Cooperation Tendencies and Evaluation of Games

Francesc Carreras[1] and Maria Albina Puente[2]
[1]*ETSEIAT, Technical University of Catalonia, Colom 11, Terrassa, Spain*
[2]*EPSEM, Technical University of Catalonia, Manresa, Spain*

Keywords: Cooperative Game, Shapley Value, Probabilistic Value, Binomial Semivalue.

Abstract: Multinomial probabilistic values were first introduced by one of us in reliability and later on by the other, independently, as power indices. Here we study them on cooperative games from several viewpoints, and especially as a powerful generalization of binomial semivalues. We establish a dimensional comparison between multinomial values and binomial semivalues and provide two characterizations within the class of probabilistic values: one for each multinomial value and another for the whole family. An example illustrates their use in practice as power indices.

1 INTRODUCTION

Weber's general model for assessing cooperative games (Weber, 1988) is based on *probabilistic values*, a family of values axiomatically characterized by means of linearity, positivity, and the dummy player property. Every probabilistic value allocates, to each player in each game of its domain, a weighted (convex) sum of the marginal contributions of the player in the game. These allocations can be interpreted as a measure of players' bargaining relative strength. The most conspicuous member of this family (in fact, the inspiring one) is the *Shapley value* (Shapley, 1953). In the present paper we study a subfamily of probabilistic values that we call *multinomial (probabilistic) values*.[1] Technically, their main characteristic is the *systematic generation of the weighting coefficients in terms of a few parameters (one parameter per player)*.

Our research group has been studying *semivalues* (Dubey et al., 1981), a subfamily of probabilistic values characterized by anonymity and including the Shapley value as the only *efficient* member. In the analysis of certain cooperative problems we have successfully used *binomial semivalues* (Freixas and Puente, 2002), a monoparametric subfamily that includes the Banzhaf value (Owen, 1975).

From this experience, we feel that multinomial values (n parameters, n being the number of players) offer a deal of flexibility clearly greater than binomial semivalues (one parameter), and hence many more possibilities to introduce additional information when evaluating a game. Fig. 1 describes the relationships between the above values and families of values and the main characteristics of each one of them.

Probabilistic values provide tools to study not only games *in abstracto* (i.e. from a merely structural viewpoint) but also the influence of *players' personality* on the issue. They are assessment techniques that do not modify the game but only the criteria by which payoffs are allocated. Parameters will be addressed here to cope with different attitudes the players may hold when playing a given game, even if they are not individuals but countries, enterprises, parties, trade unions, or collectivities of any other kind. We will attach to parameter p_i the meaning of *generical tendency of player i to form coalitions*, assuming p_i and p_j independent of each other if $i \neq j$.

Multinomial values are a consistent alternative or complement to classical values (Shapley, Banzhaf). They represent a wide generalization of binomial semivalues, whose monoparametric condition implies a quite limited capability of analysis of cooperation tendencies. Of course, these tendencies can neither be analyzed, without modifying the game, by means of the classical values, which can be concerned only with the structure of the game.

The organization of the paper is as follows. Section 2 includes a minimum of preliminaries. In Section 3, we introduce multinomial values. Section 4

[1]They were introduced in reliability (Freixas and Puente, 2002) with the name of "multibinary probabilistic values" and independently defined for simple games only —i.e. as power indices— in a work on decisiveness (Carreras, 2004), where they were called "Banzhaf α–indices."

Carreras F. and Puente M..
Cooperation Tendencies and Evaluation of Games.
DOI: 10.5220/0004414104150422
In *Proceedings of the 15th International Conference on Enterprise Information Systems* (ICEIS-2013), pages 415-422
ISBN: 978-989-8565-59-4

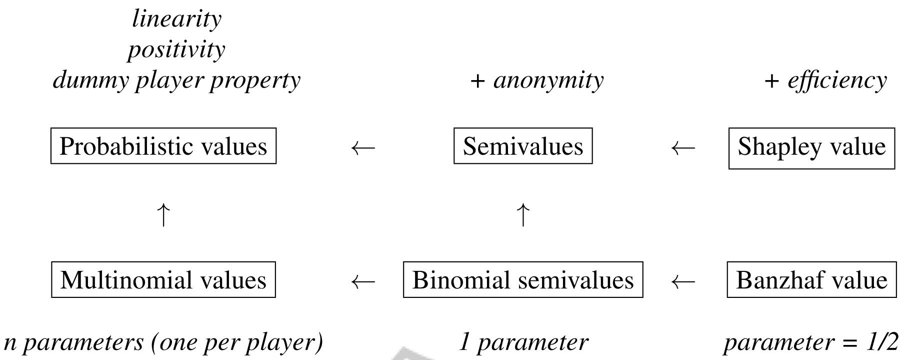

Figure 1: Inclusion relationships between values and families of values.

contains a result on the dimension of the subspace spanned by multinomial values and two characterizations: one, individual, for each multinomial value; another, collective, for the whole subfamily they form. For space reasons, proofs have been omitted. Section 5 is devoted to analyze a political problem by using these values.

2 PRELIMINARIES

Let N be a finite set of *players*, usually denoted as $N = \{1,2,\dots,n\}$. A *(cooperative) game* in N is a function v that assigns a real number $v(S)$ to each *coalition* $S \subseteq N$, with $v(\emptyset) = 0$. This number is understood as the utility that coalition S can obtain by itself, that is, independently of the remaining players' behaviour.

Game v is *monotonic* if $v(S) \leq v(T)$ when $S \subset T \subseteq N$. Player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$. Players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

Endowed with the natural operations for real–valued functions, $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set of all cooperative games in N is a vector space $\mathcal{G}_N$. For every nonempty $T \subseteq N$, the *unanimity game* u_T in N is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$, so $\dim \mathcal{G}_N = 2^n - 1$.

By a *value* on $\mathcal{G}_N$ we mean a map $g : \mathcal{G}_N \to \mathbb{R}^N$, which assigns to every game v a vector $g[v]$ with components $g_i[v]$ for all $i \in N$. The *total power* of value g in v is

$$\pi^g(v) = \sum_{i \in N} g_i[v]. \tag{1}$$

Following Weber's axiomatic definition (Weber, 1988), $\phi : \mathcal{G}_N \to \mathbb{R}^N$ is a (group) *probabilistic value* if it satisfies the following properties:

(i) *linearity*: $\phi[v + v'] = \phi[v] + \phi[v']$ and $\phi[\lambda v] = \lambda\phi[v]$ for all $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;

(ii) *positivity*[2]: if v is monotonic, then $\phi[v] \geq 0$;

(iii) *dummy player property*: if $i \in N$ is a dummy in game v, then $\phi_i[v] = v(\{i\})$.

There is an interesting characterization of the probabilistic values (Weber, 1988): (a) given a set $P = \{p^i_S : i \in N,\ S \subseteq N \backslash \{i\}\}$ of $n2^{n-1}$ *weighting coefficients*, such that

$$\text{all } p^i_S \geq 0 \quad \text{and} \quad \sum_{S \subseteq N \backslash \{i\}} p^i_S = 1 \text{ for each } i, \tag{2}$$

the expression

$$\phi_i[v] = \sum_{S \subseteq N \backslash \{i\}} p^i_S [v(S \cup \{i\}) - v(S)] \tag{3}$$

for all $i \in N$ and $v \in \mathcal{G}_N$ defines a probabilistic value ϕ on $\mathcal{G}_N$; (b) conversely, every probabilistic value can be obtained in this way; (c) the correspondence given by $P \mapsto \phi$ is one–to–one. Thus, the payoff that a probabilistic value allocates to each player in any game is a weighted sum of the marginal contributions of the player in the game. We quote (Weber, 1988):

> "Let player i view his participation in a game v as consisting merely of joining some coalition S and then receiving as a reward his marginal contribution to the coalition. If p^i_S is the probability that he joins coalition S, then $\phi_i[v]$ is his expected payoff from the game."

The action of a probabilistic value ϕ on the basis of unanimity games is as follows: if $\emptyset \neq T \subseteq N$ then

$$\phi_i[u_T] = \sum_{\substack{S \subseteq N \backslash \{i\}: \\ T \backslash \{i\} \subseteq S}} p^i_S \quad \text{if } i \in T \tag{4}$$

and $\phi_i[u_T] = 0$ otherwise.

[2] Weber calls *monotonicity* to this property, but we prefer to call to it *positivity* (Dubey et al., 1981).

Among probabilistic values, *semivalues* (Dubey et al., 1981) are characterized by the *anonymity property*: there is a vector $\{p_s\}_{s=0}^{n-1}$ such that $p_S^i = p_s$ for all $i \in N$ and all $S \subseteq N\backslash\{i\}$, where $s = |S|$, so that all coalitions of a given size share a common weight and Eq. (3) reduces therefore to

$$\phi_i[v] = \sum_{S \subseteq N\backslash\{i\}} p_s[v(S \cup \{i\}) - v(S)]$$

for all $i \in N$ and $v \in \mathcal{G}_N$. The weighting coefficients $\{p_s\}_{s=0}^{n-1}$ of any semivalue ϕ satisfy two characteristic conditions, derived from Eq. (2): each $p_s \geq 0$ and $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$.

Among semivalues, the *Shapley value* (Shapley, 1953), denoted here by φ and defined by $p_s = 1/\binom{n-1}{s}n$ for all s, is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$. The *Banzhaf value* (Owen, 1975), denoted here by β and defined by $p_s = 1/2^{n-1}$ for all s, is the only semivalue satisfying the *total power property*: for every $v \in \mathcal{G}_N$,

$$\sum_{i \in N} \beta_i[v] = \frac{1}{2^{n-1}} \sum_{S \subseteq N} \sum_{i \notin S} [v(S \cup \{i\}) - v(S)]. \quad (5)$$

The *multilinear extension* (Owen, 1972) of a game $v \in \mathcal{G}_N$ is the real–valued function defined in $\mathbb{R}^n$ by

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N\backslash S} (1 - x_j) v(S).$$

As is well known, both the Shapley and Banzhaf values of any game v can be obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0,1]^n$ (Owen, 1972) while the partial derivatives of that multilinear extension, evaluated at point $(1/2, 1/2, \ldots, 1/2)$, give $\beta[v]$ (Owen, 1975).

3 MULTINOMIAL VALUES

Definition 3.1. Set $N = \{1, 2, \ldots, n\}$ and let a *profile* $\mathbf{p} \in [0,1]^n$, that is, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ with $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$, be given. Then the coefficients

$$p_S^i = \prod_{j \in S} p_j \prod_{\substack{k \in N\backslash S \\ k \neq i}} (1 - p_k) \quad (6)$$

for all $i \in N$ and $S \subseteq N\backslash\{i\}$ (where the empty product, arising if $S = \emptyset$ or $S = N\backslash\{i\}$, is taken to be 1) define (Freixas and Puente, 2002) a probabilistic value on $\mathcal{G}_N$ that we call the **p**–*multinomial value* $\lambda^{\mathbf{p}}$. Thus,

$$\lambda_i^{\mathbf{p}}[v] = \sum_{S \subseteq N\backslash\{i\}} \prod_{j \in S} p_j \prod_{\substack{k \in N\backslash S \\ k \neq i}} (1 - p_k)[v(S \cup \{i\}) - v(S)]$$

for all $i \in N$ and $v \in \mathcal{G}_N$.

As was announced in Section 1, we will attach to p_i the meaning of *generical tendency of player i to form coalitions*, and thus we will say that **p** is a *tendency* profile on N. According to Eq. (6), coefficient p_S^i, the probability of i to join S, will depend on the positive tendencies of the members of S to form coalitions and also on the negative tendencies in this sense of the outside players, i.e. the members of $N\backslash(S \cup \{i\})$.

Remarks 3.2. (a) For example, for $n = 2$ we have $\mathbf{p} = (p_1, p_2)$ and, if $i \neq j$,

$$\lambda_i^{\mathbf{p}}[v] = (1 - p_j)[v(\{i\}) - v(\emptyset)] + p_j[v(N) - v(\{j\})].$$

Hence, the allocation given by $\lambda^{\mathbf{p}}$ to player i does not depend on p_i but only on p_j. If player j is not greatly interested in cooperating (say, p_j tends to 0), player i's allocation will tend to his individual utility $v(\{i\})$. Instead, if player j is highly interested in cooperating (say, p_j tends to 1), player i's allocation will tend to his marginal contribution to the grand coalition $v(N) - v(\{j\})$.

(b) It is easy to check that the action of $\lambda^{\mathbf{p}}$ on any unanimity game u_T is given by:

$$\lambda_i^{\mathbf{p}}[u_T] = \prod_{\substack{j \in T \\ j \neq i}} p_j \quad \text{if } i \in T \quad (7)$$

and $\lambda_i^{\mathbf{p}}[u_T] = 0$ otherwise. Using Eq. (7), it readily follows that, for $n \geq 2$, $\mathbf{p} \neq \mathbf{p}'$ implies $\lambda^{\mathbf{p}} \neq \lambda^{\mathbf{p}'}$ (if $n = 1$ all profiles give rise to a unique multinomial value).

(c) Whenever, in particular, $p_1 = p_2 = \cdots = p_n = q$ for some $q \in [0,1]$, coefficients p_S^i reduce, for all $i \in N$, to

$$p_S^i = p_s = q^s(1-q)^{n-s-1} \quad \text{for} \quad s = 0, 1, \ldots, n-1,$$

where $s = |S|$ and $0^0 = 1$ by convention in cases $q = 0$ and $q = 1$. These coefficients $\{p_s\}_{s=0}^{n-1}$ define the *q–binomial semivalue* ψ^q (Freixas and Puente, 2002) and, obviously, $\lambda^{\mathbf{p}} = \psi^q$. If, moreover, $q = 1/2$ then we obtain $\psi^{1/2} = \beta$, the Banzhaf value.

(d) The multilinear extension procedure extends well to all binomial semivalues and even to any multinomial value $\lambda^{\mathbf{p}}$ (Freixas and Puente, 2002): if f is the multilinear extension of game $v \in \mathcal{G}_N$ then

$$\lambda_i^{\mathbf{p}}[v] = \frac{\partial f}{\partial x_i}(p_1, p_2, \ldots, p_n) \quad \text{for all } i \in N.$$

4 THEORETICAL RESULTS

We devote this section to extending three results stated in the previous literature on binomial semivalues. In all cases the extension is not straightforward and reveals new features of multinomial values. We assume $n \geq 2$ because for $n = 1$ all is trivial.

4.1 About Dimensions

Let $\mathcal{L}(\mathcal{G}_N, \mathbb{R}^n)$ denote the space of all linear maps from $\mathcal{G}_N$ to $\mathbb{R}^n$, which includes most values studied in the literature. It is clear that $\dim \mathcal{L}(\mathcal{G}_N, \mathbb{R}^n) = n(2^n - 1)$. Let $\mathcal{BS}(\mathcal{G}_N, \mathbb{R}^n)$ denote the subspace spanned by binomial semivalues. It is known that $\dim \mathcal{BS}(\mathcal{G}_N, \mathbb{R}^n) = n$ (Freixas and Puente, 2002). Moreover, it coincides with the subspace spanned by all semivalues, and any n different binomial semivalues $\psi^{q_1}, \psi^{q_2}, \ldots, \psi^{q_n}$ form a basis.

Now, let $\mathcal{MV}(\mathcal{G}_N, \mathbb{R}^n)$ denote the subspace spanned by multinomial values. We shall determine its dimension. To this end, the following auxiliar notion is useful (and a basis for this subspace is found during the proof).

Definition 4.1. A value g on $\mathcal{G}_N$ satisfies the property of *neutrality (for unanimity games)* if, for each $T \subseteq N$ with $0 \leq |T| \leq n-2$,

$$g_i[u_{T\cup\{i\}}] = g_j[u_{T\cup\{j\}}] \quad \text{for any} \quad i, j \notin T.$$

This property is satisfied by any multinomial value[3] since, by Remark 3.2(b), we have

$$\lambda_i^{\mathbf{p}}[u_{T\cup\{i\}}] = \prod_{k\in T} p_k = \lambda_j^{\mathbf{p}}[u_{T\cup\{j\}}].$$

Theorem 4.2. *Let $\mathcal{MV}(\mathcal{G}_N, \mathbb{R}^n)$ be the subspace spanned by multinomial values within the space $\mathcal{L}(\mathcal{G}_N, \mathbb{R}^n)$ of linear maps. Then* $\dim \mathcal{MV}(\mathcal{G}_N, \mathbb{R}^n) = 2^n - 1$.

The difference between $n = \dim \mathcal{BS}(\mathcal{G}_N, \mathbb{R}^n)$ and $2^n - 1 = \dim \mathcal{MV}(\mathcal{G}_N, \mathbb{R}^n)$ reflects the much greater versatility of multinomial values.

4.2 Individual Characterization of each Multinomial Value

The notion of total power given by Eq. (1) has been proven to be fruitful in absence of efficiency. For example, when applying a normalization process to a value. The total power property of the Banzhaf value given by Eq. (5) was the natural substitute of efficiency in well–known axiomatic characterizations of this value (e.g. Feltkamp, 1995). It was extended to all binomial semivalues (Carreras and Puente, 2012), giving rise to the *q–binomial total power property*:

$$\sum_{i\in N} \psi_i^q[v] = \sum_{S\subseteq N} q^s (1-q)^{n-s-1} \sum_{i\notin S} [v(S\cup\{i\}) - v(S)]$$

[3]In particular, all binomial semivalues, but also the Shapley value, satisfy this property.

for every $v \in \mathcal{G}_N$.

For each $q \in [0,1]$, this property characterizes the q–binomial semivalue ψ^q among semivalues, and this characterization can be alternatively stated as follows: if ψ is a semivalue such that $\sum_{i\in N} \psi_i[v] = \sum_{i\in N} \psi_i^q[v]$ for all $v \in \mathcal{G}_N$ then $\psi = \psi^q$. The natural extension of the property to probabilistic values must be carried out in the following terms.

Definition 4.3. Let $\mathbf{p} \in [0,1]^n$ be a profile on N. A (probabilistic or not) value g on $\mathcal{G}_N$ satisfies the **p**–*multinomial total power property* if, for all $v \in \mathcal{G}_N$,

$$\sum_{i\in N} g_i[v] = \sum_{S\subseteq N} \sum_{i\notin S} \prod_{j\in S} p_j \prod_{\substack{k\in N\setminus S \\ k\neq i}} (1-p_k)[v(S\cup\{i\}) - v(S)]. \tag{8}$$

However, this property, clearly equivalent to $\sum_{i\in N} g_i[v] = \sum_{i\in N} \lambda_i^{\mathbf{p}}[v]$ and hence obviously satisfied by the **p**–multinomial value $\lambda^{\mathbf{p}}$, does not characterize it within the class of probabilistic values. Indeed, it is easy to see, e.g. for $n = 2$ and using Eqs. (4) and (7), that in general not only $\lambda^{\mathbf{p}}$ but also infinitely many probabilistic values satisfy Eq. (8) for a given **p**.

Therefore, we need to introduce a second property in order to characterize each $\lambda^{\mathbf{p}}$ within the class of probabilistic values. The reader will notice that, due to anonymity, this property holds for all binomial semivalues and hence it was irrelevant for them.

Definition 4.4. Let $\mathbf{p} \in [0,1]^n$ be a profile on N. A value g on $\mathcal{G}_N$ satisfies the property of **p**–*weighted payoffs for unanimity games* if, for every nonempty $T \subseteq N$,

$$p_i g_i[u_T] = p_j g_j[u_T] \qquad \text{for all} \quad i, j \in T.$$

By Eq. (7) it is clear that $\lambda^{\mathbf{p}}$ satisfies this property.

Theorem 4.5. *(Characterization of each* **p**–*multino-mial value). Let* **p** *be a profile on N. Then the unique probabilistic value on $\mathcal{G}_N$ that satisfies the* **p**–*multinomial total power property and the property of* **p**–*weighted payoffs for unanimity games is the multinomial value $\lambda^{\mathbf{p}}$.*

4.3 Collective Characterization of Multinomial Values

Among semivalues, the binomial family is characterized by the monotonicity of the weighting coefficients (Alonso et al., 2007): a semivalue ψ on $\mathcal{G}_N$ is binomial if and only if its weighting coefficients $\{p_s\}_{s=0}^{n-1}$ are in geometric progression, i.e. satisfy, for some μ, the condition $p_{s+1} = \mu p_s$ for $s = 0, 1, 2, \ldots, n-2$

(maybe the simplest form of monotonicity).[4] The extension, not completely straightforward, will be given by Theorem 4.9. To this end, we need to consider two special types of players with regard to the weighting coefficients of a probabilistic value.

Definition 4.6. Let ϕ be a probabilistic value on $\mathcal{G}_N$ with weighting coefficients $\{p_S^i\}$.

- A player $h \in N$ is a ϕ–*ordinary player*[5] if there is $\mu_h \geq 0$ such that, for all $i \in N$, $p_S^i = \mu_h p_{S\setminus\{h\}}^i$ whenever $h \in S \subseteq N\setminus\{i\}$.
- A player $h \in N$ is a ϕ–*magnetic player* if $p_{S\setminus\{h\}}^i = 0$ whenever $h \in S \subseteq N\setminus\{i\}$. This condition is equivalent to saying that $p_S^i = 0$ for all $S \subseteq N\setminus\{i,h\}$.

Examples 4.7. (a) For the Banzhaf value β, all players are ordinary, with $\mu_h = 1$ for all of them. The same happens for every binomial semivalue ψ^q, with $\mu_h = q/(1-q)$, unless $q = 1$ (marginal index), in which case all players are magnetic.

(b) The Shapley value φ does not admit magnetic players. For $n = 2$ both players are ordinary, with $\mu_h = 1$. For $n > 2$ there are not ordinary players.

(c) Let $n = 3$ and assume that, for a given probabilistic value ϕ, players 1 and 2 are ordinary and player 3 is magnetic. Then we have, for some μ_1, μ_2, the links given by Table 1. Imposing Eq. (2) yields the relevant weighting coefficients in terms of μ_1, μ_2:

$$p_{\{3\}}^1 = \frac{1}{1+\mu_2}, \quad p_{\{3\}}^2 = \frac{1}{1+\mu_1},$$
$$p_{\emptyset}^3 = \frac{1}{1+\mu_1+\mu_2+\mu_1\mu_2}.$$

Choosing, for example, $\mu_1 = 1/2$ and $\mu_2 = 1$, we obtain all weighting coefficients and hence a probabilistic value.

Remarks 4.8. (a) The conditions of Definition 4.6 are incompatible. If there were a simultaneously ordinary and magnetic player h then, for any other $i \in N$, we would have $p_S^i = 0$ for all $S \subseteq N\setminus\{i\}$, contradicting that these coefficients sum up to 1.

(b) The condition of ordinary player means that the relation between p_S^i and $p_{S\setminus\{h\}}^i$ follows a pattern common to all $i \in N$ and very similar to the monotonicity in the binomial semivalue case, although the proportionality factor depends here on player h.

(c) Instead, the existence of a magnetic player h implies that none of the other players would join a coalition excluding h.

(d) Let $\phi = \lambda^{\mathbf{p}}$, a multinomial value. Then player $h \in N$ is ϕ–ordinary if $p_h < 1$, and ϕ–magnetic if $p_h = 1$. This follows from the proof of the next result.

Theorem 4.9. *(Collective characterization of all multinomial values). A probabilistic value ϕ on $\mathcal{G}_N$ is a multinomial value if and only if all players $h \in N$ are ϕ–ordinary or ϕ–magnetic. In this case, $\phi = \lambda^{\mathbf{p}}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is given by*

$$p_h = \begin{cases} \dfrac{\mu_h}{1+\mu_h} & \text{if } h \text{ is a } \phi\text{–ordinary player,} \\ 1 & \text{if } h \text{ is a } \phi\text{–magnetic player.} \end{cases}$$

The difference between monotonicity at *individual level* established in Theorem 4.9 and the *uniform* monotonicity that characterizes binomial semivalues is a new good sample of the higher versatility of the multinomial values.

Examples 4.10. (a) The Shapley value is multinomial only for $n = 2$. In fact, in this case φ and β coincide.

(b) According to Theorem 4.9, the value obtained in Example 4.7(c) is multinomial. From μ_1 and μ_2 we get the profile that defines it: $\mathbf{p} = (1/3,\ 1/2,\ 1)$.

(c) Let ϕ be the probabilistic value for $n = 3$ defined by the weighting coefficients

$$p_{\emptyset}^1 = 0, \quad p_{\{2\}}^1 = 0, \quad p_{\{3\}}^1 = 0.2,$$
$$p_{\{2,3\}}^1 = 0.8, \quad p_{\emptyset}^2 = 0, \quad p_{\{1\}}^2 = 0,$$
$$p_{\{3\}}^2 = 0.8, \quad p_{\{1,3\}}^2 = 0.2, \quad p_{\emptyset}^3 = 0.4,$$
$$p_{\{1\}}^3 = 0.1, \quad p_{\{2\}}^3 = 0.4, \quad p_{\{1,2\}}^3 = 0.1.$$

It is easy to check that, with regard to ϕ, player 1 is ordinary (with $\mu_1 = 1/4$) and player 3 is magnetic, but player 2 is neither ordinary nor magnetic. Then, using again Theorem 4.9, ϕ is not a multinomial value.

5 A POLITICAL EXAMPLE: IDEOLOGICAL CONSTRAINTS

The model based on multinomial values is able to encompass additional information due to ideological re-

[4] Strictly speaking, the condition is as follows: (i) $p_{s+1} = \mu p_s$ for all s or (ii) $p_s = \mu' p_{s+1}$ for all s. The dictatorial index ψ^0 satisfies (i) only, with $p_0 = 1$ and $\mu = 0$. The marginal index ψ^1 satisfies (ii) only, with $p_{n-1} = 1$ and $\mu' = 0$. Any other binomial semivalue, with $q \neq 0, 1$, satisfies (i) and (ii) because $\mu = \dfrac{1-q}{q} \neq 0$; thus, $q = \dfrac{\mu}{1+\mu}$ and $p_0 = \dfrac{1}{(1+\mu)^{n-1}}$.

[5] We use this term to emphasize that exceptionality corresponds to the next option, that of magnetic player.

Table 1: Links between weighting coefficients in Example 4.7(c).

$$p^1_\emptyset = 0 \xrightarrow{\mu_2} p^1_{\{2\}} = \mu_2 p^1_\emptyset = 0, \qquad p^1_{\{3\}} \xrightarrow{\mu_2} p^1_{\{2,3\}} = \mu_2 p^1_{\{3\}},$$

$$p^2_\emptyset = 0 \xrightarrow{\mu_1} p^2_{\{1\}} = \mu_1 p^2_\emptyset = 0, \qquad p^2_{\{3\}} \xrightarrow{\mu_1} p^2_{\{1,3\}} = \mu_1 p^2_{\{3\}},$$

$$p^3_\emptyset \xrightarrow{\mu_1} p^3_{\{1\}} = \mu_1 p^3_\emptyset, \qquad p^3_{\{1\}} \xrightarrow{\mu_2} p^3_{\{1,2\}} = \mu_2 p^3_{\{1\}},$$
$$p^3_\emptyset \xrightarrow{\mu_2} p^3_{\{2\}} = \mu_2 p^3_\emptyset, \qquad p^3_{\{2\}} \xrightarrow{\mu_1} p^3_{\{1,2\}} = \mu_1 p^3_{\{2\}}.$$

strictions. We will discuss here a political problem described by a simple game.[6]

We recall that v is a *simple game* if it is monotonic and $v(S) = 0$ or 1 for all $S \subseteq N$. It is determined by the set $W(v) = \{S \subseteq N : v(S) = 1\}$ of *winning coalitions* and even by the subset $W^m(v) = \{S \in W(v) : T \notin W(v) \text{ if } T \subset S\}$ of *minimal winning coalitions*. In particular, v is a *weighted majority game* if there exist a *quota* $q > 0$ and *weights* $w_1, w_2, \ldots, w_n \geq 0$ such that $S \in W(v)$ if and only if $\sum_{i \in S} w_i \geq q$. We denote this fact by $v \equiv [q; w_1, w_2, \ldots, w_n]$.

Example 5.1. We consider a 50–member parliamentary body with $n = 4$ parties and a seat distribution of 21, 18, 7 and 4 seats, respectively. Assuming that voting is ruled by absolute majority and voting discipline holds within each party, its structure is described by the weighted majority game $v \equiv [26; 21, 18, 7, 4]$. The family of minimal winning coalitions is $W^m(v) = \{\{1,2\},\{1,3\},\{2,3,4\}\}$, so that $W(v) = \{\{1,2\},\{1,3\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}, \{1,2,3,4\}\}$ is the family of all winning coalitions. The expression of game v in terms of unanimity games is

$$v = u_{\{1,2\}} + u_{\{1,3\}} - u_{\{1,2,3\}} + u_{\{2,3,4\}} - u_{\{1,2,3,4\}}. \quad (9)$$

Let us assume that the basic ideological feature is defined by a classical left–to–right axis[7] where parties can be precisely located as for example in Fig. 2.

The Shapley value yields the following evaluation of the game:

$$\varphi[v] = (5/12, 3/12, 3/12, 1/12).$$

It is clear that the Shapley value strictly represents the relative strength of each party in the game, disregarding the effect, in the coalition formation process, due to the ideological positions of the involved parties. We wish to incorporate this exogenous information to the evaluation of the game by using a suitable probabilistic value.

Any probabilistic value ϕ is defined by a set $\{p^i_S\}$ of weighting coefficients for all $i \in N$ and all $S \subseteq N \backslash \{i\}$. For each $i \in N$, coefficients $\{p^i_S\}$ must provide a probability distribution on the family of coalitions $S \subseteq N \backslash \{i\}$. In our case ($n = 4$), 32 coefficients p^i_S are needed in principle. However, since the game is simple, we only have to define p^i_S when i is *crucial* for $S \cup \{i\}$ in v, i.e. when $S \notin W(v)$ but $S \cup \{i\} \in W(v)$ (we will set $S \in C_v(i)$ to denote this fact). This reduces the set to 12 coefficients,

$$p^1_{\{2\}}, p^1_{\{3\}}, p^1_{\{2,3\}}, p^1_{\{2,4\}}, p^1_{\{3,4\}},$$

$$p^2_{\{1\}}, p^2_{\{1,4\}}, p^2_{\{3,4\}}, p^3_{\{1\}}, p^3_{\{1,4\}}, p^3_{\{2,4\}}, p^4_{\{2,3\}},$$

and there are restrictions in choosing these coefficients for each $S \in C_v(i)$:

$$\text{all } p^i_S \geq 0 \quad \text{and} \quad \sum_{S \in C_v(i)} p^i_S \leq 1 \ \text{ for each } \ i.$$

Once the coefficients are chosen, we will simply have, from Eq. (3),

$$\phi_i[v] = \sum_{S \in C_v(i)} p^i_S. \quad (10)$$

Note that: (a) $\phi_i[v] \leq 1$ for all i, and (b) the total power is $\sum_{i \in N} \phi_i[v] \leq n$.

Given $\{p^i_S\}$, let $q^i(v)$ be the probability that i joins any coalition $S \notin C_v(i)$, i.e. such that i is not crucial in $S \cup \{i\}$. This is the amount of irrelevant probability that we may leave undefined. Then, from Eq. (10) it follows that $\phi_i[v] = 1 - q^i(v)$. Thus, *the more is probability $q^i(v)$ the less is the allocation that player*

[6] As to the additional information given by ideological constraints in politics, it is worthy of mention, at least incidentally, a singular example. In the general elections held in Greece in May 7 and June 17, 2012, the willingness of the parties to form *any* coalition was being, due to Greek economy's dramatic situation, much more decisive than the ideological constraints. Our model might well apply to study this situation. The profile components after May 7 were very low and led to an impasse, whereas they increased after June 17 and gave rise, finally, to a coalition government.

[7] A similar scheme could be applied if the relevant notion were nationalism (vs. centralism), as for example in regions like Catalonia or the Basque Country. Higher–dimensional ideological spaces might be treated in a similar but more complicated way.

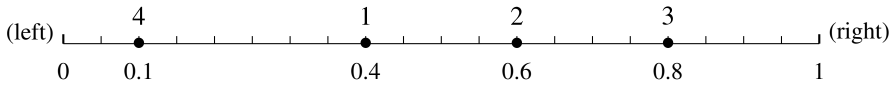

Figure 2: Party–distribution on a left–to–right axis.

i will get according to the corresponding probabilistic value.

How should we take into account the ideological constraints? At this point, it is worthy of mention that, in Weber's general model, p_S^i may well depend not only on i's interest in forming coalition $S \cup \{i\}$ but also on the opinion of the members of S as to joining (accepting) i. In other words, *coefficient p_S^i needs not being only a choice of i himself*. The multinomial values offer a reasonable solution to this since, given a profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, the corresponding value $\lambda^{\mathbf{p}}$ is defined by means of Eq. (6).

Thus, we will use multinomial values. It remains only to choose the profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ in terms of Fig. 2. An alternative interpretation of the profile in simple games is as follows (Carreras, 2004). There is a status quo Q and a proposal P to modify it. The action of the parliamentary members reduces to vote for or against P. Then *each p_i can be viewed as the probability that player i votes for P*. Since the result of a voting is essentially equivalent to forming a coalition (the coalition of players that vote for P), this interpretation of p_i agrees with that of "tendency to form a coalition" that we are using in this paper.

Step 1. Additional Assumption. Initially, we will assume that the coalition to form will also have an *ideological degree μ*, such that $0 \le \mu \le 1$. Then, it is natural to take p_i as the level of agreement of party i with this "coalitional" degree, i.e.

$$p_i = 1 - |\mu - \mu_i|, \qquad (11)$$

where μ_i is the position of party i in Fig.2. This is a simple but not too radical assumption. If $\mu_i \le \mu$ then p_i can vary between $1-\mu$ and 1, whereas if $\mu \le \mu_i$ then p_i can vary between μ and 1. As extreme cases, $p_i = 0$ if and only if either $\mu = 0$ and $\mu_i = 1$ or $\mu_i = 0$ and $\mu = 1$, and $p_i = 1$ if and only if $\mu = \mu_i$.

Step 2. A Particular Case. E.g., let us take $\mu = 0.5$. Then, by Eq. (11),

$$p_1 = 0.9, \quad p_2 = 0.9, \quad p_3 = 0.7, \quad p_4 = 0.6.$$

The weighting coefficients are given by Eq. (6). To compute $\lambda^{\mathbf{p}}[v]$ we can use Eq. (10) or, directly, Eq. (9), linearity, and the action of a multinomial value on unanimity games, given by Eq. (7). Then we obtain

$$\lambda_1^{\mathbf{p}}[v] = 0.592, \quad \lambda_2^{\mathbf{p}}[v] = 0.312,$$

$$\lambda_3^{\mathbf{p}}[v] = 0.144, \quad \lambda_4^{\mathbf{p}}[v] = 0.063.$$

Table 2: Parameters p_1, p_2, p_3, p_4 as functions of μ.

μ	p_1	p_2	p_3	p_4
$[0, 0.1]$	$0.6+\mu$	$0.4+\mu$	$0.2+\mu$	$0.9+\mu$
$[0.1, 0.4]$	$0.6+\mu$	$0.4+\mu$	$0.2+\mu$	$1.1-\mu$
$[0.4, 0.6]$	$1.4-\mu$	$0.4+\mu$	$0.2+\mu$	$1.1-\mu$
$[0.6, 0.8]$	$1.4-\mu$	$1.6-\mu$	$0.2+\mu$	$1.1-\mu$
$[0.8, 1]$	$1.4-\mu$	$1.6-\mu$	$1.8-\mu$	$1.1-\mu$

These allocations are the result of combining both the strategic position of each party in the game and its ideological relevance in forming a "politically balanced" coalition ($\mu = 0.5$). Notice that the symmetry of 2 and 3 in the game, reflected by the Shapley value, has been broken. The total power is $\pi^{\lambda^{\mathbf{p}}}(v) = 1.111$.

Looking at $q^i(v)$ we find

$$q^1(v) = 0.408, \quad q^2(v) = 0.688,$$
$$q^3(v) = 0.856, \quad q^4(v) = 0.937.$$

These amounts represent the probability wasted by each party in joining coalitions where it is not crucial. For example, party 1 is not crucial in $\{1\}$, $\{1,4\}$ and $\{1,2,3,4\}$, and $q^1(v)$ is therefore the probability that party 1 joins $\emptyset$, $\{4\}$ or $\{2,3,4\}$. This waste of probability is the effect of the choice of p_1 but also of p_2, p_3, p_4.

Step 3. Arbitrary Ideological Degree. Now we proceed for a general μ. Then, from Eq. (11), we have the results displayed in Table 2. So we get the multinomial value $\lambda^{\mathbf{p}}[v]$ in terms of μ:

$$\lambda_1^{\mathbf{p}}[v] = \begin{cases} -\mu^3 - 2.5\mu^2 + 0.78\mu + 0.448, \\ \mu^3 - 1.5\mu^2 + 0.82\mu + 0.432, \\ -\mu^3 + 3.5\mu^2 - 2.62\mu + 1.128, \\ \mu^3 - 5.5\mu^2 + 8.02\mu - 2.648, \end{cases}$$

if, respectively,

$$0 \le \mu \le 0.1, \qquad 0.1 \le \mu \le 0.6,$$
$$0.6 \le \mu \le 0.8, \quad 0.8 \le \mu \le 1,$$

and similar expressions for the remaining values $\lambda_i^{\mathbf{p}}[v]$ for $i = 2, 3, 4$.

Finally, if we wish to aggregate these results and obtain a single evaluation of the relative strength of each party in the coalition formation process *in abstracto*, i.e. without prescribing any ideological degree μ to the coalition, it suffices to integrate the multinomial value of each party with respect to μ, thus getting

$$\xi_1[v] = \int_0^1 \lambda_1^{\mathbf{p}}[v] d\mu \approx 0.6333$$

and, similarly,

$$\xi_2[v] \approx 0.3365, \quad \xi_3[v] \approx 0.2681, \quad \xi_4[v] \approx 0.1393.$$

Remark 5.2 An important difference between the Shapley value assessment and the result of applying a (multinomial or not) probabilistic value is that the former is efficient whereas the latter, in general, is not (Weber, 1988). For this reason we speak of *relative* strength. If the results have to be applied to sharing political responsibilities, they can be directly applied in the Shapley value case by efficiency, whereas a *normalization process*, similar to that of the original Banzhaf power index, is needed in the multinomial case, by defining

$$\Lambda_i^{\mathbf{p}}[v] = \frac{\lambda_i^{\mathbf{p}}[v]}{\pi^{\lambda^{\mathbf{p}}}(v)} = \frac{\lambda_i^{\mathbf{p}}[v]}{\sum_{j \in N} \lambda_j^{\mathbf{p}}[v]}$$

for each $i \in N$ and any $v \in \mathcal{G}_N$ for which this normalization makes sense. The normalization may of course be applied also to the single evaluation $\xi[v]$ obtained in Step 3, giving normalized values

$$\overline{\xi}_1[v] \approx 0.4598, \quad \overline{\xi}_2[v] \approx 0.2443,$$
$$\overline{\xi}_3[v] \approx 0.1947, \quad \overline{\xi}_4[v] \approx 0.1012.$$

Which is therefore the meaning of the results obtained in Step 3? In the same way as one accepts the Shapley value of the game as an a priori evaluation of the relative strength of each player in the coalition formation bargaining, the values just obtained represent an analogous a priori evaluation of this relative strength *when the political relationships between the parties are taken into account*. The differences between our (normalized) assessment and the mere evaluation of the game provided by the Shapley value are interesting: if $\Delta_i[v] = \overline{\xi}_i[v] - \varphi_i[v]$ for all i, then

$$\Delta_1[v] = 0.0431, \quad \Delta_2[v] = -0.0057,$$
$$\Delta_3[v] = -0.0553, \quad \Delta_4[v] = 0.0179.$$

This indicates that the political relationships in this particular game improve party 1 strongly (around 10.34%) and party 4 very strongly (around 21.37%), while they damage party 2 very slightly (around 2.28%) and party 3 very strongly (around 22.12%).

ACKNOWLEDGEMENTS

Research supported by Grant SGR 2009–01029 of the Catalonia Government (*Generalitat de Catalunya*) and Grant MTM 2012–34426 of the Economy and Competitiveness Spanish Ministry.

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