the following basic requirements for a linearity measure of open curve segments.

- The straightness index varies through the interval $[0,1]$;
- The straightness index equals 1 only for straight line segments;
- The straightness index is invariant with respect to translation, rotation and scaling transformation on a curve considered.

Also, the straightness index is simple to compute and its behavior can be clearly predicted, i.e. we can see easily which curves have the same linearities, measured by the straightness index. It is obvious that

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those curves whose end points and the length coincide, have the same straightness index. But the diversity of such curves is huge and the straightness index cannot distinguish among them, which could be a big drawback in certain applications. Some illustrations using simple polygonal curves are shown in Fig. 1.

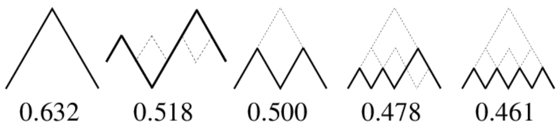

Figure 1: Five displayed curves (solid lines) have different linearities measured by $\mathcal{L}(\mathcal{C})$. The straightness index has the same value for all three curves.

In this paper we define a new linearity measure $\mathcal{L}(\mathcal{C})$ for open curve segments. The new measure satisfies the basic requirements (listed above) which are expected to be satisfied for any curve linearity measure. Since it considers the distance of the end points of the curve to the centroid of the curve, the new measure is also easy to compute. The fact that it uses the curve centroids implies that it takes into account a relative distribution of the curve points.

The paper is organized as follows. Section 2 gives basic definitions and denotation. The new linearity measure for planar open curve segments is in Section 3. Several experiments which illustrate the behavior and the classification power of the new linearity measure are provided in Section 4. Concluding remarks are in Section 5.

2 DEFINITIONS AND DENOTATIONS

Without loss of generality, throughout the paper, it will be assumed (even if not mentioned) that every curve $\mathcal{C}$ has length equal to 1 and is given in an arc-length parametrization. I.e., planar curve segment $\mathcal{C}$ is represented as:

$$x = x(s), \qquad y = y(s), \quad \text{where} \quad s \in [0,1].$$

The parameter s measures the distance of the point $(x(s), y(s))$ from the curve start point $(x(0), y(0))$, along the curve $\mathcal{C}$.

The centroid of a given planar curve $\mathcal{C}$ will be denoted by $(x_\mathcal{C}, y_\mathcal{C})$ and computed as

$$(x_\mathcal{C}, y_\mathcal{C}) \; = \; \left(\int_\mathcal{C} x(s)\, ds, \; \int_\mathcal{C} y(s)\, ds \right). \qquad (1)$$

Taking into account that the length of $\mathcal{C}$ is assumed to be equal to 1, we can see that the coordinates of the curve centroid, as defined in (1), are the average values of the curve points.

As usual,

$$d_2(A,B) = \sqrt{(x-u)^2 + (y-v)^2}$$

will denote the Euclidean distance between the points $A = (x,y)$ and $B = (u,v)$.

As mentioned, we introduce a new linearity measure $\mathcal{L}(\mathcal{C})$ which assigns a number from the interval $(0,1]$. The curve $\mathcal{C}$ is assumed to have the length 1. More precisely, any appearing curve will be scaled by the factor which equals the length of it before the processing. So, an arbitrary curve $\mathcal{C}$ would be replaced with the curve $\mathcal{C}$ defined by

$$\mathcal{C} \; = \; \frac{1}{\int\limits_{\mathcal{C}_a} ds} \cdot \mathcal{C}_a \; = \; \left\{ \left(\frac{x}{\int\limits_{\mathcal{C}_a} ds}, \; \frac{y}{\int\limits_{\mathcal{C}_a} ds} \right) \mid (x,y) \in \mathcal{C}_a \right\}$$

Shape descriptors/measures are very useful for discrimination among the objects – in this case open curve segments. Usually the shape descriptors have a clear geometric meaning and, consequently, the shape measures assigned to such descriptors have a predictable behavior. This is an advantage because the suitability of a certain measure to a particular shape-based task (object matching, object classification, etc) can be predicted to some extent. On the other hand, a shape measure assigns to each object (here curve segment) a single number. In order to increase the performance of shape based tasks, a common approach is to assign a graph (instead of a number) to each object. E.g. such approaches define *shape signature* descriptors, which are also 'graph' representations of planar shapes, often used in shape analysis tasks (El-ghazal et al., 2009; Zhang and Lu, 2005), but they differ from the idea used here.

We will apply a similar idea here as well. To compare objects considered we use *linearity plots* to provide more information than a single linearity measurement. The idea is to compute linearity incrementally, i.e. to compute linearity of sub-segments of $\mathcal{C}$ determined by the start point of $\mathcal{C}$ and another point which moves along the curve $\mathcal{C}$ from the beginning of to the end of $\mathcal{C}$. The linearity plot $\mathrm{P}(\mathcal{C})$, associated with the given curve $\mathcal{C}$ is formally defined as follows.

Definition 1. *Let $\mathcal{C}$ be a curve given in an arc-length parametrization: $x = x(s)$, $y = y(s)$, and $s \in [0,1]$. Let $\mathbf{A}(s)$ be the part of the curve $\mathcal{C}$ bounded by the starting point $(x(0), y(0))$ and by the point $(x(s), y(s)) \in \mathcal{C}$. Then, for a linearity measure $\mathcal{L}$, the linearity plot $\mathrm{P}(\mathcal{C})$ is defined by:*

$$\mathrm{P}(\mathcal{C}) = \{(s, \mathcal{L}(\mathbf{A}(s)) \mid s \in [0,1]\}. \qquad (2)$$

We also will use the *reverse linearity plot* $\mathrm{P}_{rev}(\mathcal{C})$ defined as:

$$\mathrm{P}_{rev}(\mathcal{C}) = \{(s, \mathcal{L}(\mathbf{A}_{rev}(1-s)) \mid s \in [0,1]\}, \qquad (3)$$

where $\mathbf{A}_{rev}(1-s)$ is the segment of the curve $\mathcal{C}$ determined by the end point $(x(1),y(1))$ of $\mathcal{C}$ and the point which moves from the end point of $\mathcal{C}$, to the start point of $\mathcal{C}$, along the curve $\mathcal{C}$. In other words, $\mathrm{P}_{rev}(\mathcal{C})$ is the linearity plot of the curve $\mathcal{C}'$ which coincides with the curve $\mathcal{C}$ but the start (end) point of $\mathcal{C}$ is the end (start) point of $\mathcal{C}'$. A parametrization of $\mathcal{C}'$ can be obtained by replacing the parameter s, in the parametrization of $\mathcal{C}$, by a new parameter s' such that $s' = 1-s$. Obviously such a defined s' measures the distance of the point $(x(s'),y(s'))$ from the starting point $(x(s'=0),y(s'=0))$ of $\mathcal{C}'$ along the curve $\mathcal{C}'$, as s' varies through the interval $[0,1]$.

3 NEW LINEARITY MEASURE FOR OPEN CURVE SEGMENTS

In this section we introduce a new linearity measure for open planar curve segments.

We start with the following theorem which says that amongst all curves having the same length, straight line segments have the largest sum of distances between the curve end points to the curve centroid. This result will be exploited to define the new linearity measure for open curve segments.

Theorem 1. *Let $\mathcal{C}$ be an open curve segment given in an arc-length parametrization $x=x(s)$, $y=y(s)$, and $s\in[0,1]$. The following statements hold:*

(a) *The sum of distances of the end points $(x(0),y(0))$ and $(x(1),y(1))$ from the centroid $(x_\mathcal{C},y_\mathcal{C})$ of the curve $\mathcal{C}$ is bounded from above by 1, i.e.:*

$$\begin{aligned} & d_2((x(0),y(0)),(x_\mathcal{C},y_\mathcal{C})) \\ + \;& d_2((x(1),y(1)),(x_\mathcal{C},y_\mathcal{C})) \\ \leq \;& 1. \end{aligned} \tag{4}$$

(b) *The upper bound established by the previous item is reached by the straight line segment and, consequently, cannot be improved.*

Proof. Let $\mathcal{C}$ be a curve given in an arc-length parametrization: $x=x(s)$ and $y=y(s)$, with $s\in[0,1]$, and let $S=(x(0),y(0))$ and $E=(x(1),y(1))$ be the end points of $\mathcal{C}$. We can assume, without loss of generality, that the curve segment $\mathcal{C}$ is positioned such that

- the end points S and E belong to the x-axis (i.e. $y(0)=y(1)=0$), and
- S and E are symmetric with respect to the origin (i.e. $-x(0)=x(1)$),

as illustrated in Fig. 2.

Furthermore, let

$$\mathcal{E} = \{X=(x,y) \mid d_2(X,S)+d_2(X,E) \; = \; 1\}$$

be an ellipse which consists of points whose sum of distances to the points S and E is equal 1. Now, we prove **(a)** in two steps.

(i) First, we prove that the curve $\mathcal{C}$ and the ellipse $\mathcal{E}$ do not have more than one intersection point (i.e. $\mathcal{C}$ belongs to the closed region bounded by $\mathcal{E}$).
This will be proven by a contradiction. So, let us assume the contrary, that $\mathcal{C}$ intersects $\mathcal{E}$ at k ($k\geq 2$) points:

$$(x(s_1),y(s_1)),\;(x(s_2),y(s_2)),\;\ldots,\;(x(s_k),y(s_k)),$$

where $0<s_1<s_2<\ldots<s_k<1$. Let $A=(x(s_1),y(s_1))$ and $B=(x(s_k),y(s_k))$. Since the sum of the lengths of the straight line segments $[SA]$ and $[AE]$ is equal to 1, the length of the polyline $SABE$ is, by the triangle inequality, bigger than 1. Since the length of the arc $\widehat{SA}$ (along the curve $\mathcal{C}$) is not smaller than the length of the edge $[SA]$, the length of the arc $\widehat{AB}$ (along the curve $\mathcal{C}$) is not smaller than the length of the straight line segment $[AB]$, and the length of the arc $\widehat{BE}$ (along the curve $\mathcal{C}$) is not smaller than the length of the straight line segment $[BE]$, we deduce that the curve $\mathcal{C}$ has length bigger than 1, which is a contradiction. A more formal derivation of the contradiction $1<1$ is

$$\begin{aligned} 1 &= d_2(S,A)+d_2(A,E) \\ &< d_2(S,A)+d_2(A,B)+d_2(B,E) \\ &\leq \int_{\widehat{SA}} ds+\int_{\widehat{AB}} ds+\int_{\widehat{BE}} ds=\int_{\mathcal{C}} ds \\ &= 1. \end{aligned} \tag{5}$$

So, $\mathcal{C}$ and $\mathcal{E}$ do not have more than one intersection point, implying that $\mathcal{C}$ lies in the closed region bounded by $\mathcal{E}$.

(ii) Second, we prove that the centroid of $\mathcal{C}$ does not lie outside of $\mathcal{E}$.
The proof follows easily:
   - the convex hull $CH(\mathcal{C})$ of $\mathcal{C}$ is the smallest convex set which includes $\mathcal{C}$ and, consequently, is a subset of the region bounded by $\mathcal{E}$;
   - The centroid of $\mathcal{C}$ lies in the convex hull $CH(\mathcal{C})$ of $\mathcal{C}$ because it belongs to every half plane which includes $\mathcal{C}$ (the intersection of such half planes is actually the convex hull of $\mathcal{C}$ (see (Valentine, 1964)));
   - the two items above give the required: $(x_\mathcal{C},y_\mathcal{C})\in CH(\mathcal{C})\subset region\_bounded\_by\_\mathcal{E}$.

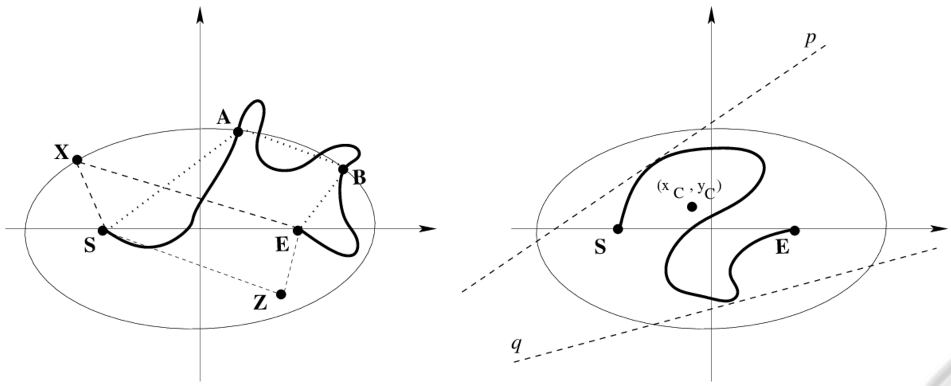

Figure 2: Denotations in the proof of Theorem 1 are illustrated above.

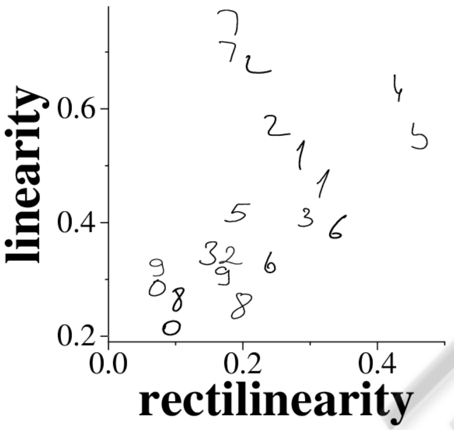

Figure 3: Handwritten digits ordered by linearity and rectilinearity.

Finally, since the centroid of $\mathcal{C}$ does not lie outside $\mathcal{E}$, the sum of the distances of the centroid $(x_\mathcal{C}, y_\mathcal{C})$ of $\mathcal{C}$ to the points S and E may not be bigger than 1, i.e.

$$d_2((x(0),y(0)),(x_\mathcal{C},y_\mathcal{C})) + d_2((x(1),y(1)),(x_\mathcal{C},y_\mathcal{C}))$$

$$= d_2(S,(x_\mathcal{C},y_\mathcal{C})) + d_2(E,(x_\mathcal{C},y_\mathcal{C}))$$

$$\leq 1.$$

This establishes **(a)**.

To prove **(b)** it is enough to notice that if $\mathcal{C}$ is a straight line segment of length 1, then the sum of its end points to the centroid of $\mathcal{C}$ is 1. □

Now, motivated by the results of Theorem 1, we give the following definition for a new linearity measure $\mathcal{L}(\mathcal{C})$ for open curve segments.

Definition 2. *Let $\mathcal{C}$ be an open curve segment. Then, the linearity measure $\mathcal{L}(\mathcal{C})$ of $\mathcal{C}$ is defined as the sum of distances between the centroid $(x_\mathcal{C}, y_\mathcal{C})$ of $\mathcal{C}$ and the end points of $\mathcal{C}$. I.e.:*

$$\mathcal{L}(\mathcal{C}) = \sqrt{(x(0)-x_\mathcal{C})^2 + (y(0)-y_\mathcal{C})^2} + \sqrt{(x(1)-x_\mathcal{C})^2 + (y(1)-y_\mathcal{C})^2}$$

where $x = x(s)$, $y = y(s)$, $s \in [0,1]$ is an arc-length representation of $\mathcal{C}$.

The following theorem summarizes desirable properties of $\mathcal{L}(\mathcal{C})$.

Theorem 2. *The linearity measure $\mathcal{L}(\mathcal{C})$ has the following properties:*

(i) *$\mathcal{L}(\mathcal{C}) \in (0,1]$, for all open curve segments $\mathcal{C}$;*
(ii) *$\mathcal{L}(\mathcal{C}) = 1 \Leftrightarrow \mathcal{C}$ is a straight line segment;*
(iii) *$\mathcal{L}(\mathcal{C})$ is invariant with respect to the similarity transformations.*

Proof. Item **(i)** is a direct consequences of Theorem 1.

To prove **(ii)** we will use the same notations as in the proof of Theorem 1 and give a proof by contradiction. So, let assume the following:
– the curve $\mathcal{C}$ differs from a straight line segment, and
– the sum of distances between the end points, and the centroid of $\mathcal{C}$ is 1.
Since $\mathcal{C}$ is not a straight line segment, then $\mathbf{d}_2(S,E) < 1$, and the centroid $(x_\mathcal{C}, y_\mathcal{C})$ lies on the ellipse $\mathcal{E} = \{X = (x,y) \mid \mathbf{d}_2(X,S) + \mathbf{d}_2(X,E) = 1\}$. Further, it would mean that there are points of the curve $\mathcal{C}$ belonging to both hyperplanes determined by the tangent on the ellipse $\mathcal{E}$ passing through the centroid of $\mathcal{C}$. This would contradict the fact that $\mathcal{C}$ and $\mathcal{E}$ do not have more than one intersection point (what was proven as a part of the proof of Theorem 1).

To prove item **(iii)** it is enough to notice that translations and rotations do not change the distance between the centroid and the end points. Since we assume that $\mathcal{C}$ is represented by using an arc-length parametrization: $x = x(s)$, $y = y(s)$, with the parameter s varying through $[0,1]$, the new linearity measure

Figure 4: Filtering connected edges by linearity. Left/first column: connected edges (minimum length: 25 pixels); second column: sections of curve with $\mathcal{L}(\mathcal{C}) < 0.5$; third column: sections of curve with $\mathcal{L}(\mathcal{C}) > 0.9$; fourth column: sections of curve with $\mathcal{L}(\mathcal{C}) > 0.95$.

$\mathcal{L}(\mathcal{C})$ is invariant with respect to scaling transformations as well. □

4 EXPERIMENTS

In this section we provide several experiments in order to illustrate the behavior and efficiency of the linearity measure introduced here.

First Experiment: Behavior Illustration. To demonstrate how various shapes produce a range of linearity values, figure 3 shows the first two samples of each handwritten digit (0–9) from the training set captured by Alimoğlu and Alpaydin (Alimoğlu and Alpaydin, 2001) plotted in a 2*D* feature space of linearity $\mathcal{L}(\mathcal{C})$ versus rectilinearity $\mathcal{R}_1(\mathcal{C})$ (Žunić and Rosin, 2003).

Despite the variability of hand writing, most pairs of the same digit are reasonably clustered. The major separations occur for:

- “2” since only one instance has a loop in the middle;
- “4” since the instance next to the pair of “7”s is missing the vertical stroke;
- “5” since the uppermost right instance is missing the horizontal stroke.

Second Experiment: Filtering Edges. Figure 4 shows the application of the linearity to filtering edges. The edges were extracted from the images using the Canny detector (Canny, 1986), connected into curves, and then thresholded according to total edge magnitude and length (Rosin, 1997). Linearity was measured in local sections of curve of length 25, and sections above (or below) a linearity threshold were retained. It can be seen that retaining sections of curve with $\mathcal{L}(\mathcal{C}) < 0.5$ finds small noisy or corner sections. Keeping sections of curve with $\mathcal{L}(\mathcal{C}) > 0.9$ or $\mathcal{L}(\mathcal{C}) > 0.95$ identifies most of the significant structures in the image.

Experiments are also shown in which Poisson image reconstruction is performed from the image gradients (Pérez et al., 2003). In figure 5b all the connected edges with minimum length of 25 pixels (shown at the bottom of the first column in figure 4) are used as a mask to eliminate all other edges before reconstruction. Some fine detail is removed as expected since small and weak edges have been removed in the pre-processing stage. When linearity filtering is applied, and only edges corresponding to sections of curve with $\mathcal{L}(\mathcal{C}) > 0.95$ are used (see bottom of the fourth column in figure 5) then the image reconstruction retains only regions that are locally linear structures (including sections of large circular objects).

Figure 5: Reconstructing the image from its filtered edges. (a) original intensity image; (b) image reconstructed using all connected edges (minimum length: 25 pixels); (c) image reconstructed using sections of curve with $\mathcal{L}(\mathcal{C}) > 0.95$.

Third Experiment: Signature Verification. For the second application we use data from Munich and

Perona (Munich and Perona, 2003) to perform signature verification. The data consists of pen trajectories for 2911 genuine signatures taken from 112 subjects, plus five forgers provided a total of 1061 forgeries across all the subjects. Examples of corresponding genuine and forged signatures are shown in Figure 6. To compare signatures we use the linearity plots defined by (2) and (3) to provide more information than a single linearity measurement. Linearity plot examples are in figure 7.

Figure 6: Examples of genuine (first and third rows) and forged (second and fourth rows) signatures.

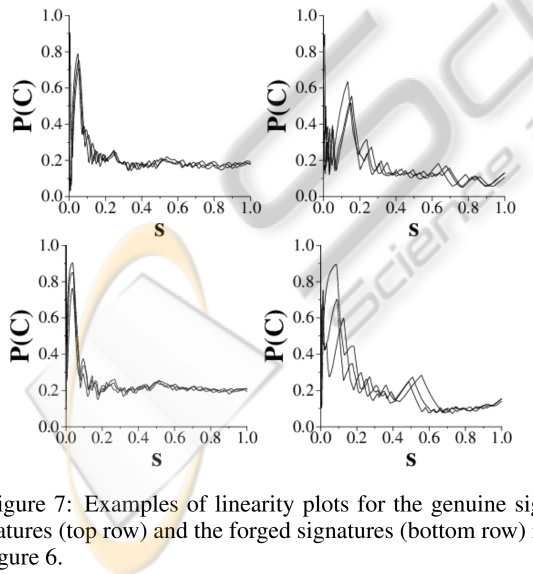

Figure 7: Examples of linearity plots for the genuine signatures (top row) and the forged signatures (bottom row) in figure 6.

The quality of match between signatures $\mathcal{C}_1$ and $\mathcal{C}_2$ is measured by the similarity between the linearity plots $P(\mathcal{C}_1)$ and $P(\mathcal{C}_2)$. This similarity is measured by the area bounded by the linearity plots $P(\mathcal{C}_1)$ and $P(\mathcal{C}_2)$ and by the vertical lines $s = 0$ and $s = 1$. Figure 7 demonstrates the linearity plots for the signatures shown in figure 6. The plots in the first (second respectively) row contain the three genuine (forged respectively) signatures from figure 6.

Nearest neighbour matching is then performed on all the data using the leave-one-out strategy. Signature verification is a two class (genuine or fake) problem. Since the identity of the signature is already known, the nearest neighbour matching is only applied to the set of genuine and forged examples of the subject's signature. Computing linearity of the signatures using $\mathcal{L}(\mathcal{C})$ produces 96.9% accuracy. This improves the results obtained by using the linearity measure defined in (Žunić and Rosin, 2011) which achieved 93.1% accuracy.

5 CONCLUSIONS

This paper has described a new shape descriptor $\mathcal{L}(\mathcal{C})$ for computing the linearity of open curve segments. It is both extremely simple to implement and efficient to compute. In addition, it satisfies the basic requirements for a linearity measure:

- $\mathcal{L}(\mathcal{C})$ is in the interval $(0,1]$;
- $\mathcal{L}(\mathcal{C})$ equals 1 only for straight line segments;
- $\mathcal{L}(\mathcal{C})$ is invariant with respect to translation, rotation and scaling transformation on a curve considered.

Finally, experiments show the effectiveness of the new linearity measure on a variety of tasks.

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