Lagrangian Road Pricing

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Abstract: We consider the problem of trajectory-based road pricing with the objective of reducing congestion on a road network. It is well-known that traffic conditions resulting from typical non-cooperative behavior of selfish drivers do not minimize total travel time spent on the road network. In the context of real-time GPS data collection from all vehicles, drivers can be charged differently based on their origin and destination, and according to the path they take from that origin to that destination. In this work, we propose a new formulation of the set of multi-commodity prices based on a price potential, and describe an efficient algorithm to construct such multi-commodity prices. We provide an analysis of the subset of valid prices satisfying several specific user-driven constraints. The numerical performances of the method proposed are assessed on a benchmark network, and the social benefits resulting from the commodity-based potential pricing scheme introduced in this article are discussed.

1 INTRODUCTION

Congestion pricing, which dates back to 1969 with the work of (Vickrey, 1969), is motivated by the fundamental difference existing between the natural state of traffic, typically assumed to be a *user equilibrium* or *Wardrop equilibrium*, and the best possible allocation of traffic, or "social optimum". The difference between these two states of traffic has been called the "price of anarchy" (Papadimitriou, 2001). A user equilibrium can be arbitrarily far from the social optimum, as soon as instances of Braess paradox (Braess, 1968) exist in the network.

Pricing schemes aim at designing a tolled user equilibrium coinciding with the social optimum of the network. A basic solution consists of "internalizing externalities" by charging users with the marginal costs they occur to the network (Pigou, 1920). However, more specific operator or user-driven constraints such as profit maximization, or fairness guarantees, have historically motivated the search for other pricing schemes.

The recent improvement in real-time positioning capabilities using GPS devices (Mobile Millenium, 2008) creates new possibilities and challenges for congestion pricing. Charging users based on their path properties, and not only at sparse locations on the road network, is now technically feasible.

In this article, we consider the problem of characterizing feasible Lagrangian (trajectory-based) toll sets achieving a given flow allocation on the network. We allow different prices for different origin - destination pairs (multi-commodity pricing), and provide an extension of the toll set formulation first proposed by (Hearn and Ramana, 1998), as the translation of the set of prices generated by a potential on the nodes of the network. In particular, we show that the set of feasible prices obtained for trajectory-based pricing is convex (Boyd and Vandenberghe, 2004).

We provide an algorithm to construct a pricing scheme for different strategies (Hearn and Ramana, 1998) and assess the performance of our pricing schemes against benchmark pricing algorithms. Numerical simulations and performance analysis are performed in Python using a convex optimization library (Dahl and Vandenberghe, 2008).

The rest of the article is organized as follows. In section 2, we introduce notations. In section 3, we present our main results, on the characterization and properties of a commodity-based potential pricing scheme. Section 4 presents a numerical study of the problem. Finally, section 5 provides concluding remarks and discusses extensions to this work.

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2 PRELIMINARIES

In this section, we introduce the notations and preliminary results leading to our problem formulation and the main results of the following section.

2.1 Notations

Let graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ represent the road network, and K denote the set of *origin-destination (OD) pairs* (if $k \in K$, $k = (p,q), p,q \in \mathcal{N}$), and $(d_k)_{k \in K}$ the travel demand for pair k. Several representations of the traffic flow on the network can be used.

Path Flows. For $k = (p,q) \in K$, let $\mathcal{R}_k$ denote the set of directed acyclic paths connecting the origin p with the destination q, and let $\mathcal{R} := \cup_{k \in K} \mathcal{R}_k$. $h = (h_r)_{r \in \mathcal{R}}$ is the path flow. h is feasible if it is positive and satisfies demand. H is the set of feasible path flows.

$$\sum_{r \in \mathcal{R}_k} h_r = d_k \ \forall k \in K \text{ and } 0 \leq h_r \ \forall r \in \mathcal{R}. \quad (1)$$

If we note Γ the OD-paths incidence matrix, then equation (1) can be written as :

$$\Gamma^{\mathsf{T}} h = d \text{ and } 0 \leq h.$$

Let $h \in H$ be a feasible flow. We note $\mathcal{R}^{k,h-\text{eff}} = \{r \in \mathcal{R}_k | h_r > 0\}$, and $\mathcal{R}^{h-\text{eff}} = \cup_{k \in K} \mathcal{R}^{k,h-\text{eff}}$ the set of paths for which the flow h is positive.
We note $\mathcal{R}_{i,j}$ the set of paths from i to j.

Commodity Arc Flows. Let w_a^k be the flow of vehicles from OD pair $k = (p,q)$ going through arc a, and w^k the vector of arc flows for OD pair k. In this context, k is a commodity and w^k an OD-flow or commodity-flow. $w_a^k = \sum_{r \in \mathcal{R}_k, a \in r} h_r$.

We note E the node-arc incidence matrix, and for $k = (p,q)$, $i^k \in \Re^{|\mathcal{N}|}$ the node-OD incidence vector, *i.e.* the vector such that for $n \in \mathcal{N}$, $i_n^k = +1$ if $n = q$, $i_n^k = -1$ if $n = p$ and 0 otherwise. w is feasible if and only if:

$$E w^k = d_k i^k \ \forall k \in K \text{ and } 0 \leq w^k \ \forall k \in K. \quad (2)$$

W is the set of feasible commodity flows.

We denote by $\tilde{E}$ the matrix $\begin{pmatrix} E & 0 & .. & 0 \\ 0 & E & .. & 0 \\ & & ... & \\ 0 & 0 & .. & E \end{pmatrix}$ such that $\tilde{E} w = (d_k i_k)_{k \in K}$.

Arc Flows. An aggregate flow $f = (f_a)_{a \in A}$ is defined on each arc a as the total traffic flow on each arc of the network. It can be computed from h or w:

$$f_a = \sum_{r \in \mathcal{R}, a \in r} h_r \qquad \forall a \in \mathcal{A} \quad \text{or } f = \Lambda h \quad (3)$$

$$f_a = \sum_k w_a^k \qquad \forall a \in \mathcal{A} \quad \text{or } f = \sum_k w^k \quad (4)$$

where Λ is the arc-path incidence matrix $(\Lambda)_{a,r} = \delta_{a \in r}$.

A flow f is *feasible* if it is the sum of feasible commodity flows or of a feasible path flow. F is the set of feasible flows f.

We call the three flows f, $w^k, k \in K$, h *equivalent* if f and w can be computed from h and if one of the three given flows is feasible.

Finally, we use the generic notation $v = (v_i)_{i \in I}$ to denote the arc, path or commodity formulation. We denote by V the set of feasible flows v, $n = |I|$ the number of traces on which v is defined, A the matrix of linear constraints ($\tilde{E}$ if $v = w$ and Γ^{T} if $v = h$) and p the number of rows of A (resp. $|\mathcal{N}|$ and $|K|$).

If v denotes a flow with only one OD pair, $\mathcal{R}_{i,j}^{v-\text{eff}} = \{r \in \mathcal{R}_{i,j} | h_r > 0\}$ where h is the path flow associated to v.

2.2 Tolled User Equilibrium

A standard approach to congestion pricing consists in creating a tolled user equilibrium (UE), coinciding with the social optimum. In this section we assume that all users have the same value of time, so that in a pure user equilibrium (UE), enforcing a toll at a road of the network is equivalent to adding a certain delay to the experienced travel time on the corresponding arc of the graph. Let ρ be the vector of prices (expressed as a delay) on every element of the flow representation, $v \in V$ the corresponding flow vector, and $l(v)$ the latency function, the effective travel cost experienced by a user is $l(v) + \rho$.

A tolled UE v^* is solution of the following convex optimization problem, extension of Beckmann's formula (Beckmann et al., 1956):

$$\min_{f \geq 0, v \in V} \sum_a \int_0^{f_a} l_a(x) \mathrm{d}x + \rho^{\mathsf{T}} v.$$

where $l_a(\cdot)$ is the arc latency function, assumed non-decreasing. The tolled UE can also be expressed as a variational inequality problem (VIP), (Dafermos, 1980), as:

$$\forall v \in V, \ \ (l(v^*) + \rho)^{\mathsf{T}} (v - v^*) \geq 0. \quad (5)$$

Remark. *When the latency functions are strictly monotonic, the solution v to the tolled UE problem is unique and the set of valid flow vectors is a singleton.*

Assuming that the latency functions are strictly monotonic, we define the *toll problem*:

$$\text{Find } v^*, \rho \text{ such that: } \begin{cases} v^* \text{is a solution of } \min_{v\in V} v^\top l(v) \\ \forall v \in V,\ (l(v^*)+\rho)^\top (v-v^*) \geq 0. \end{cases}$$

3 LAGRANGIAN PRICING SCHEMES AND POTENTIAL PRICING

In this section, we present our results on the mathematical theory of commodity-based potential pricing. Our main result states that, for a given flow allocation, the set of valid multi-commodity tolls (such that the tolled user equilibrium corresponds to this flow) can be expressed as the sum of three terms: 1 - the opposite of travel time on the arcs, 2 - any potential field defined on the nodes of the network for each commodity (or, more precisely, the potential difference between the end node and the start node of the arc), 3 - any positive toll on the arcs where the flow vanishes (Theorem 3.3).

3.1 Set of Feasible Prices

We denote by w^* the commodity flow at which we want to stabilize the tolled user equilibrium. We use the characterization of prices introduced in equation (5):

$$Q(w^*) = \{\rho \geq 0 \mid (l(w^*)+\rho)^\top (w-w^*) \geq 0\ \forall w \in W\}.$$

The two following results are due to (Hearn and Ramana, 1998):

Lemma 3.1. *$-l(f^*)$ is a valid pricing to reach the flow w^*; $-l(f^*) \in Q(w^*)$.*

Lemma 3.2. *The toll set $Q(w^*)$ is a shifted polyhedral cone: if f is the arc flow corresponding to w,*

$$Q(w^*) = -l(f^*) + N(w^*, W), \tag{6}$$

where $N(w^*,W) = \{u \geq 0 \mid u^\top (w-w^*) \geq 0\ \forall w \in W\}$.

Definition (Potential pricing)**.** $p \in \Re^{|\mathcal{A}|}$ is a *potential pricing* if there exists a vector $\pi \in \Re^{|\mathcal{N}|}$ defined on the nodes of the network such that $p = E^\top \pi$, with E the node-arc incidence matrix defined in section 2.1. π is the *price potential*. For $a = (i, j)$ an arc joining the nodes i and j, $p_a = \pi_j - \pi_i$.
p is a *shifted potential pricing* associated with v if p is the sum of a potential pricing $E\pi$, of the opposite of the travel times on the arcs $-l(v)$ and of positive scalars μ_a on the arcs for which $v_a = 0$.

We extend this definition to the case of multicommodity arcs, $p = \tilde{E}^\top \pi$ is a potential pricing composed of the commodity potential pricings $E^\top \pi^k$. The total tolls levied between two nodes i and j for commodity k are $\pi^k_j - \pi^k_i$: tolls do not depend on the path taken from i to j. With $k = (p,q)$, we define $\Lambda_k = \pi^k_q - \pi^k_p$ the sum along a path of $\mathcal{R}^k$ of the commodity arc potential pricing. It does not depend on the path chosen.

Theorem 3.3. *The toll set $Q(w^*)$ is the set of all shifted potential pricings:*

$$Q(w^*) = -l(f^*) + \mathrm{Im}(\tilde{E}^\top) + \mathrm{vect}^+(e^k_a,\ k,a\ \ s.t.\ w^k_a = 0)$$
$$Q(w^*) = \{(-l_a(f^*_a) + (E^\top \pi^k)_a + \mu^k_a \delta_{w^k_a=0})_{k\in K, a\in\mathcal{A}} \mid \mu^a_k \geq 0\ \forall k \in K,\ a \in \mathcal{A}\}.$$

Proof. An expression of $Q(w)$ is given in equation (6). $Q + l(f)$ is the intersection of an affine set with the set of positive flows : $Q + l(f) = \{w \mid \tilde{E}w = \tilde{d}\} \cap (\Re^+)^{|K|.|\mathcal{A}|}$.

If a, k are such that $(w^*)^k_a > 0$, then it is not possible to find $\lambda \neq 0$ such that $\lambda(w^k_a - (w^*)^k_a) > 0$ for every $w \in W$, as it is possible to choose $w \in W$ such that $w^k_a > (w^*)^k_a$ and $u \in W$ such that $u^k_a < (w^*)^k_a$. Hence, if $w^* > 0$, then $\{u \geq 0 \mid (w-w^*)^\top (u) = 0\ \ \forall w \in W\} = \{u \geq 0 \mid w^\top u = 0\ \ \forall w, \tilde{E}w = 0\} = \ker(\tilde{E})^\perp$. Hence, $N(w^*, W) = (\ker(\tilde{E})^\perp)^+$. $N(w^*,W)$ is the restriction to positive vectors of a vector space of dimension $(|K|.|\mathcal{A}| - \dim(\ker \tilde{E}))$.

Let $I_0 = \{(a,k) \mid w^k_a = 0\}$ and suppose I_0 is not empty. Then the space of solutions is larger. More precisely, for $i \in I_0$, $\forall w \in W, w_i - w^*_i \geq 0$. Hence, every $u_i \geq 0$ is a valid toll for trace i. Then $N(w^*,W) = \ker(\tilde{E})^\perp + \{\sum \mu_i e_i \mid i \in I_0, \lambda_i \geq 0\}$. As w^*_i reaches a physical limit $w^*_i = 0$, the control operated through prices is relaxed and the i^{th}-component of the valid prices only has to be positive.

We now have $Q(w^*) = -l(w^*) + \ker(\tilde{E})^\perp + \mathrm{vect}^+(e^k_a, (a,k) \in I_0)$. A well known result of linear algebra states that $\ker(\tilde{E})^\perp = \mathrm{Im}(\tilde{E}^\top) = \{\tilde{E}^\top \pi, \pi \in \Re^{|\mathcal{N}|.|K|}\}$. □

This theorem provides new alternative tolls for pricing users in order to reach the desired flow w^*. Additionally, those tolls depend of the values of the flow in the network only through $-l(f)$, hence provide greater robustness to measurement error. Finally, the set of valid tolls set is simply expressed as an unconstrained vector space.

In the following section, we extend these results to the case of positive pricing in $Q(v^*)$. We also show that this pricing technique allows one to stabilize the network allocation around flows that are not stabilizable using a traditional positive arc pricing.

3.2 Positive Prices

The objective of this section is to determine under which conditions we can find a pricing scheme such that a user of the network is never charged a negative toll.

Definition (Acyclic Flow)**.** Let v be a flow defined on the arcs of a network. v is *acyclic* if v is not positive on all the arcs of a directed cycle of the graph $\mathcal{G}$. It is equivalent to saying that the subgraph $\mathcal{G}_v$ of $\mathcal{G}$ consisting of all arcs for which v is positive is acyclic. In the following we say that w is acyclic if the related commodity flows w^k are acyclic.

Algorithm 1 (Construction of a Potential Pricing $\sigma(v)$)**.** *Let v be an acyclic arc flow and $\mathcal{G}_v$ the associated subgraph, hence acyclic. l_a, $a \in \mathcal{A}$ are the latencies on the arcs of the network for flow v. $\mathcal{N}_v$ is the set of nodes of $\mathcal{G}_v$ with the partial order $\prec$ such that $i \prec j$ if there exists a directed path from i to j in $\mathcal{G}_v$.*
We note $i = \mathrm{prec}(j)$ if there exists an arc $a = i \rightarrow j$.
We define $\sigma(v)$ through the following steps:

1. *Let $E = \emptyset$.*
2. *Choose i a minimal element of $\mathcal{N}_v \setminus E$*
3. *Define $\pi_i = \max\{0, \{\pi_j + l_a \mid j = \mathrm{prec}(i), a = j \rightarrow i\}\}$*
4. *Update $E = E \cup \{i\}$.*
5. *If $\mathcal{N}_v \setminus E$ is not empty go to step 2.*
6. *For $a \in \mathcal{G}_v$, $\mu_a = 0$.*
7. *For $a \in \mathcal{G} \setminus \mathcal{G}_v$, $a = (i, j)$, $\mu_a = \pi_i - \pi_j + l_a$.*
8. *$\sigma(v) = E^{\mathsf{T}}\pi + \mu$.*

As $\mathcal{N}_v$ is partially ordered, the iteration in this algorithm terminates, when $E = \mathcal{N}_v$.
We have the following invariant: $\forall i \in E, \pi_i = \max_{r \in \mathcal{R}_{p,i}^{v-\mathrm{eff}}} l_r(v)$.

Theorem 3.4. *Let v be a positive flow defined on the arcs, there exists a potential pricing in $Q(v)$ strictly positive everywhere if and only if v is acyclic.*

Proof. Let $\varepsilon > 0$. We modify $\sigma(v)$ defined in Algorithm 1 by modifying step 3: $\pi_i = \max\{\varepsilon, \{\pi_j + l_a \mid j = \mathrm{prec}(i), a = i \rightarrow j\}\}$ and step 7: $\mu_a = \pi_i - \pi_j + l_a + \varepsilon$. For $a \in \mathcal{G}_v$, $\sigma(v)_a = \pi_j - \pi_i > 0$ because $a \in \mathcal{G}_v \Rightarrow i \prec j$. For $a \in \mathcal{G} \setminus \mathcal{G}_v$, $\sigma(v) \geq \varepsilon > 0$.

Now suppose that there exists a potential pricing $E\pi + \mu$ in $Q(v)$ strictly positive everywhere, and suppose $\mathcal{G}_v$ contains a directed cycle $i_0 \rightarrow .. \rightarrow i_k \rightarrow i_0$. Then μ vanishes on the arcs of this directed cycle (as it belongs to $\mathcal{G}_v$) and π is such that $\pi_0 < .. < \pi_k < \pi_0$ which is absurd. Hence, if there exist a potential pricing strictly positive everywhere, $\mathcal{G}_v$ is acyclic and the same holds for v. ☐

Corollary 3.5. *Let v be a feasible flow defined on the arcs of the network, acyclic, then there exists a positive shifted potential pricing in $Q(v)$.*

Proof. Let $\sigma(v)$ be the potential pricing defined in Algorithm 1. $\tilde{s}$ defined as $\tilde{s} = -l(v) + \sigma(v)$ is a positive shifted potential pricing in $Q(v)$. ☐

Corollary 3.6. *If w^{SO} is a multi-commodity flow solution of a SO problem, then the set of positive commodity toll vectors leading to w^{SO} is given by $Q(w^{\mathrm{SO}})^+ = Q(w^{\mathrm{SO}}) \cap \Re^+$ and $Q(w^{\mathrm{SO}})^+ \neq \emptyset$.*

Proof. As for each commodity k, w^k is acyclic, corollary 3.5 prooves that the set is non empty. Characterization of the set comes from theorem 3.3. ☐

Algorithm 2 (Construction of Multicommodity Pricing Vector $s(w)$)**.** *Let w be an acyclic multicommodity flow. For $k \in K$, let $\sigma^k = \sigma(w^k)$ be the potential pricing associated with arc commodity flow w_a^k and arc travel times $l_a(f_a)$. We define $s(w)$: $s(w)_a^k = -l_a(f_a) + \sigma_a^k$.*

Remark. *The potential field associated with $s(w)$ is such that $\pi_i^k \geq \max_{r \in \mathcal{R}_{j,i}^{w^k-\mathrm{eff}}, j \prec i} l_r(v) + \pi_j^k$, $\forall i \in \mathcal{N}$.*

Proposition 3.7. *Let $s(w)$ denote the pricing scheme constructed in Algorithm 2, $s(w)$ is the minimum of $Q(w)^+$: every positive pricing leading to commodity flow w^k is the sum of $s(w)$ and of positive prices on every arc of the network.*

$$Q(w)^+ = s(w) + \{\tilde{E}^{\mathsf{T}}\pi \mid (E^{\mathsf{T}}\pi^k)_a \geq 0 \ \forall a,\ k \text{ s.t. } w_a^k > 0\} + \mathrm{vect}^+(e_a^k,\ k,\ a \text{ s.t. } w_a^k = 0).$$

Proof. Every positive pricing must satisfy the following inequality: $\forall i \in \mathcal{N}, \pi_i^k - \pi_p^k - \max_{r \in \mathcal{R}_{p,i}^{w^k-\mathrm{eff}}} l_r(v) \geq 0$. The case of equality, as for $s(w)$, gives the minimum of the valid positive prices. ☐

Corollary 3.8. *The pricing scheme $s(w)$ is the positive toll vector that realizes the minimum of the total tolls levied.*

Remark (Path Tolls)**.** *The total amount of tolls charged to a user on path r (on which the flow does not vanish at equilibrium) is the sum of the commodity arc tolls charged to him,* i.e. $\Lambda_k - l_r(h)$.

Pricing based on Origins. As the aggregation of all commodity flows having a same origin is still an acyclic flow, it is possible to apply a positive pricing scheme depending only on the origins of the users and not on their destinations. This speeds up the computation, as it decreases the size of the pricing vector.

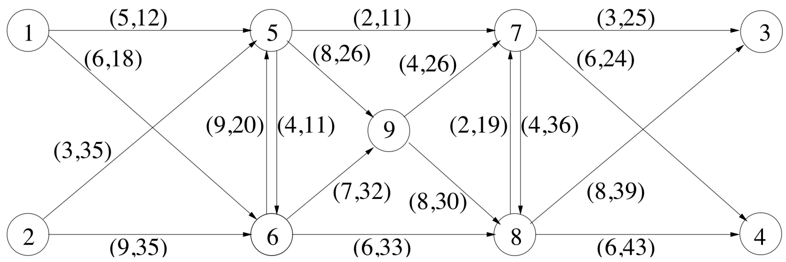

Figure 1: Nine Node Network.

4 NUMERICAL RESULTS

In this section, we present results obtained on a benchmark network. In (Hearn and Ramana, 1998), the authors propose different secondary objectives for a toll problem. The problem MINSYS consists in minimizing the total amount of toll.

In this section, we provide a comparison of our multicommodity MINSYS algorithm, or MMINSYS, introduced in this article, with the MINSYS algorithm, on a synthetic network. We present results for the Nine Node network described in (Hearn and Ramana, 1998), and illustrated in figure 1. The arcs are given "Bureau of Public Roads" (BPR) latency functions. The tuple near an arc denotes its free flow travel time followed by its capacity in the sense of BPR. There are four OD pairs, with four different travel demands.

OD pair:	[1,3]	[1,4]	[2,3]	[2,4]
Demand:	10	20	30	40

Results are presented in table 1. The aggregate flow and commodity flows corresponding to the different OD pairs, at social optimum, are explicited for each arc of the network. Origin based-arc tolls, solution of the algorithm 2, are also listed.

The Λ_k, potential difference between the destination and the origin, are respectively 30.59, 29.21, 32.95, 31.57 for the above OD pairs. It is also an upper bound to the tolls charged to one user of the OD pair. As journeys for the different OD pairs are similar in terms of travel times, the fixed part (which does not depend on the path choosen by the user) is approximately the same for each commodity.

A comparison of general properties of MMINSYS program with other secondary objectives is presented in table 2. The toll vectors are computed as explained in the previous section.

Let us focus on the new formulation MMINSYS compared to the other classical programs.

- MMINSYS solution gives a lower total of tolls than MINSYS. This is mathematically evident, as MINSYS is a problem restriction of MMINSYS. MMINSYS total tolls are 9.5 % lower: allowing multicommodity tolls helps minimizing the total number of tolls raised.
- MMINSYS solution gives also interesting values for the number of tolled arcs or for the maximum arc toll: there are respectively 4 and 5 arcs tolled for origins 0 and 1, which represents 5 arcs tolled to the operator point of view. The maximum arc toll is 8.00 for one commodity, 12.00 for the other, which is greater than MINSYS solution and MINTB solution (identical in this problem).
- The main benefit of this new algorithm is that the pricing vector has an analytical expression and can be computed in a linear time in the product of the number of arcs of the network and the number of different origins of the OD pairs. It does not need numerical solvers. On the contrary, MINSYS is the solution of a linear program with polynomial complexity, not expected to be linear in general.

5 CONCLUSIONS

In this article, we introduce the notion of commodity-based potential pricing in order to design optimal OD-pair differentiated congestion charges, in the context of real-time GPS sensing. Our contributions include the mathematical construction and analysis of commodity-based potential pricing schemes, the design of algorithmic methods for efficient computation of these potentials, and the theoretical and numerical analysis of their properties.

We show that our potential-based formulation provides a new characterization of the set of pricing schemes such that the charge incurred on each arc is positive, whose existence is equivalent to the acyclic property of the commodity flows. The proof of this equivalence result is constructive, and is based on a

Table 1: Nine Node Network - Flows and tolls for each commodity (tolls are identical for OD pairs with same origin).

Arc	From	to	Aggr. flow	Travel time	Pair [1 3]	Pair [1 4]	Tolls	Pair [2 3]	Pair [2 4]	Tolls
0	1	5	9.411	5.283	3.976	5.435	0.	0.000	0.000	0.
1	1	6	20.589	7.540	6.024	14.565	0.	0.000	0.000	0.
2	2	5	38.334	3.648	0.000	0.000	0.	22.185	16.149	0.
3	2	6	31.666	9.905	0.000	0.000	0.	7.815	23.851	0.
4	5	6	0.000	9.000	0.000	0.000	0.	0.000	0.000	0.
5	5	7	21.303	6.220	2.104	3.029	8.	7.569	8.602	12.
6	5	9	26.442	9.283	1.872	2.406	0.	14.617	7.547	4.
7	6	5	0.000	4.000	0.000	0.000	0.	0.000	0.000	0.
8	6	8	39.474	7.843	2.361	11.145	7.2	4.501	21.467	7.2
9	6	9	12.781	7.027	3.664	3.420	0.	3.313	2.384	0.
10	7	3	29.608	3.885	6.042	0.000	7.2	23.566	0.000	7.2
11	7	4	20.757	6.503	0.000	6.390	3.2	0.000	14.367	3.2
12	7	8	0.000	2.000	0.000	0.000	0.	0.000	0.000	0.
13	8	3	10.392	8.007	3.958	0.000	0.	6.434	0.000	0.
14	8	4	39.243	6.625	0.000	13.610	0.	0.000	25.633	0.
15	8	7	0.000	4.000	0.000	0.000	0.	0.000	0.000	0.
16	9	7	29.062	4.937	3.938	3.361	0.	15.997	5.765	0.
17	9	8	10.162	8.015	1.597	2.465	0.	1.933	4.166	0.

Table 2: Nine Node Network - Alternative Tolls.

SO total time	2253.92				
UE total time	2455.84 (8.95% greater)				
Problem solution	MSCP	MINSYS	MINMAX	MINTB	MMINSYS
Total tolls	1493.46	887.57	1167.57	887.57	803.6
Total tolls / SO total time (%)	66.38	39.38	51.80	66.38	35.65
Number of toll booths	14	5	7	5	6
Max. arc toll	16.88	11.20	8.00	11.20	12.00

linear algorithm for the definition of a minimal positive pricing scheme.

Further directions of research include applications of this new formulation to efficient computation of more complex constrained network pricing problems: robustness of pricing schemes around the social optimum, introduction of multi-class users or more sophisticated modeling of users preferences.

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