Faustmann Optimal Pine Stands Stochastic Rotation Problem

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Abstract: The Faustmann optimal rotation harvesting pine stands models under Logistic and Gompertz wood stock and Brown price stochastic diffusion processes are reformulated as stochastic one dimensional optimal stopping problem, which are solvable with the Hamilton-Jacobi-Bellman equations. The stochastic models predict a significant increase of the deterministic optimal cut, with 47.0% and 48.0% in the cases of the Logistical and Gompertz wood stock diffusion respectively. The application of these models to a Chilean forest company shows discrepancies due to the absence of consideration to wood stock and price uncertainties that the company actual cut policy shows. The experimental data significantly validate the Faustmann stochastic logistic model. They give a better approximation of the company cut policy, underestimating it by 8.09% and producing a more reliable saturation volume than the Gompertz model. The sensitivity analysis shows that both volatilities have a similar linear effect in the optimal cut, but the wood stock volatility volume elasticity of 0.687 almost doubles the stumpage price volume elasticity of 0.350, showing the importance of this uncertainty.

1 INTRODUCTION TO RADIATA PINE STANDS EXPLOITATION

The need to incorporate uncertainty in wood stock and price is not new. Samuelson (1976) not only validated Faustmann's deterministic formula (1995) as the correct one, but also considered that the forestry economist's "simple notion of stationary equilibrium needs to be replaced by the notion of perpetual Brownian motion". The majority of the early papers considered only price stochastic diffusion and simple harvest rotation see (Clark and Reed, 1989); Others like (Morck & Schwartz,1989); (Insley, 2002; and Alvarez et al., 2006) considered also wood stochastic diffusion. Few of them (Insley & Rollins, 2005; Willasen, 1998) formulated these problems as stochastic impulsive control and considered price and wood stock stochastic diffusion for the multiple rotation or Faustmann model. In a previous paper Navarrete (2011) extended the single and multiple optimal rotations harvesting pine stands models without the stands' regeneration cost for Logistic wood stock and Brown price stochastic diffusion processes, and reformulated it as an optimal stopping problems with only one stochastic diffusion, solvable with the Hamilton-Jacobi-Bellman differential equations.

The objective of this paper is to extend those stochastic results to the Faustmann formula with the stands regeneration cost for Logistic and Gompertz wood stock and Brown price diffusions, to solve the stochastic rotation of even aged pine stands harvesting and to validate these results by applying them to a Chilean forest company.

2 METHODOLOGY

2.1 Model Formulation

Given the following variable and parameters

Vt	= Wood stock at time t
μ(Vt)	= Wood stock diffusion drift parameter
σ(Vt)	= Wood stock volatility parameter
Pt	= Wood stumpage spot price at time t
Po	= Initial stumpage wood price
α	= Wood price diffusion drift rate
β	= Wood price volatility
W	= Wiener diffusion
C	= Stands regeneration cost
c	= C/Po
R, Q	= Probabilistic metrics
F	= Functional Objective

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The model considers ITO diffusion for the wood stock and a geometric Brown diffusion for the wood price respectively, given by equations (1) and (2).

$$dV_t = \mu(V_t)dt + \sigma(V_t)\, dW \tag{1}$$

$$dP_t = \alpha P_t dt + \beta P_t dW \tag{2}$$

Under the assumption of a weak solution (Vt, t) for the diffusion equations (1, 2) and initial conditions (V0 ≥ 0, P0 ≥ 0), the multiple actualized harvest value or Faustmann model (3), (see Johnson, 2006) is given by objective functional (3).

$$F^F(V_0, P_0) = \frac{\sup}{\forall(t \geq t_o)}\left(\frac{E^R(e^{-rt}P_t V_t) - C}{1 - e^{-rt}}\right) \tag{3}$$

2.2 Reformulation of the Multiple Harvest Rotation Problem

The stochastic model (1, 2, and 3) is difficult to solve. The following theorem reduces this model to a one dimensional stopping problem that is more amenable.

Theorem 1: A probabilistic measure Q exists and is equivalent to the actual metric R, such that: (see, Appendix A)

$$F(V_0, P_0) = \frac{\sup}{\forall(t \geq t_o)} E^R\left(\frac{e^{-rt}P_t V_t}{1 - e^{-rt}}\right) = P_0 \sup\left[E^Q\left(\frac{e^{-(r-\alpha)t}V_t}{1 - e^{-rt}}\right)\right. \tag{4}$$

Furthermore, under the metric Q, the process V_t follows the diffusion (5).

$$dV_t = \{\mu(V_t) + \beta\sigma(V_t)\}dt + \sigma(V_t)d\overline{W} \tag{5}$$

An optimization strategy previously developed by the author (see Navarrete, 2011) was used. The functional objective (3) was parameterized for different time values $t=t_n$, generating a family of n stochastic optimization problems. Since $C/(1-e^{-rt_n})$ is constant for each t_n, we can apply theorem 1 and reformulate each of these problems as the following optimal stopping problem with one dimensional ITO diffusion.

$$\begin{aligned} F^F(V) &= \sup\left(E^P[e^{-rt_n} P_t V_t] - C)/(1 - e^{-rt_n})\right) \\ &= P_o \sup\{E^Q[e^{-(r-\alpha)t_n} V_t/(1 - e^{-rt_n})] - C/(1 - e^{-rt_n})\} \end{aligned} \tag{6}$$

Dividing by the constant Po, this objective is reformulated as

$$\begin{aligned} F(V) = \text{Max}\ \{ & E^Q[e^{-(r-\alpha)t_n} V_t/(1 - e^{-rt_n})] \\ & - c/(1 - e^{-rt_n})\} \end{aligned} \tag{7}$$

with the following wood stock diffusion under the metric Q.

$$dV_t = \{\mu(V_t) + \beta\sigma(V_t)\}dt + \sigma(V_t)d\overline{W} \tag{8}$$

The formulation of the Hamilton Jacobi Bellman equation for this problem is given by the following inequation with the capitalized interest rate $r_t = r/(1-e^{-rt_n})$.

$$\begin{aligned} \text{Max}\{ & \tfrac{1}{2}\sigma^2 V^2 F''(V) + [\mu(V) + \beta\sigma(V)]\, F'(V) \\ & - [r_t - \alpha]\, F(V) - c\, r_t,\ (V-c)/(1 - e^{-rt_n}) - F(V)\} = 0 \end{aligned} \tag{9}$$

In this case the differential equation for the continuation region (V< V*) is given by the non homogenous differential equation (10).

$$\begin{aligned} & \tfrac{1}{2}\sigma^2 V^2 F''(v) + [\mu(V) + \beta\sigma(V)]\, F'(V) \\ & - (r_t - \alpha)\, F(V) - c\, r_t = 0 \end{aligned} \tag{10}$$

with $F(0) = -(r_t/r)c$.
And by equation (11) for the stopping zone (V< V*).

$$(V-c)/(1 - e^{-rt_n}) - F(V) = 0 \tag{11}$$

The solution of this ordinary differential equation under the initial condition for a given capitalized interest r_t is given in (12), with ψ(V) the solution of the homogenous part and $[r_t/r]c$ the particular solution of equation (10).

$$F^F(V, r_t) = \begin{cases} A\Psi(V) - [r_t/r]c & V < V^* \\ (V - c)/(1 - e^{-rt}) & V \geq V^* \end{cases} \tag{12}$$

In this case the smooth pasting condition for each parameter r_t is given by:
$A\Psi(V^*) - (r_t/r)\, c = (V^*-c)/(1-e^{-rt}) = (r_t/r)(V^*-c)$ and $A\psi'(V^*) = r_t/r$. So V^*_t must fulfill a similar smooth-pasting condition to the Vicksell model for each parameter r_t.

$$\Psi(V^*) = V^*\, \psi'(V^*) \tag{13}$$

This solution series is then optimized for the Faustmann functional objective equation (7) under metric Q by inspection of its values.

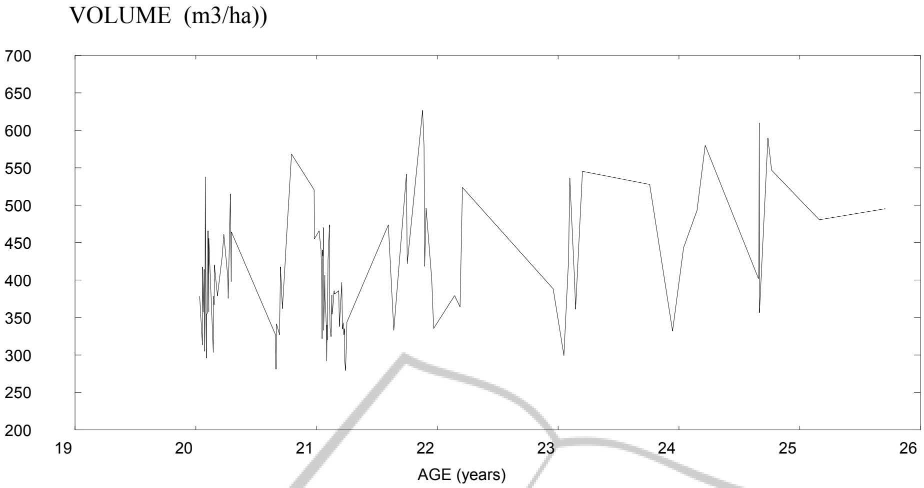

Figure 3.1: VOLUME (m3/ha) versus years, single plot.

3 EXPERIMENTAL DATA AND PARAMETERS FITTING

3.1 Logistic Diffusion Fitting

The experimental data was provided by a Chilean forest company. These data belongs to 128 harvest stock of its pine stocks stands between 1999 and 2005 and came from different sample plots which belong to site indexes between 30 and 35 meters and represent sites with high forest aptitude and a tree average initial volume of 32 m3/ha after the end of the first 4 years initial seed cultivation period. This information is located outside the 95% range of confidence for the logistic adjusted figure, forming an initial series of 122 data points, which are plotted in figure 3.1.

$$dV = \mu\, V\, (1- \gamma\, V)\, dt \; + \sigma\, Vdw \qquad (14)$$

The basic requirement of a pine stand growing diffusion is its sigmoid pattern (Garcia, 2005). The logistic diffusion, equation (14) is a special case of the sigmoid model given by $\mu(V) = \mu V(1\text{-}\gamma V)$ and $\sigma(V) = \sigma\, V$, where μ and γ are the drift and saturation parameters and σ is the volatility parameter.

$$V_t = \frac{V_0 Exp[(\mu - \frac{\sigma^2}{2})t + \sigma W}{1 + \mu\gamma V_0 \int_{t_m}^{t} [Exp[(\mu - \frac{\sigma^2}{2})s + \sigma W]ds} \qquad (15)$$

The integration of the value of V is given by equation (15) (Kloeden& Platen, 1991, page 125) and its expected value is given by equation (16)

$$E(V_t) = \frac{1/\gamma}{1 + e^{-\mu(t - t_m)}} \qquad (16)$$

With; $1/\gamma$ = saturation volume, μ = growth rate parameter = L_n (81)/Δ_t, Δ_t = time necessary to increase volume from 10% to 90% of saturated volume and t_m= time to achieve the midpoint of the saturation volume.

The standard deviation $Sd(\infty)$ at the saturation zone is constant and σ can be easily estimated by equation (17).

$$\sigma = Sd(\infty)/V_s$$
$$= (95\% \text{ saturation confidence interval})/(2 * 1.96 * Vs) \qquad (17)$$

The logistic diffusion model was fitted using a logistical nonlinear regression and a Monte Carlo/Bootstrap simulation sampling method, implemented by Meyer et al. (Loglet Lab.1 software, 1999).

The result is presented in figure 3.2, showing the drift parameter and its 95% confidence interval for the whole series and for its saturation zone. The summary of the parameter fitting is shown in table 3.1.

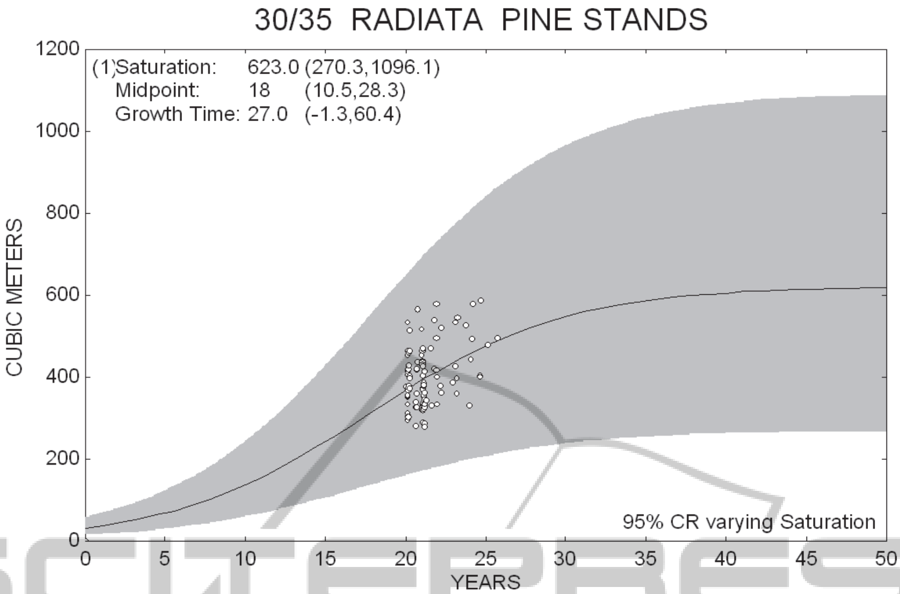

Figure 3.2: Expected logistic fitting for the stands 30/35 site.

Table 3.1: Logistic fitting parameters.

Site Index mts.	Drift Parameter μ	Drift Saturation Parameter γ	Saturation Standard Deviation	Volatility σ
30/35	0.163	0.00161	210.66	0.339
Deterministic	0.163	0.00161	0.00	0.00

See Navarrete 2011

3.2 Gompertz Diffusion Fitting

Another important sigmoid diffusion is the Gompertz geometrical diffusion, which is given by the following equation (18)

$$dV = kV[\theta - \ln(V)]dt + \sigma V dW \qquad (18)$$

This equation is integrated to the following expression, (see Gutierrez 2009)

$$V(t) = \exp[\ \ln(V_0)e^{-kt} + \{(k\theta - \sigma^2/2)/k\}\ (1 - e^{-kt}) + \sigma e^{-kt} \int dW] \qquad (19)$$

The expected value takes the following expression

$$E[V(t)] = \exp\ [\ln(V_0)e^{-kt} + \{(\theta - \sigma^2/(2k)\}(1 - e^{-kt}) + (\sigma^2/(4k))(1 - e^{-2kt})] \qquad (20)$$

Taking natural logarithm and rearranging it, we get

$$\ln E[V(t)] = A - Bx - Cx^2 \qquad (21)$$

with $A = \theta - \sigma^2/(4k)$, $B = \theta - \sigma^2/(2k) - \ln(V_0)$, $C = \sigma^2/(4k)$ and $x = e^{-kt}$

Given a value for k, a quadratic fitting for e^{-kt} and e^{-2kt} can be done estimating the value of A, B and C until a common value for θ can be obtained from A and B, determining the estimation for θ, k and σ. The deterministic parameter only requires a linear fitting with e^{-kt}. Both fittings were done for the initial value V_0 = 32 (m3/ha.) and the results are summarized in table 3.2.

Table 3.2: Gompertz Diffusions Parameters Estimations.

Models	V_0	V_s	k	θ	σ
Gompertz	32.00	1046.68	0.058	7.0	0.171
Deterministic	32.00	1083.09	0.058	6.984	0.000

3.3 Gompertz versus Logistical Diffusion Fitting

The expected drift growing pattern of the Logistic and Gompertz Wood Stock diffusion are shown in figure 3.3.

Both models have fitting advantages and disadvantages.

The Logistic model is a better representation of the sigmoid growth pattern of the tree stands and produces a more reliable estimation of the saturation zone. Unfortunately they cannot be adjusted by maximum likelihood estimates, and must be adjusted by a Bootstrap simulating sampling method, see (Beskos et al., 2006), similar as the one used.

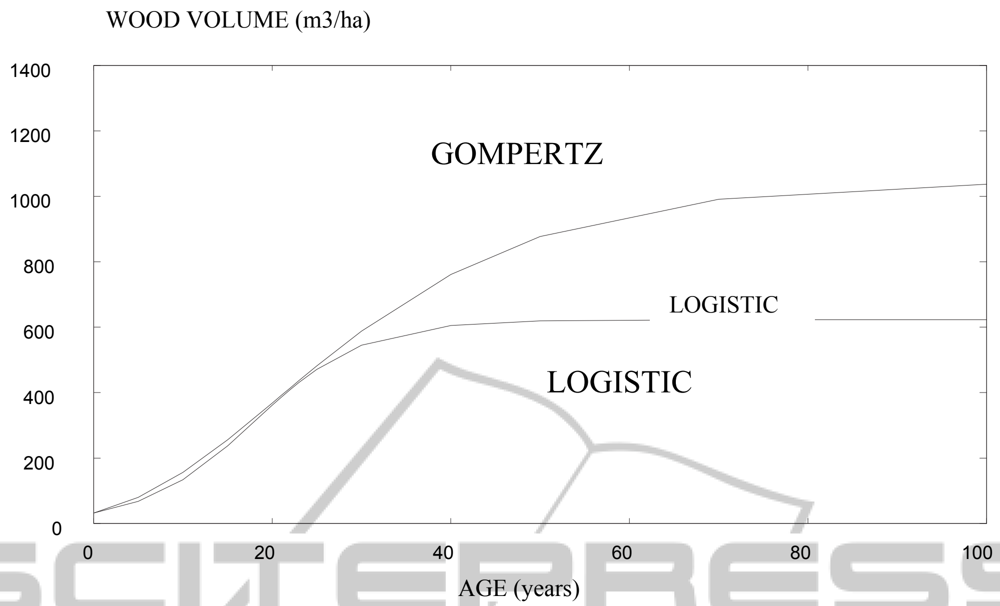

Figure 3.3: Wood Stock growing diffusion pattern.

The Gompertz model can be fitted by common statically features, such as Maximum likelihood. See Gutierrez, et al, (2008) or a Quadratic fitting, which were the methods used. It also presents a better adjustment to the experimental data given its lower volatility parameter, but it produces a worst estimation equal to 1046.6 m3/ha of the saturation zone for the tree growth stands, which was not validated by the experimental data used. Given its higher initial estimation of the growth parameter and its lower volatility parameter it will always produce a lower stochastic optimal solution than the Logistic model.

3.4 Wood Price Diffusion Fitting

The stumpage stands price Brown diffusion parameters were estimated by Navarrete, (2011). The summary of Brown diffusion parameters for the pulp commercial and stumpage prices is given in Table 3.3 and for actual stumpage price in table 3.4.

Table 3.3: Stumpage Price Diffusion Parameters.

Summary	Stumpage logs	Saw logs	Pulp logs
Percentage	100 %	83.9	16.1
Price drift α	2.9%	3.08	1.79
Volatility β	15.9%	16.52	12.74

See Navarrete 2011

Table 3.4: Stumpage Actual Price Estimation.

YEARS	Saw log price (83.9%) US$/m3	Pulp log price (16.1%) US$/m3	Stumpage log price (100%) US$/m3
2007	43	20	39.30
2008	46	22	42.14
2009	41	21	37.78
Average			39.74

Source: IFOP Anuario Forestal 2010

The regeneration costs of Radiata Pine Stands in 2009 are given in table 3.5

Table 3.5: Radiata Pine Stands Regeneration Cost.

Stands regeneration cost C	US$/ha	882
Actual stumpage log price P_T	US$/ha	39.74
Initial stumpage price P_0	US$/ha	21.43
$c=C/P_0$		41.16

Source: CEFOR-UACH

3.5 Capital Cost Estimation

The capital cost is estimated by using the CAPM model for the Chilean Forest industrial sector. The risky rate of return "r" was estimated as the international Weight Average Cost of Capital WACC given the high volatility of actual financial markets.

Chilean equity capital cost $K_e = R_f + \beta(E(R_m) - R_f) = 3.3+(1.01)\ 6 = 9.4$

Chilean Company WACC= 0.76 (12.4) +0.24(7.6) (1-0.17) = 11%

International Company WACC = r = 11.8 ~12%

Source: CMPC Corp Search June 2009

4 STOCHASTIC RADIATA PINE HARVESTING RESULTS

4.1 Wood Stock Logistic and Brown Price Diffusion

Two of the more common sigmoid diffusion processes used in this area (see Garcia, 2005) are the Gompertz and the Logistic geometric diffusion. The logistic geometric diffusion wood stock diffusion parameters are; $\mu(V) = \mu V\ (1- \gamma V)$ and $\sigma_V = \sigma\ V$.

The Faustmann deterministic optimum is given by the optimization of the deterministic functional objective in equation (22).

$$\frac{(P_t V_t)'}{P_t V_t - C} = \frac{r}{(1-e^{-rt})} = r_t \qquad (22)$$

Replacing it in the equation (22) $P_t = P_o e^{\alpha t}$ and $V_t' = \mu V_t(1\text{-}\gamma V_t)$ we finally obtain

$$V_t = \{(\alpha+\mu-r_t) + \sqrt{[(\alpha+\mu-r_t)^2 + 4\mu\gamma c r_t\, e^{-\alpha t}]}\} / (2\mu\gamma) \qquad (23)$$

In the stochastic case, the positive function ψ (V), is the solution of the homogenous component (24) of the differential equation (10), (see Navarrete 2011)

$$\tfrac{1}{2}\, \sigma^2 V^2 F''(v) + [\mu V(1\text{-}\gamma V) + \beta\, \sigma V]\, F'(V) - (r_t\text{-}\alpha)\, F(V) = 0 \qquad (24)$$

The solution of equation (24) is given by the Kummer expression (25)

$$\psi(V) = V^{\theta}\ \mathit{KummerM}\ \{\frac{2\mu\gamma V}{\sigma^2},\ \theta,\ 2\theta + \frac{2(\mu+\beta\sigma)}{\sigma^2}\} \qquad (25)$$

with θ the positive root is given by equation (26)

$$\theta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \frac{\beta}{\sigma} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2} - \frac{\beta}{\sigma})^2 + \frac{2(r-\alpha)}{\sigma^2}} \qquad (26)$$

The Faustmann deterministic optimum is obtained by intersecting curve (24) with the logistic curve (16). The solution is programmed in Maple 15 for both curves, and the optimum obtained is $V^* = 245.7\ m^3/h$. The Faustmann stochastic solution, in (13), for different values of the capitalized interest $r_t = r/\ (1\text{-}e^{-rt})$ is programmed in Maple 15, using its *KummerM* function. The optimum is obtained by evaluating the Faustmann functional objective under the Q metric (20) for the different V^*_t solutions of equation (13). The summary of all optimal cuts results for the aggregate 30/35 site index series of the multiple rotation harvest or Faustmann formula is given in table 4.1.

These results show that the Faustmann deterministic optimum underestimates the actual policy cut by 37.47% and its stochastic optimum also underestimates the actual average cut by 8.09%, so that the Stochastic optimum is 47.0 % bigger than the deterministic value.

Table 4.1: Multiple Harvest Rotation Optimal Results Wood stock Logistic diffusion.

Optimum	Stands cuts m^3/ha	Percentage Increase %	Percentage Increase %
Deterministic	245.7	-37.47	100
Actual	392.9	100	
Stochastic	361.12	-8.09	47.0

4.2 Wood Stock Gompertz and Brown Price Diffusion

In this case the parameters of the diffusion are: $\mu(V) = k\ V\ (\theta\text{-}\ \ln(V))$, and $\sigma(V) = \sigma\ V$. The deterministic optimum is obtained by replacing $P_t = P_0\ e^{\alpha t}$ and$V = \exp\ (\ln(V_0)e^{-kT} + \theta(1\text{-}e^{-kT})$ in equation (22) resulting equation (28). Which intersection with equation (20) gives the optimal volume V^{opt}.

$$V = \{r_t c e^{-\alpha t} + k(\theta - \ln(V_0))\ \exp(\theta - kT - (\theta - \ln(V_0))e^{-kT}\} / \{r_t - \alpha\} \qquad (27)$$

The stochastic increasing function $\psi(V)$, in this case, is given by the solution of the homogenous part of the differential equation (10) or equation (28).

$$\tfrac{1}{2}\, \sigma^2 V^2 F''(v) + [kV(\theta - \ln(V)) + \beta\, \sigma V]\ F'(V) - (r_t - \alpha)\ F(V) = 0 \qquad (28)$$

Choosing $\theta' = \theta - \sigma^2/(2k) + \beta\sigma/k$ and $r = r_t - \alpha$, the equation is similar to the exponential Ornstein Ulhembeck equation whose positive solution $\psi(V)$ is given by equation (29), (see Johnson, 2005)

$$\psi(V) = \{ \begin{matrix} \{\Gamma(a+1-b)/\Gamma(1-b)\}\mathit{KummerU}(a,b,z) & V \le e^{\theta} \\ \mathit{KummerM}(a,b,z) & V \ge e^{\theta} \end{matrix} \qquad (29)$$

with $a = (r_t - \alpha)/(2k)$ $b = 0.5$ and $z = (k/\sigma^2)\ [\theta - \sigma^2/(2k) + \beta\sigma/k\text{-}\ln(V)]^2$.

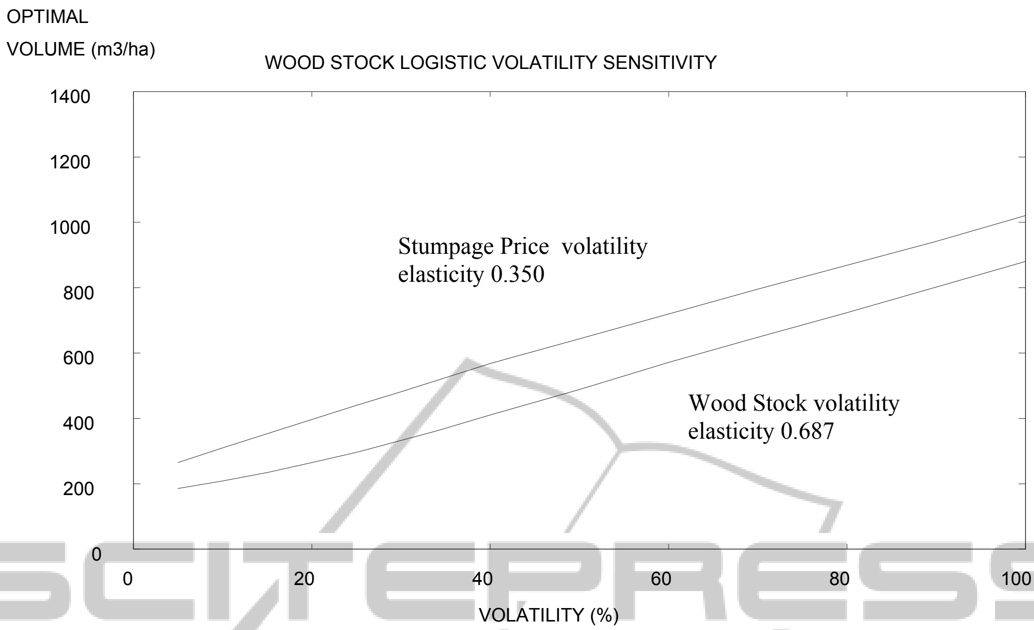

Figure 4.4: Faustmann stochastic optimal cut sensitivity.

The deterministic and stochastic optimum were programed in Maple 15, using in this case the *KummerU* function of the program, the results are summarized in table 4.2.

Table 4.2: Multiple Harvest Rotation Optimal Results Wood Stock Gompertz diffusion.

Optimum	Stands cuts m^3/ha	Percentage Increase %	Percentage Increase %
Deterministic	182,6	-53.45	100
Actual	392.9	100	
Gompertz	211.13	-46.26	15.62

4.3 Growing Pattern of the Wood Stock Logistic Diffusion Process

The sensitivity of the Faustmann model shows similar effects for both volatilities in the optimal cut, being the Inventory elasticity lower than the stumpage price elasticity. Obviously, this is due to the higher volatility of inventory 33.9% over price 15.9%.

4.4 Summary

Table 4.3 shows the summary of the results.

Table 4.3: Results summary.

Optimum	Logistic Diffusion %	Gompertz Diffusion %
Deterministic Optimum	-37.47	-53.45
Actual Policy	100	100
Stochastic Optimum	-8.09	-46.26
Stochastic optimum increment	47.0	15.62

5 CONCLUSIONS

The effects of the wood Stock and price stochastic diffusion processes are important for the optimal cut. The Logistic diffusion increases the deterministic optimum by 47.0%, and the Gompertz diffusion by 15.62%. The difference is due to the lower volatility estimation of the Gompertz model.

The deterministic optimums in both cases significantly underestimate the company actual average cut, and the stochastic optima, being higher, also underestimate the Company actual average. The discrepancy in the theoretical and practical cut policy can be explained by the absence of consideration that the Company gives to the Faustmann model and the stochastic behavior of price and wood stock.

The experimental data significantly validate the Faustmann stochastic logistic model. They give a better approximation of the company cut policy (-

8.09%) and produce a more reliable saturation volume than the Gompertz model.

The sensitivity analysis of both volatilities of the Logistic models shows similar linear relations with the stochastic optimal cut. The wood stock volatility elasticity of 0.687 almost double the stumpage price volatility elasticity of 0.350 due to its lower actual volatility.

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APPENDIX A

Proof of Lemma 1

Theorem 1: A probabilistic measure Q exists and is equivalent to the actual metric R, such that it is proven (see, Jacco J.J. Thijssen, 2010)

$$W^{V}(V_0,P_0) = \frac{\sup}{\forall(t \geq t_o)} E^{R}(e^{-rt}P_tV_t)/(1-e^{-rt})$$
$$= P_0 \sup\{E^{Q}(e^{-(r-\alpha)t}V_t)/(1-e^{-rt})\} \quad \text{(A1)}$$

Furthermore, under the metric Q, the process Vt follows the diffusion (A2)

$$dV_t = \{\mu V)_t + \beta\sigma(V)_t\}dt + \sigma(V)_t d\overline{W} \quad \text{(A2)}$$

Proof.

Replacing the integral solution of (2) in this last expression (A1), Pt = P0 eαt exp {βWt - 1/2β2t], since Mt = exp {βWt - 1/2β2t] is a martingale, a new metric Q (dQ/dR = Mt) can be defined via the Radon-Nikodym derivative. Considering that, in this case, β is positive, a straightforward application of Girsanov´s theorems I and II (Oksendal, 2000, pages155-157) yields the equivalent objective for metric Q, and the ITO diffusion (A2)