Enhancing Estimation Skills with GeoGebra
Volume Ratios of Essential Solids

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Abstract: The first part of this article reports the results of a survey focusing on estimation skills relating to the concept of volume. The survey tested pre-service and in-service math teachers of various nationalities and various school types, and investigated their skills in estimating volume ratios of essential solids (a cylinder, a ball, and a cone). The second part of the article analyzes the mathematical background of cone and ball cases. The third part of the article shows the possibilities of GeoGebra software in creating test materials for similar surveys, and — in accordance with the results of the survey — presents dynamic models designed to enhance estimation skills in volume ratios. The text gives detailed instruction on how to create such kind of GeoGebra materials.

1 INTRODUCTION

Estimation is a process whereby one approximates, through rough calculations, the worth, size, or amount of an object or quantity that is present in a given situation. The approximation, or estimate, is a value that is deemed close enough to the exact value or measurement to answer the question being posed (NCES, 1999). The importance of estimation in the school curriculum was acknowledged for instance in the 1986 yearbook of National Council of Teachers of Mathematics, see (Schoen and Zweng, 1986). The acquiring of estimation skills in schools is said to provide an essential practical means of operating within many mathematical and everyday situations in which precise calculation or measurement are contextually defined as either impossible or unnecessary (Levine, 1982).

This article focuses on estimation skills related to the concept of volume, that means on the measurement-type estimation skills. It is particularly devoted to estimating volume ratios. The issue can be represented by the question

> *"What are the corresponding height and volume ratios of a given solid?"*

Precisely, for a given solid and for a given volume ratio m/n we explore the level to which the solid should be filled with water in order to fill exactly m/n of the solid volume. This level is specified relatively, as a ratio of the height of the solid. We may also study the issue conversely — fill the solid to a given height ratio α, and look for the volume of the filled part expressed as a ratio of the volume of the whole solid.

The issue of volume ratios is not a common part of school mathematics, due to difficult calculations backgrounding the problem. On the other side, volume ratios are an integral part of everyday reality. Together it makes the issue an ideal candidate for engaging estimates.

The first part of this article reports the results of a survey focusing on volume ratios of three essential solids: a cylinder, a ball, and a cone. See Figure 1. The survey tested 80 pre-service and in-service math teachers of various nationalities and various school types, and investigated their skills in estimating volume ratios of these solids.

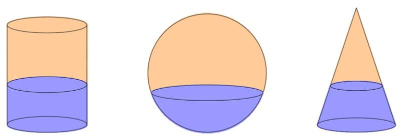

Figure 1: Solids filled with water to a certain level.

The second part of the article analyzes the mathematical background of cone and ball cases.

The third part of the article shows the possibilities of GeoGebra software in creating test materials for similar surveys, and — in accordance with the results of the survey — presents dynamic models designed to enhance estimation skills in volume ratios. The text

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gives detailed instruction on how to create such kind of GeoGebra materials.

2 THE SURVEY

2.1 The Sample

We tested 37 in-service math teachers, namely 11 university teachers and teacher trainers from Czechia, Germany, Serbia and Bulgaria, and 26 teachers from Czech, German and Serbian primary, lower-secondary and upper-secondary schools. These teachers were participants in workshops at some conferences and training seminars held between November 2011 and October 2012.

Concurrently we tested 43 pre-service math teachers from Czech and German universities. These teacher students were not individually selected; the survey was conducted in whole classes.

2.2 The Test

All surveyed teachers got to fill the same worksheet, consisting of 17 quick-answer questions with a common instruction. These questions were open-ended. The worksheet is precisely shown in Figures 2-5.

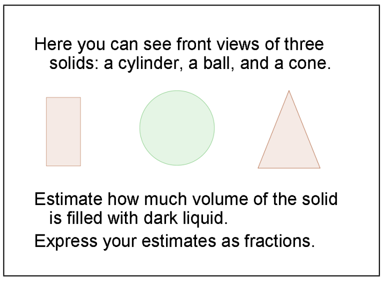

Figure 2: The worksheet, a title page.

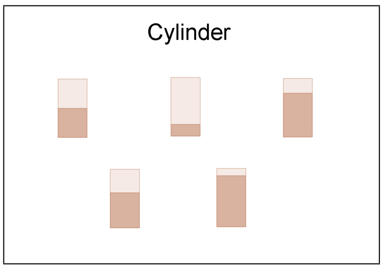

Figure 3: The worksheet, a cylinder page.

The content of the worksheet is of escalating difficulty. It begins with a cylinder case, which serves as a kind of calibration — all volume ratios of a cylinder are identical to height ratios. Then the worksheet

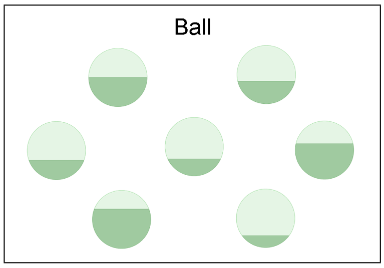

Figure 4: The worksheet, a ball page.

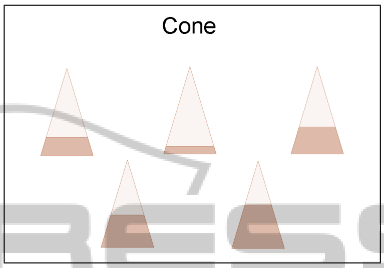

Figure 5: The worksheet, a cone page.

continues with a ball, whose height and volume ratios coincide only in $1/2$ case. The final part of the worksheet is devoted to a cone, in which all height ratios differ from their corresponding volume ratios.

2.3 The Evaluation

We divided respondents into 2 groups according to their status (in-service, pre-service).

We focused on relative errors of estimates calculated as

$$\frac{\text{estimate} - \text{exact answer}}{\text{exact answer}} \qquad (1)$$

so that the sign can tell us if the estimate is bigger than the exact answer (+ sign) or smaller (− sign).

As the first evaluation method we ascertained relative errors of all outcome estimates, and determined their arithmetic mean and median. A detailed overview can be found in Table 1.

The table shows that both in-service and pre-service teacher respondents have good estimation skills in a cylinder case, and also in a ball case — except 1/10 of the ball volume. Both groups overrated 1/10 of the ball volume, in-service performed a little better, their mean relative error is 44 %. Also both medians are overrated in this case.

On the other side, cone estimates are generally underrated, for all tested volume ratios. The worst cone ratio was 1/2 (with mean relative error -33 %, resp. -32 %), closely followed by 1/4 (with both mean relative errors -29 %).

Since the variability of answers was limited by the requirement to express volume ratios as fractions,

Table 1: Relative errors of in-service and pre-service teachers' estimates; in-service: $N = 37$, pre-service: $N = 43$.

Solid & volume ratio	In-service		Pre-service	
	Mean	Median	Mean	Median
Cylin 1/2	-1 %	0 %	0 %	0 %
Cylin 1/5	-2 %	0 %	0 %	0 %
Cylin 3/4	1 %	0 %	1 %	0 %
Cylin 3/5	3 %	0 %	3 %	0 %
Cylin 7/8	-7 %	0 %	0 %	0 %
Ball 1/2	0 %	0 %	0 %	0 %
Ball 1/3	8 %	0 %	5 %	0 %
Ball 1/4	8 %	0 %	14 %	6 %
Ball 1/5	15 %	0 %	7 %	0 %
Ball 2/3	-2 %	0 %	-1 %	0 %
Ball 3/4	1 %	0 %	-1 %	0 %
Ball 1/10	44 %	34 %	48 %	55 %
Cone 1/2	-33 %	-33 %	-32 %	-33 %
Cone 1/4	-29 %	-20 %	-29 %	-33 %
Cone 2/3	-30 %	-25 %	-28 %	-25 %
Cone 3/4	-19 %	-20 %	-19 %	-20 %
Cone 7/8	-15 %	-14 %	-14 %	-14 %

Table 2: Modus of estimates.

Solid, volume ratio	In-service	Pre-service
Ball 1/10	1/6	1/5
Cone 1/2	1/3	1/3
Cone 1/4	1/5	1/6
Cone 2/3	1/2	1/2
Cone 3/4	3/5, 1/2	3/5
Cone 7/8	3/4	3/4

we may also focus on modus of our data. A detailed overview of cases whose modus differs from the exact answer is in Table 2.

We may also analyze the difference between groups by independent two-sample t-test. Take for example the samples of relative errors belonging to Ball 1/10 picture. Formula

$$\frac{S_X^2}{S_Y^2} = 1,443 \in \left\langle \frac{1}{F_{42,36}(0,025)}, F_{42,36}(0,025) \right\rangle \quad (2)$$

means that we do not reject the hypothesis of equal variances at 0,05 level, and

$$T = 0,399 \leqq t_{78}(0,05) \quad (3)$$

means that we do not reject the hypothesis of equal means at 0,05 level either.

As the last method of evaluation we use the scoring method for Estimation Interview Test used in (Montague and van Garderen, 2003): an estimate is considered accurate if it is within 50 % of the exact answer. From this perspective, the case of 1/10 of the ball volume appears as the one with the highest failure rate: it has 17 inaccurate estimates among in-service teachers' answers (which means 47 % answers being inaccurate), and 23 among pre-service teachers' answers (54 % inaccurate). The second one in terms of failure is the case of 1/4 of the cone volume with 25 %, resp. 28 % inaccurate answers. A detailed overview can be found in Table 3.

Table 3: Percentage of inaccurate estimates.

Solid, volume ratio	In-service	Pre-service
Cylinder 1/2	0 %	0 %
Cylinder 1/5	3 %	0 %
Cylinder 3/4	0 %	0 %
Cylinder 3/5	0 %	2 %
Cylinder 7/8	8 %	2 %
Ball 1/2	0 %	0 %
Ball 1/3	6 %	9 %
Ball 1/4	11 %	23 %
Ball 1/5	11 %	7 %
Ball 2/3	0 %	9 %
Ball 3/4	0 %	7 %
Ball 1/10	47 %	54 %
Cone 1/2	6 %	21 %
Cone 1/4	25 %	28 %
Cone 2/3	6 %	7 %
Cone 3/4	8 %	19 %
Cone 7/8	6 %	9 %

2.4 The Summary

The survey showed that both pre-service and in-service math teachers had significant difficulties with estimating volume ratio from a picture of a cone, and in some cases also from a picture of a ball.

It would be expedient to enhance this kind of estimation skills, for instance through a suitable ICT environment.

3 MATHEMATICAL BACKGROUND

This section shall reveal the mathematical background of the problem of finding the height ratio α for a given volume ratio m/n, and vice versa.

3.1 The Cone

The volume of a cone with base radius r and height k is given by a formula $\frac{1}{3}\pi r^2 k$. The water in the cone reaches an unknown height h, expressed as an α-multiple of the height of the cone, i.e., $h = \alpha \cdot k$.

The complement of the water in the cone is also a cone, with height $k(1-\alpha)$, and base radius $r(1-\alpha)$.

See Figure 6. Thus, the volume of the water can be expressed as

$$\begin{aligned} V_{water} &= \frac{1}{3}\pi r^2 k - \frac{1}{3}\pi r^2 (1-\alpha)^2 k(1-\alpha) \\ &= \frac{1}{3}\pi r^2 k\big(1-(1-\alpha)^3\big) \qquad (4) \end{aligned}$$

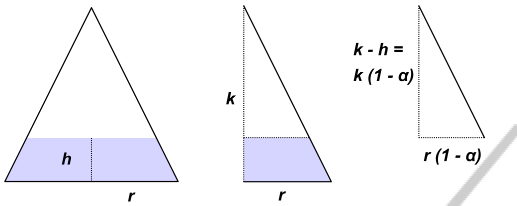

Figure 6: The cone (left), similar triangles (middle, right).

We are looking for a water level corresponding to m/n of the cone volume:

$$\begin{aligned} V_{water} &= \frac{m}{n} \cdot V_{cone} \\ 1-(1-\alpha)^3 &= \frac{m}{n} \qquad (5) \\ \alpha &= 1 - \sqrt[3]{1-\frac{m}{n}} \qquad (6) \end{aligned}$$

3.2 The Ball

The volume of a ball with radius r is given by a formula $\frac{4}{3}\pi r^3$. The height of the ball equals $2r$. The water in the ball reaches an unknown height $h = \alpha \cdot 2r$.

The water in the ball occupies a spherical cap with height h, for details see Figure 7.

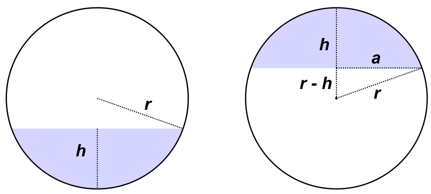

Figure 7: The ball situation in detail.

The volume of the water equals the volume of the spherical cup, that means

$$V_{water} = \frac{1}{6}\pi h(3a^2+h^2) = \frac{4}{3}\pi r^3 \alpha^2 (3-2\alpha) \qquad (7)$$

We are looking for a water level corresponding to m/n of the ball volume:

$$\begin{aligned} V_{water} &= \frac{m}{n} \cdot V_{ball} \\ \alpha^2(3-2\alpha) &= \frac{m}{n} \qquad (8) \\ 2\alpha^3 - 3\alpha^2 + \frac{m}{n} &= 0 \qquad (9) \end{aligned}$$

This cubic equation has three real solutions, one of them belonging to an interval $\langle 0,1 \rangle$:

$$\alpha = \frac{1}{2} - \cos\left(\frac{\pi + \arccos\left(1-\frac{2m}{n}\right)}{3}\right) \qquad (10)$$

Detailed solution of (9) leading to (10) can be found in (Samkova, 2012).

4 ICT SUPPORT

We shall show the possibilities of enhancing estimation skills with help of ICT, both in passive and active ways. We shall use GeoGebra, free mathematics dynamic software for teaching and learning mathematics at all school levels. GeoGebra is currently available in about 55 languages, it has received several educational software awards in Europe and the USA. For more about GeoGebra see (GeoGebra, 2012).

4.1 Creating Illustrations with GeoGebra

At first we shall demonstrate the passive way of GeoGebra support — dynamic illustrations of transparent hollow essential solids partially filled with water.

4.1.1 The Cylinder

The GeoGebra construction begins with sliders for n, m, cylinder radius r, and cylinder height k. Then we create a front view of the cylinder, which is actually a rectangle with base $2r$ and height k:

```
poly1= Polygon[(r,0),(r,k),(-r,k),(-r,0)]
```

The filled part of the cylinder is also a cylinder, its front view is another rectangle:

```
h=k*m/n
poly2=Polygon[(r,0),(r,h),(-r,h),(-r,0)]
```

The construction is almost done, we just have to design it properly. We change LineThickness of `poly1` to 6, Color of `poly2` to blue, LineThickness of `poly2` to 0, Opacity of `poly2` to 50. The preview of the construction is in Figure 8.

With this dynamic illustration we may prepare various pictures of cylinders filled with water to a certain level, and export them as PNG or EPS files.

We may also export the dynamic worksheet as a webpage, and make it available to students. In this case, we label the r slider with “base radius”, and the k slider with “cylinder height”. We add an interactive text to the worksheet.

The final form of the worksheet is in Figure 9.

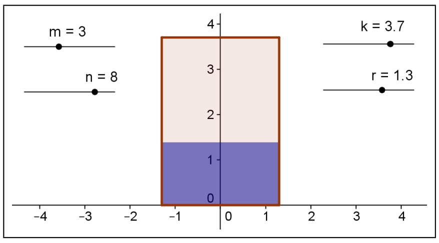

Figure 8: The preview of the cylinder case construction.

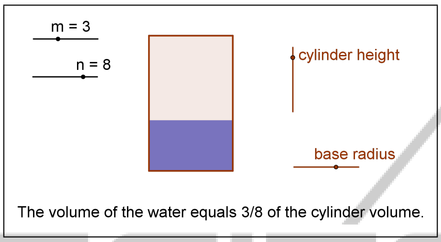

Figure 9: The dynamic worksheet for a cylinder.

4.1.2 The Cone

The construction begins again with sliders for n, m, base radius r, and cone height k. Then we create a front view of the cone, which is actually an isosceles triangle with base $2r$ and height k:

```
poly1= Polygon[(r,0),(0,k),(-r,0)]
```

The filled part of the cone is a truncated cone, its front view is an isosceles trapezoid:

```
alpha=1-(1-m/n)^(1/3)
h=alpha*k
d=Line[(0,h),xAxis]  ... the water level
Intersect[d,poly1]  ... points A and B
poly2=Polygon[(r,0),B,A,(-r,0)]
```

Now we just have to design the picture the same way as in the cylinder case. The preview of the construction is in Figure 10.

Note that the illustration is not correct for $m = n$. We have to create a blue triangle

```
poly3=Polygon[(r,0),(0,k),(-r,0)]
```

with a Condition to Show Object `m=n`.

4.1.3 The Ball

The construction begins again with sliders for n, m, and ball radius r. Then we create a front view of the ball, which is a circle with radius r:

```
circ1=Circle[(0,r),r]
```

As the next step we solve graphically the equation (9): we define a left side as a function, and find where its graph intersect $\langle 0,1 \rangle$ at the x-axis:

```
f(x)=2x^3-3x^2+m/n
```

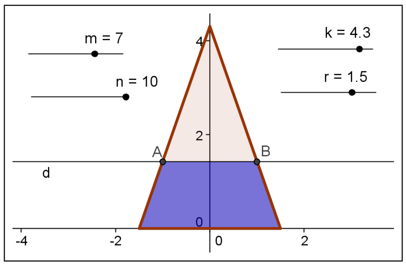

Figure 10: The preview of the cone case construction.

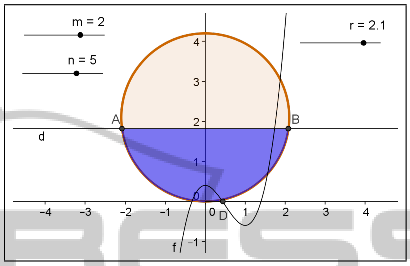

Figure 11: The preview of the ball case construction.

```
a=Segment[(0,0),(1,0)]
D=Intersect[f,a]
alpha=x(D)
```

The filled part of the ball is an upside-down oriented spherical cup, its front view is a circular segment with a chord parallel to x-axis:

```
h=alpha*2r
d=Line[(0,h),xAxis]  ... the water level
Intersect[d,circ1]  ... points A and B
circ2=Arc[circ1,A,B]
```

As previously, we design the picture, and solve separately the situation for $m = n$. The preview of the construction is in Figure 11.

4.2 Interactive Estimation Training with GeoGebra

Now we shall demonstrate the active way of GeoGebra support — a dynamic GeoGebra worksheet for interactive estimation training of volume ratios. This tool focuses on the process of finding the right estimate of water level for a given volume ratio. The worksheet randomly generates volume ratios m/n, waits for the user to draw his estimate to the picture, and evaluates the estimate.

We shall show the construction in a cone case. The construction begins with random sliders for n, m, with sliders for r, k, and with `poly1`, `alpha`, `h` as in 4.1.2. Then we create the exact water level:

```
d=Segment[(-2*r,h),(2*r,h)]
```

This exact answer should be hidden if needed, so that we create a check box `q` with `d` as its selected object, and label it `Exact answer`.

The next step will prepare the picture for user's estimation process:

```
p=Segment[(0,0),(0,k)]
P=Point[p]
o=Line[P,xAxis]
Intersect[o,poly1]  ... points A and B
poly2=Polygon[(r,0),B,A,(-r,0)]
```

The preview of the construction is in Figure 12.

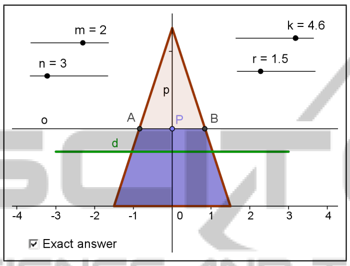

Figure 12: The preview of the training construction.

As a final activity we have to manage the process of generating random values of *n*, *m*: we create a button with label `New task`, and with GeoGebra script

```
UpdateConstruction[]
q=false
```

Pressing this button will load new random values for *n* and *m*, and hide the segment with exact answer. The user can move point `P` to a position where he thinks the corresponding water level should be, then mark the check box `Exact answer`, and compare his estimate with the line of exact answer.

We may also determine relative error of the estimate: define `err=round((y(P)-h)/h*100)`, and incorporate it into an interactive text with a Condition to Show Object `q=true` to it.

The final form of the worksheet can be seen in Figures 13 and 14.

5 CONCLUSIONS

The issue of volume ratios is a remarkable component of the concept of volume. Our survey showed that even math teachers had difficulties with estimating volume ratios of some essential solids. GeoGebra software offers an interesting way how to enhance this kind of estimation skills — through a dynamic GeoGebra environment we can create illustrations related to the volume ratio issue, or an interactive estimation training tool. Future surveys may focus on non-teacher respondents or more deeply on the particular role of GeoGebra in enhancing estimation skills.

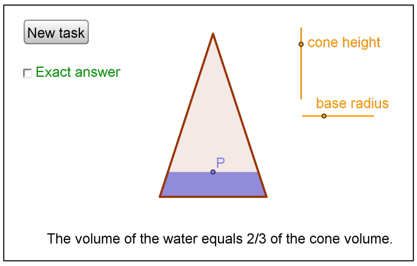

Figure 13: The worksheet ready for user's estimation.

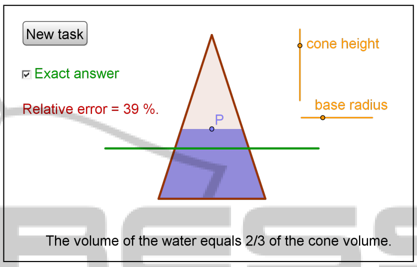

Figure 14: The evaluation of the user's estimate.

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